Current Density Impedance Imaging

Adrian I. Nachman
University of Toronto

Banff Oct. 2009

1. CDI
M. Joy, G. Scott, R. Henkelman

2. CDII (two currents)
with M. Joy, A. Ma, K. Hasanov.

3. CDII (magnitude of one current)
with A. Tamasan and A. Timonov.

Partially supported by MITACS, NSERC, CITO, Phillips Heartstream, ORDCF, FMI.

1. How do you make a CD image using an MR imager?

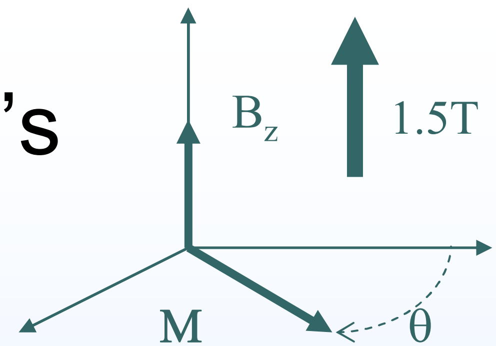
- The imager yields an array of complex numbers related to the nuclear magnetization at points inside the object.
- The magnitude, *m*, forms the standard MR image .
- An applied low-frequency current creates a magnetic field which affects the phase image.

$$Z = me^{j\theta}$$

Magnitude MR Image

Phase Image

LF CDI is based on Ampere's law $\mathbf{J} = \nabla \times \mathbf{B} / \mu_0$



Where **B** is the magnetic field produced by the current density **J**

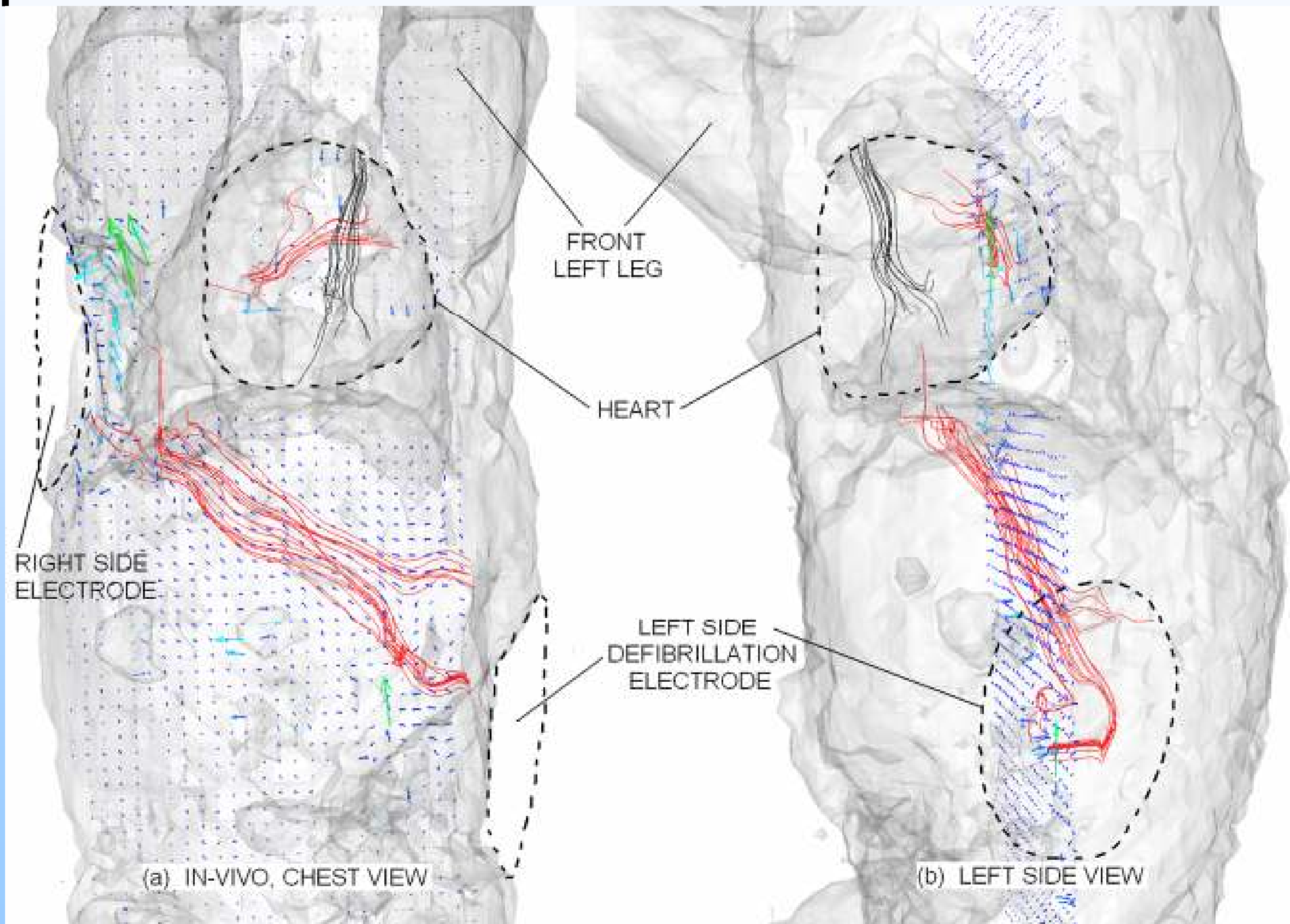
- The phase θ depends linearly on the magnetic field component B_Z produced by the current density **J** and the duration of the current pulse T_C.

