Complex Analysis and Complex Geometry

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1 Overview of the Field

Complex analysis and complex geometry can be viewed as two aspects of the same subject. The two are inseparable, as most work in the area involves interplay between analysis and geometry. The fundamental objects of the theory are complex manifolds and, more generally, complex spaces, holomorphic functions on them, and holomorphic maps between them. Holomorphic functions can be defined in three equivalent ways as complex-differentiable functions, as sums of complex power series, and as solutions of the homogeneous Cauchy-Riemann equation. The threefold nature of differentiability over the complex numbers gives complex analysis its distinctive character and is the ultimate reason why it is linked to so many areas of mathematics.

Plurisubharmonic functions are not as well known to nonexperts as holomorphic functions. They were first explicitly defined in the 1940s, but they had already appeared in attempts to geometrically describe domains of holomorphy at the very beginning of several complex variables in the first decade of the 20th century. Since the 1960s, one of their most important roles has been as weights in a priori estimates for solving the Cauchy-Riemann equation. They are intimately related to the complex Monge-Ampère equation, the second partial differential equation of complex analysis. There is also a potential-theoretic aspect to plurisubharmonic functions, which is the subject of pluripotential theory.

In the early decades of the modern era of the subject, from the 1940s into the 1970s, the notion of a complex space took shape and the geometry of analytic varieties and holomorphic maps was developed. Also, three approaches to solving the Cauchy-Riemann equations were discovered and applied. First came a sheaf-theoretic approach in the 1950s, making heavy use of homological algebra. Hilbert space methods appeared in the early 1960s, and integral formulas around 1970 through interaction with partial differential equations and harmonic analysis. The complex Monge-Ampère equation came to the fore in the late 1970s with Yau's solution of the Calabi conjectures and Bedford and Taylor's work on the Dirichlet problem.

Most current work in complex analysis and complex geometry can be seen as being focused on one or both of the two fundamental partial differential equations, Cauchy-Riemann and Monge-Ampère, in the setting of Euclidean space or more general complex manifolds. The past ten years have seen an increasing thrust towards extending both the theory and its applications to singular spaces, to almost complex manifolds, and to infinite-dimensional manifolds.

Today, as before, complex analysis and complex geometry is a highly interdisciplinary field. The foundational work described above has been followed by a broad range of research at the interfaces with a number of other areas, such as algebraic geometry, functional analysis, partial differential equations, and symplectic geometry, to name a few. Complex analysts and complex geometers share a common toolkit, but find inspiration and open problems in many areas of mathematics.

2 Recent Developments and Open Problems

1. Analytic methods in complex algebraic geometry are based on increasingly sophisticated ways of solving the Cauchy-Riemann equation (often also called the $\bar{\partial}$-equation) with L^2-estimates using plurisubharmonic weights in geometric settings. Yum-Tong Siu has long been a leader in this area. His announcement of an analytic proof of the finite generation of the canonical ring of a smooth complex projective variety of general type [27] came on the heels of an algebraic proof by Birkar, Cascini, Hacon, and McKernan. This is a milestone in algebraic geometry.

Bo Berndtsson and Mihai Paun use analytic methods to obtain a nearly optimal criterion for the pseudo-effectivity of relative canonical bundles and give several applications in algebraic geometry [2]. Shigeharu Takayama uses analytic techniques, including multiplier ideal sheaves, to extend Siu's celebrated result on the invariance of plurigenera from the smooth case to the case of fibres with canonical singularities [28]. So far, there is no known algebraic proof of the full result.

Currents are differential forms with distribution coefficients; closed currents satisfying a certain positivity condition are objects of fundamental importance that generalize both Kähler forms and analytic subvarieties. New work of Tien-Cuong Dinh and Nessim Sibony advances the basic theory of currents (intersections, pullbacks, etc.) and has many potential applications ([12], [13]).