$$\theta = \gamma B_Z T_C$$

$$\text{where } \vec{\mathbf{B}} = \begin{pmatrix} B_X & B_Y & B_Z \end{pmatrix}$$

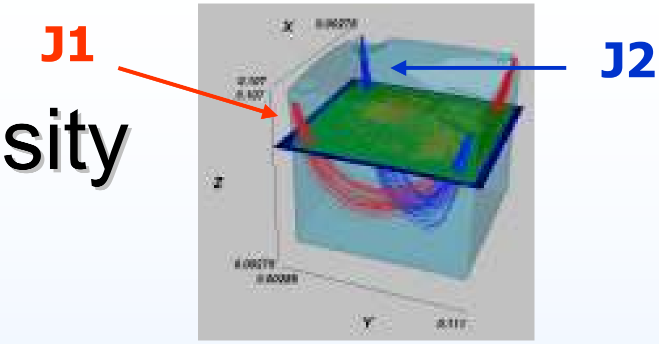
- To measure B_X and B_Y we rotate the object. (Seo et al. use only B_Z)

Defibrillator Currents in a Live Pig



- 1.5 Tesla GE Signa
- 85 mA x 4.6 ms
- 3.8mm cubical

2. **CDII** (Current Density Impedance Imaging)



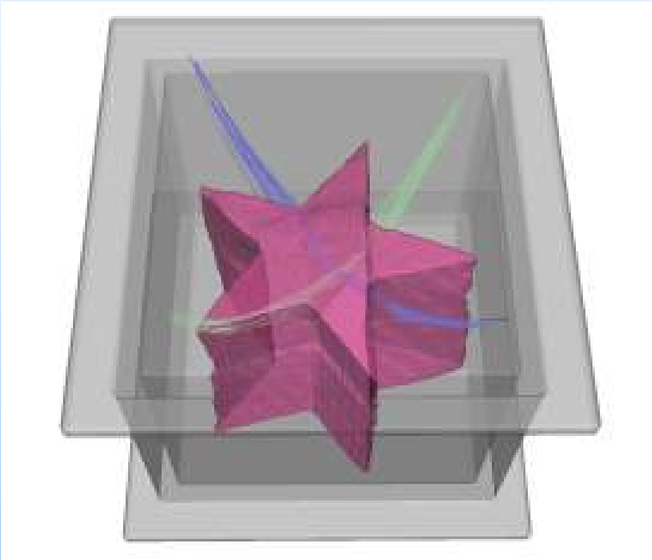
$$\nabla \ln(\sigma) = \left[\begin{array}{r} \left(\nabla \times \mathbf{J}_2 \bullet \left(\mathbf{J}_1 \times \mathbf{J}_2\right)\right) \dfrac{\mathbf{J}_1}{\left|\mathbf{J}_1 \times \mathbf{J}_2\right|^2} \\ -\left(\nabla \times \mathbf{J}_1 \bullet \left(\mathbf{J}_1 \times \mathbf{J}_2\right)\right) \dfrac{\mathbf{J}_2}{\left|\mathbf{J}_1 \times \mathbf{J}_2\right|^2} \\ +\left(\nabla \times \mathbf{J}_1 \bullet \mathbf{J}_2\right) \dfrac{\mathbf{J}_1 \times \mathbf{J}_2}{\left|\mathbf{J}_1 \times \mathbf{J}_2\right|^2} \end{array} \right]$$

where σ = electrical conductivity.