2. Pluripotential theory and the Monge-Ampère equation. Pluripotential theory on compact Kähler manifolds, based on the notion of a quasiplurisubharmonic function, has been developed by Vincent Guedj and Ahmed Zeriahi since 2004 (starting with [18]). Whereas plurisubharmonic functions have a positive Levi form by definition and are constant on compact manifolds, quasiplurisubharmonic functions are allowed to have a negative Levi form down to a fixed lower bound and lead to a fruitful pluripotential theory in a compact setting. Guedj, Zeriahi, and Philippe Eyssidieux posted a major application of this theory [14] (posted in March 2006). They extended the work of Aubin and of Yau on the complex Monge-Ampère equation to certain singular settings and proved that the canonical model of a smooth complex projective variety of general type (proved to exist soon afterwards by Birkar et al. and by Siu—this is equivalent to finite generation of the canonical ring) has a Kähler-Einstein metric of negative Ricci curvature. This is only one example, albeit a very important one, of current work on the complex Monge-Ampère equation.

The highly nonlinear nature of the Monge-Ampère operator presents many challenges. Proper understanding of its maximal domain of definition in the local case of a domain in Euclidean space was obtained only recently in work of Zbigniew Błocki [3]. Surprisingly, very recent work of Guedj, Zeriahi, and Dan Coman has shown the domain of definition to be much larger in the global case of a compact Kähler manifold [8].

Coman and Evgeny Poletsky have derived new Bernstein, Bezout, and Markov inequalities using pluripotential theory and applied them to transcendental number theory [9]. Their results have been generalized by Alexander Brudnyi [7]. In a series of papers, the earliest posted in 2002, Charles Favre and Mattias Jonsson have made a deep study of the singularities of plurisubharmonic functions and multiplier ideals in two dimensions using the novel concept of a tree of valuations [15]. In a new paper [6] with Sebastien Boucksom, they have extended some of their work to higher dimensions. A connection with probability theory appears in recent work of Thomas Bloom and Bernard Shiffman [4] and of Robert Berman [1], who use various techniques of pluripotential theory to study zeros of random polynomials and, more generally, random sections of holomorphic line bundles.

3. The Cauchy-Riemann equation on singular spaces. Existing methods for solving the Cauchy-Riemann equation are largely restricted to smooth spaces. Consequently, the central problem of classical several complex variables, the Levi problem, which asks whether Steinness is a local property and was solved for manifolds decades ago, is still open for singular spaces. Progress in this area has been slow and difficult. Recently, John Erik Fornæss, Nils Øvrelid, and Sophia Vassiliadou have been able to solve the Cauchy-Riemann equations with L^2-estimates in certain singular settings ([16], [17]).

4. Almost complex geometry. In a seminal paper of 1985, Mikhail Gromov introduced almost complex structures and pseudoholomorphic curves into symplectic topology. Interaction between complex geometry and symplectic geometry began in earnest with the work of Sergey Ivashkovich and Vsevolod Shevchishin in the late 1990s [20]. There is now a growing body of work concerned with extending concepts and results from complex analysis and complex geometry to the almost complex case. Often the non-integrable case requires new methods that shed light on the integrable case. Notable new work includes a paper by Bernard Coupet, Alexander Tumanov, and Alexander Sukhov on proper pseudoholomorphic discs [11], a long paper on fundamentals of local almost complex geometry by Coupet, Sukhov, and Hervé Gaussier [10], a paper by Xianghong Gong and Jean-Pierre Rosay on removable singularities of pseudoholomorphic maps [19], and a paper by Ivashkovich and Shevchishin on almost complex structures that are merely Lipschitz [21].

5. Infinite-dimensional complex geometry. Complex analysis in infinite dimensions languished outside the mainstream until László Lempert commenced a major research program in the mid-1990s. Generalized loop spaces (spaces of smooth maps from a compact smooth manifold into a finite-dimensional complex manifold) are examples of infinite-dimensional complex manifolds; their importance in physics provides strong motivation for Lempert's program. Fundamental notions, including Dolbeault cohomology, coherence of analytic sheaves, and holomorphic approximation, have been brought into an infinite-dimensional setting by Lempert, in part with coauthors Endre Szabó, Ning Zhang, and Imre Patyi ([24], [23], [25]). Imre Patyi has studied the Oka principle for infinite-dimensional complex manifolds [26].

3 Presentation Highlights

Zbigniew Błocki (Jagiellonian University)
On geodesics in the space of Kähler metrics
Our main result is that geodesics in the space of Kähler metrics (as considered by Mabuchi, Donaldson and Semmes) are (fully) $C^{1,1}$, provided that the bisectional curvature is nonnegative. Existence of such geodesics (without curvature assumption) with bounded mixed complex 2nd derivatives was proved by X. X. Chen. It boils down to solving a homogeneous complex Monge-Ampere equation on a compact Kähler manifold with boundary. We also discuss slightly more general equations of this kind.

Sebastien Boucksom (Institut de mathématique de Jussieu)
Equilibrium measures and equidistribution of Fekete points on complex manifolds
Fekete points are optimal configurations of points in polynomial interpolation. It is a classical result that Fekete points confined within a given compact set of the complex plane equidistribute towards the potential-theoretic equilibrium measure of the compact set. I will present a joint work with Robert Berman where we extend this result to the higher-dimensional case by a variational principle, working in the more geometric setting of sections of a line bundle over a compact complex manifold

Debraj Chakrabarti (Notre Dame University)
CR functions on subanalytic hypersurfaces
We consider the problem of local one-sided holomorphic extension of continuous or smooth CR functions from hypersurfaces with singularities, in particular from the class of subanalytic hypersurfaces, which include the real-analytic ones. We discuss the obstructions to the existence of such extension, which turn out to be different from those in the classical smooth case.

Bruno De Oliveira (University of Miami)
Symmetric differentials, differential operators and the topology of complex surfaces
The space of symmetric differentials of order 1, i.e. holomorphic 1-forms, are intimately connected with the topology of a complex surface. On the other hand, the same does not happen for symmetric differentials of higher order. Examples of this difference are: There are families of algebraic surfaces where $h^0(X_t, S^m\Omega^1_{X_t})$ is not locally constant for $m > 1$, a simply connected surface X can have nontrivial symmetric differentials of order $m > 1$ (in fact Ω^1_X can be ample). To regain the connection with the topology we need to consider a special class of symmetric differentials, we call these differentials closed, they are locally of the form $df_1...df_m$. Opposite to closed differential forms we will show that there is no collection of differential operators characterizing closed symmetric differentials, but as we will see this can be done if we ask to be closed

around a general point. A special case of a topological result to be presented is that if a complex surface X has a nontrivial closed symmetric differential of order 2 then $\pi_1(X) \neq 0$.

Xianghong Gong (University of Wisconsin)
Regularity in the CR embedding problem
We will prove a new regularity on the local embedding of strongly pseudoconvex CR manifolds of dimension at least 7. This is joint work with Sidney Webster.

Vincent Guedj (Université Aix-Marseille)
Variational approach to complex Monge-Ampère equations
I will present a new variational approach to Monge-Ampere equations on compact complex manifolds, which enbles to construct singular solutions to the Dirichlet problem without relying on Yau's fundamental existence result. This is joint work with R. Berman, S. Boucksom and A. Zeriahi.

Gordon Heier (University of California, Riverside)
On complex projective manifolds of negative holomorphic sectional curvature
It is a long-standing open problem to show that a complex projective manifold with a Kähler metric of negative holomorphic sectional curvature has an ample canonical line bundle. In this talk, partial results towards this problem will be presented. This is joint work in progress with Bun Wong.

Alexander Isaev (Australian National University)
Infinite-dimensionality of the automorphism groups of homogeneous Stein manifolds
Let X be a Stein manifold of dimension greater than 1 homogeneous with respect to a holomorphic action of a complex Lie group. We show that the Lie algebra generated by complete holomorphic vector fields on X is infinite-dimensional, i.e. it is impossible to introduce the structure of a Lie transformation group on the group of holomorphic automorphisms of X. The well-known examples of complex linear space and affine quadric fit into this general situation. The work is joint with Alan Huckleberry.