Independently discovered by J. Y. Lee (2004)

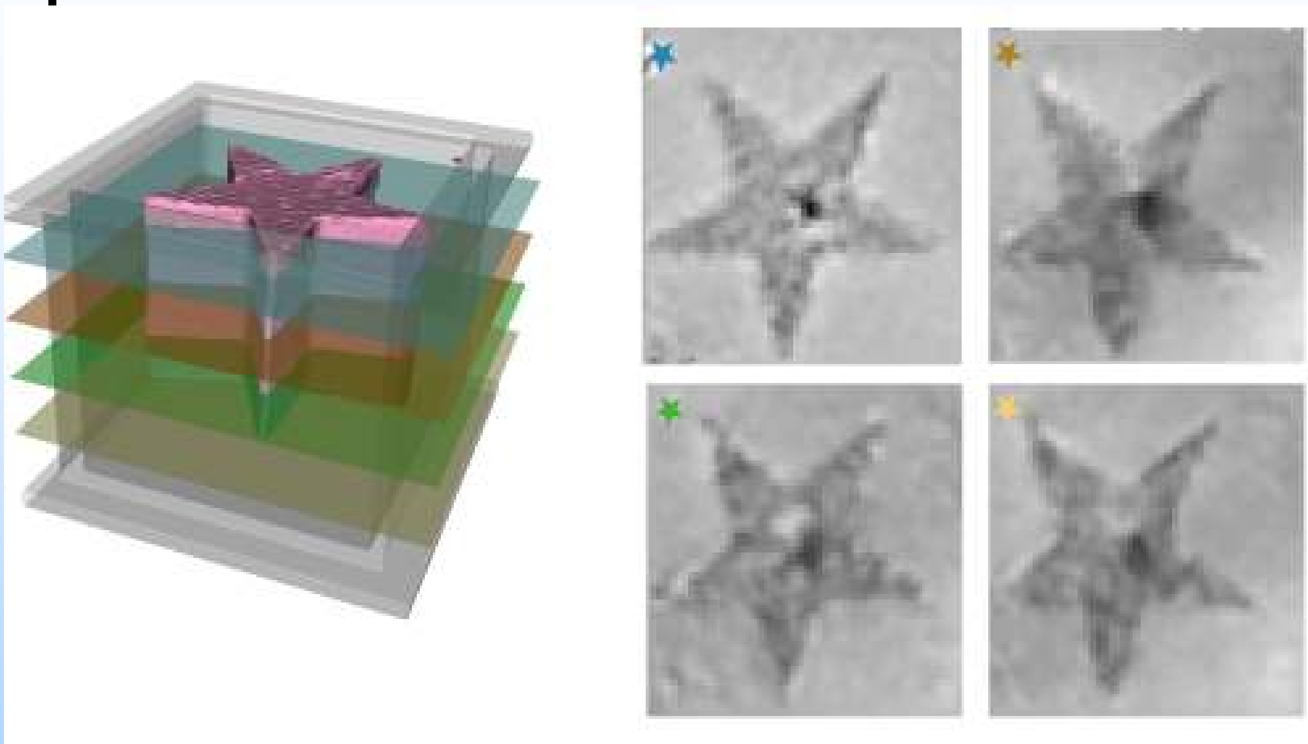
Experimental Verification

An agar + TX151 gel in saline “popsicle” phantom



Reconstructed currents

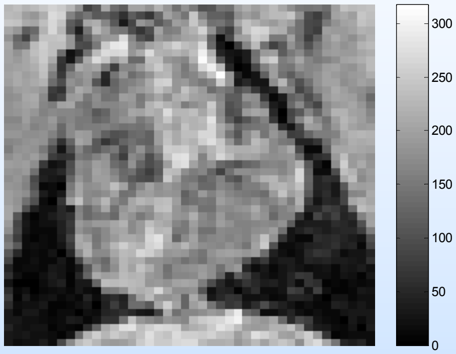
Depth Resolution in CDII



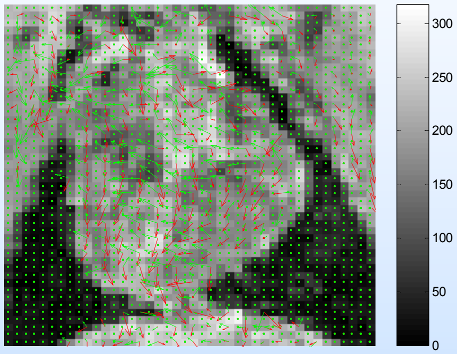
- Distance from electrodes from 30 to 80 mm.
- Average conductivity contrast ratio 1.21
- Validated with careful direct bench measurements
- Resolution is maintained with depth.

CDII of a LIVE piglet (Very Preliminary Results !)

- CDI study
  not geared towards CDII !



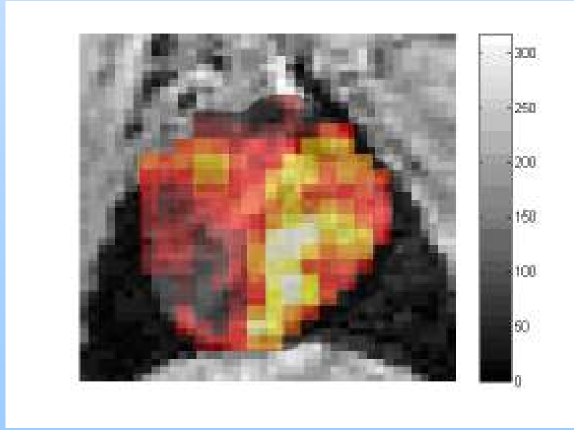
MRI image 2 mm resolution

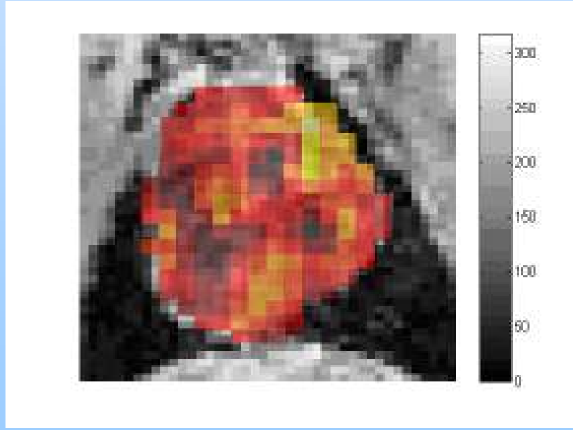


CDI -Two current vector fields

- 4mm resolution (to cut down acquisition time)
- Motion artefacts
- No averaging
- Conductivity range

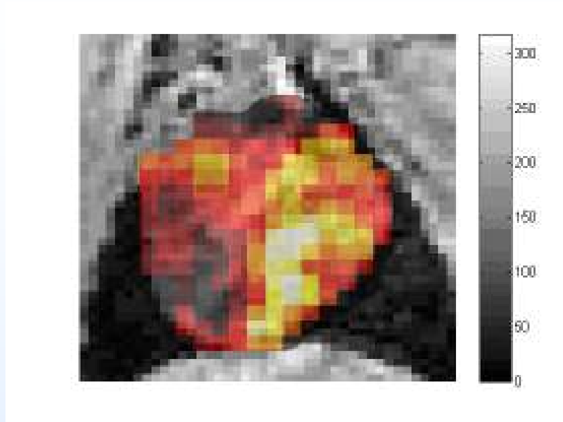
0.8 -->2.0 s/m

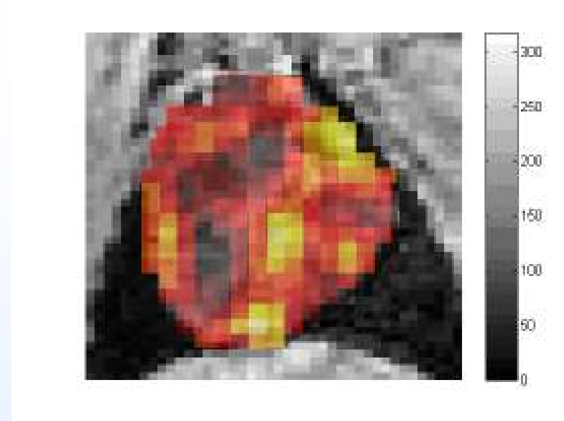


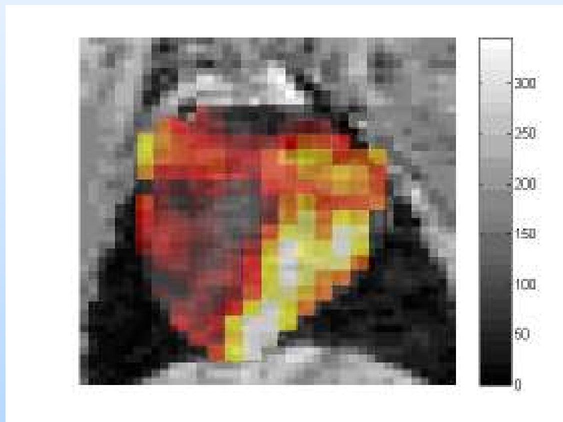


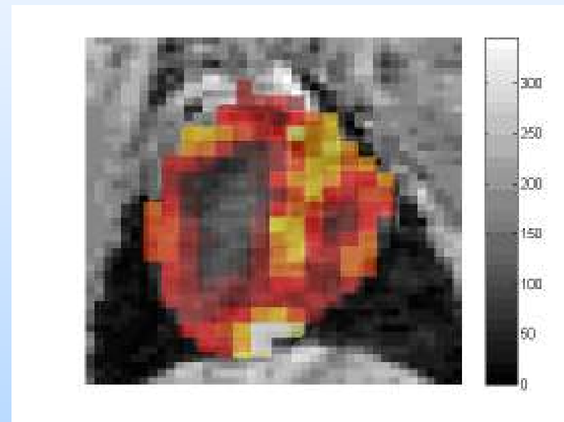
CDII overlaid on MRI (slice 39, two different times)

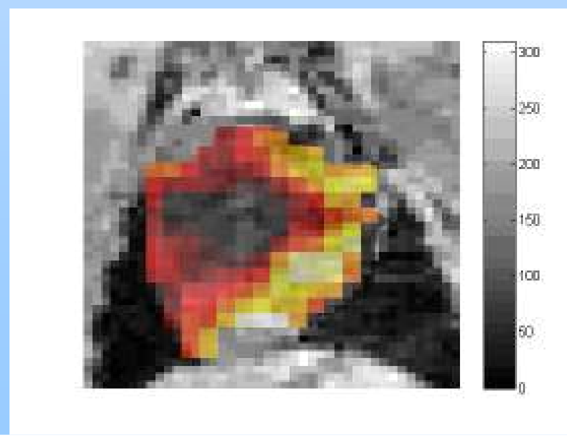
Slice 40
Slice 41
Slice 42

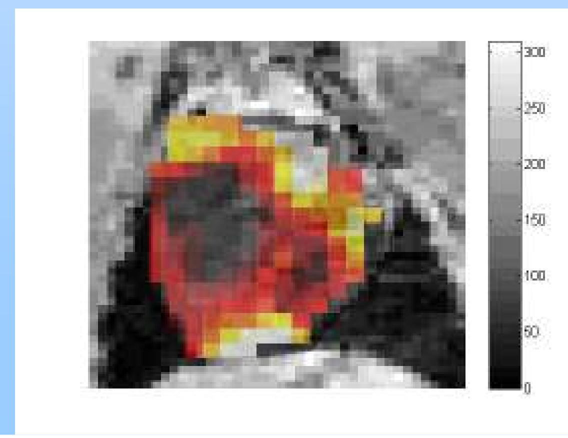












3. What if we only measure **one** current, in fact only its **magnitude |J| ?**

Motivation:

- Cuts down acquisition time.
- Math turns out to be beautiful: this brings Minimal Surfaces, Geometric Measure Theory into the field of Inverse Problems.
- Opens up the possibility of another physical approach to obtain such data directly.

Equipotential surfaces = minimal surfaces (in 2D geodesics)

Theorem 1 (N-Tamasan-Timonov '07) *If $u \in C^1(\Omega)$ is an electric potential with current density J, $|J| > 0$, then the level sets $\Sigma_c = \{x : \ u(x) = c\}$ are surfaces of zero mean curvature in the conformal metric $g = |J|^{2/(n-1)} I$; They are critical surfaces for the functional*

$$E(\Sigma) = \int_\Sigma |J| dS, \tag{1}$$

where dS is the Euclidean surface measure.

Example of non-uniqueness in the Dirichlet Problem (Sternberg& Ziemer)

$$\nabla \cdot \left(|\nabla u(x)|^{-1} \nabla u(x) \right) = 0, \ x \in D \equiv unit \ \ disk,$$
$$u(x) = (x_1)^2 - (x_2)^2, \ x \in \partial D.$$

One parameter family of viscosity solutions:

$$u^\lambda(x) = \begin{cases} 2(x_1)^2 - 1, & if \quad |x_1| \geq \sqrt{\frac{1+\lambda}{2}}, \ |x_2| \leq \sqrt{\frac{1-\lambda}{2}}, \\ \lambda, & if \quad |x_1| \leq \sqrt{\frac{1+\lambda}{2}}, \ |x_2| \leq \sqrt{\frac{1-\lambda}{2}}, \\ 1 - 2(x_2)^2, & if \quad |x_1| \leq \sqrt{\frac{1+\lambda}{2}}, \ |x_2| \geq \sqrt{\frac{1-\lambda}{2}}. \end{cases}$$

Consider a minimization problem instead!

$$\min\left\{\int_\Omega |J(x)|\cdot|\nabla u(x)|dx : \ u|_{\partial\Omega} = f\right\}$$

- Formally the Euler-Lagrange equation is

$$\nabla \cdot (\frac{|J|}{|\nabla u|}\nabla u) = 0.$$

- In the SZ example only u^0 (for $\lambda = 0$) is a minimizer of $\int_\Omega |\nabla u(x)|dx$.

Note: u^0 cannot come from a conductivity ($|J| \equiv 1$).

The voltage potential u_0 is a minimizer for $F[u]$

$\nabla \cdot \sigma \nabla u_0 = 0$, $u_0|_{\partial\Omega} = f$, $a = \sigma|\nabla u_0|$, ν=outer normal at $\partial\Omega$, and Λ_σ= Dirichlet-to-Neumann map:

$$F[u] = \int_\Omega a|\nabla u|dx = \int_\Omega \sigma|\nabla u_0| \cdot |\nabla u|dx \geq \int_\Omega \sigma|\nabla u_0 \cdot \nabla u|dx$$
$$\geq \int_\Omega \sigma \nabla u_0 \cdot \nabla u dx = \int_{\partial\Omega} \sigma \frac{\partial u_0}{\partial \nu} u ds = \langle \Lambda_\sigma f, f \rangle.$$

The lower bound is achieved when $u = u_0$.

Definition: admissible pair $(f, a) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$

$\exists \sigma(x)$ with $0 < c_- \leq \sigma(x) \leq \sigma_+$, such that, if u_σ is weak solution

$$\nabla \cdot \sigma \nabla u_\sigma = 0, \;\; u_\sigma|_{\partial\Omega} = f,$$

then

$$|\sigma \nabla u| = a.$$

σ = **generating conductivity** for the pair (f, a)

u = **corresponding potential**.

Note:

- the pair $(f, |J|)$ for ideal measurements is admissible;
- But $((x_1)^2 - (x_2)^2|_{\partial D}, 1)$ is not admissible.

A characterization of admissibility

$\Omega \subset R^n$ a domain and $(f, |J|) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$.

- If $(f, |J|)$ is admissible, say generated by some conductivity σ_0 and with u_0 is the corresponding voltage potential, then

$$u_0 \in argmin\left\{F[u] : \ u \in H^1(\Omega),\ u|_{\partial\Omega} = f\right\}$$

  and $|J|/|\nabla u_0| \in L^\infty_+(\Omega)$.

- Conversely, if $u_0 \in argmin\left\{F[u] : \ u \in H^1(\Omega),\ u|_{\partial\Omega} = f\right\}$ and $|J|/|\nabla u_0| \in L^\infty_+(\Omega)$, then $(f, |J|)$ is admissible.

Unique determination

Theorem 2 (N-Tamasan-Timonov '09)

$\Omega \subset R^n$ = *domain with connected,* $C^{1,\alpha}$*- boundary*

$(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega})$ = *admissible pair,* $|J| > 0$ *a.e. in* Ω*.*

Then $\min \int_\Omega |J||\nabla u| dx$

$$over \ \left\{ u \in W^{1,1}(\Omega) \bigcap C(\overline{\Omega}),\ |\nabla u| > 0\ a.e.,\ u|_{\partial\Omega} = f \right\}$$

has a unique solution, say u_0*;*

$\sigma = |J|/|\nabla u_0|$ *is the unique conductivity generating* $(f, |J|)$.

Remark There is also a corresponding stability result (Nashed-Tamasan' 09).

A minimization algorithm

$$\min F[u] = \min \int_\Omega a(x)|\nabla u(x)|dx.$$

Let $\Omega \subset \mathbb{R}^n$. For $u_{n-1} \in H^1(\Omega)$ given with $\frac{a}{|\nabla u_{n-1}|} \in L_+^\infty(\Omega)$ define

$$\sigma_n = \frac{a}{|\nabla u_{n-1}|}$$

and construct u_n as the unique solution to

$$\begin{cases} \nabla \cdot \sigma_n \nabla u_n = 0, \\ u_n|_{\partial\Omega} = f. \end{cases}$$

Sufficient conditions for the iterations to make sense

Ω be a $C^{1,\alpha}$ simply connected domain in R^2,

$a \in C^\alpha(\overline{\Omega})$, $a > 0$,

$f \in C^{1,\alpha}(\partial\Omega)$, almost-two-to-one.

Then

- $u_n \in C^{1,\alpha}(\overline{\Omega})$, $\sigma_n := a/|\nabla u_{n-1}| \in C^\alpha(\overline{\Omega})$.
- $F[u_n] = \int_\Omega a|\nabla u_n|dx > 0$ is decreasing,
- $\lim_{n\to\infty} F[u_n] = \lim_{n\to\infty}\langle\Lambda_{\sigma_n} f, f\rangle$;
- $\lim_{n\to\infty} \int_\Omega \sigma_n|\nabla u_{n-1} - \nabla u_n|^2 dx = 0$
- $\lim_{n\to\infty} \int_\Omega a|\nabla u_n - \nabla u_{n-1}|dx = 0$.

A sufficient condition for being a minimizing sequence

A uniform upper bound: $\sigma_n \leq M, \ \forall\, n$.

Then

$$\lim_{n\to\infty} F[u_n] = \min\{\int_\Omega a|\nabla u|dx :\ u \in H^1(\Omega),\ u|_{\partial\Omega} = f\}.$$

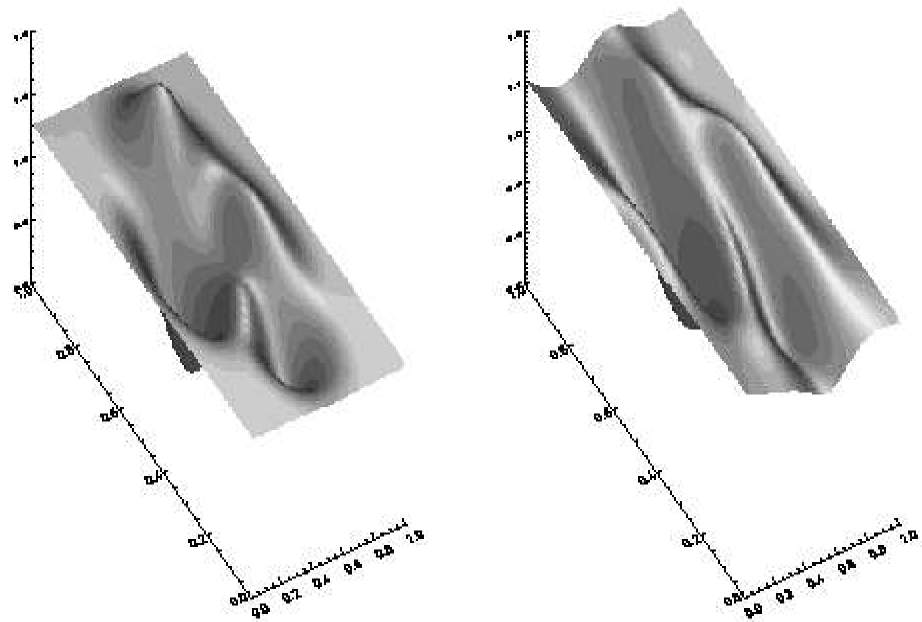


Figure 1: The original conductivity distribution (left) and the initial approximation (right)

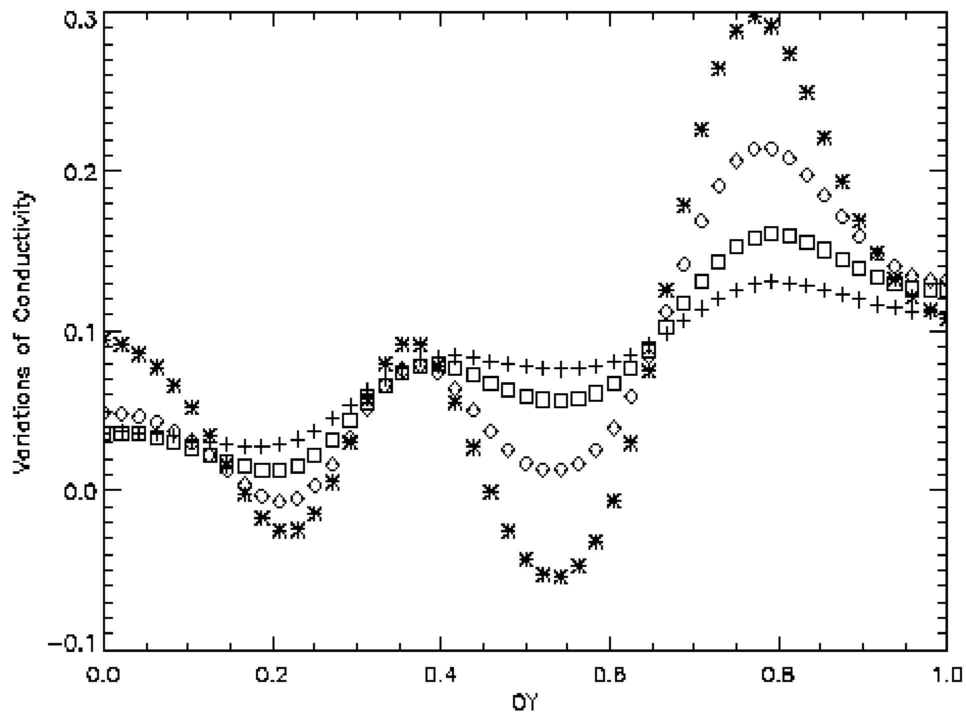


Figure 1: One slice: += initial approximation, □= 5 iter., ◇= 50 iter., and ⁂= 100 iter (indistinguishable from the simulated conductivity)

Local uniqueness from partial data

Theorem 3 (N-Tamasan-Timonov '09) *Let* $\Omega \subset \mathbb{R}^2$ *be simply connected. For* $i = 1, 2$ *let* $\sigma_i \in C^\alpha(\Omega)$ *and* u_i *be* σ_i*-harmonic with* $u_i|_{\partial\Omega} \in C^{1,\alpha}(\partial\Omega)$ *almost two-to-one. For* $a < b$ *let*

$$\Omega_{a,b} := \{x \in \overline{\Omega} : \; a < u_1(x) < b\}, \;\; \Gamma := \Omega_{a,b} \bigcap \partial\Omega.$$

Assume $u_1|_\Gamma = u_2|_\Gamma$ *and* $|J_1| = |J_2|$ *in the interior of* $\Omega_{a,b}$.

Then

$$\{x \in \overline{\Omega} : \; a < u_2(x) < b\} = \Omega_{a,b}, \tag{1}$$

$$u_1 = u_2 \; in \; \Omega_{a,b}, \;\; and \tag{2}$$

$$\sigma_1 = \sigma_2 \; in \; \Omega_{a,b}. \tag{3}$$

Reconstruction from partial data

- The two-point boundary value problem for geodesics joining equipotential points at the boundary has a unique solution.
- We solve above numerically.
- More accurate than the quasi-iteration, but slower.
- Only requires $u|_\Gamma$ and $|J|$ in the region of interest.