Sergey Ivashkovich (Université de Lille-1)
Vanishing cycles in holomorphic foliations by Riemann surfaces and foliated shells
The purpose of this talk is the study of vanishing cycles of holomorphic foliations by Riemann surfaces on compact complex manifolds. The notion of a *vanishing cycle* was implicitly introduced by S. Novikov in his proof of the existence of compact leaves in smooth foliations by surfaces on the three-dimensional sphere. Later it appeared as an obstruction to the simultaneous uniformizability of the object known as a *skew cylinder*, introduced by Ilyashenko, which is proved to be an extremely useful tool in foliation theory. Our main result consists in showing that a vanishing cycle comes together with a much richer complex geometric object—we call this object a *foliated shell.* A number of related statements will be given and several open questions will be discussed.

Burglind Juhl-Jöricke (IHES)
Envelopes of holomorphy and holomorphic discs
The envelope of holomorphy of an arbitrary domain in a Stein manifold is identified with a connected component of the set of equivalence classes of analytic discs immersed into the Stein manifold with boundary in the domain. This has several corollaries, in particular, in case of dimension two for each of its points the envelope of holomorphy contains an embedded (non-singular) Riemann surface passing through this point with boundary contained in the natural embedding of the original domain into its envelope of holomorphy. The method has applications also for the case of projective manifolds.

Nikolay Kruzhilin (Steklov Mathematical Institute)
Holomorphic maps of Reinhardt domains
Nonbiholomorphic proper maps from bounded Reinhardt domains are considered. The structure of the boundary of the source domain and the boundary behavior of the map are investigated. The role of the target domain is discussed.

Frank Kutzschebauch (Universität Bern)
A solution of Gromov's Vaserstein problem
It is standard material in a linear algebra course that the group $\mathrm{SL}_m(\mathbb{C})$ is generated by elementary matrices $E + \alpha e_{ij}, i \neq j$, i.e., matrices with 1's on the diagonal and all entries outside the diagonal are zero, except one entry. Equivalently every matrix $A \in \mathrm{SL}_m(\mathbb{C})$ can be written as a finite product of upper and lower diagonal unipotent matrices (in interchanging order). The same question for matrices in $\mathrm{SL}_m(R)$ where R is a commutative ring instead of the field $\mathbb{C}$ is much more delicate. For example if R is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a space X the problem amounts to find for a given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ a factorization as a product of upper and lower diagonal unipotent matrices

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

where the G_i are maps $G_i : X \to \mathbb{C}^{m(m-1)/2}$. Since any product of (upper and lower diagonal) unipotent matrices is homotopic to a constant map (multiplying each entry outside the diagonals by $t \in [0, 1]$ we get a homotopy to the identity matrix), one has to assume that the given map $f : X \to \mathrm{SL}_m(\mathbb{C})$ is homotopic to a constant map or as we will say null-homotopic. In particular this assumption holds if the space X is contractible. This very general problem has been studied in the case of polynomials of n variables. For $n = 1$, i.e., $f : X \to \mathrm{SL}_m(\mathbb{C})$ a polynomial map (the ring R equals $\mathbb{C}[z]$) it is an easy consequence of the fact that $\mathbb{C}[z]$ is an Euclidean ring that such f factors through a product of upper and lower diagonal unipotent matrices. For $m = n = 2$ the following counterexample was found by COHN: the matrix

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

does not decompose as a finite product of unipotent matrices. For $m \geq 3$ (and any n) it is a deep result of SUSLIN that any matrix in $\mathrm{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices. In the case of continuous complex valued functions on a topological space X the problem was studied and solved by THURSTON and VASERSTEIN. It is natural to consider the problem for rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$. Explicitly this problem was posed by GROMOV in his groundbreaking paper where he extends the classical OKA-GRAUERT theorem from bundles with homogeneous fibers to fibrations with elliptic fibers, e.g., fibrations admitting a dominating spray. In spite of the above mentioned result of VASERSTEIN he calls it the *Vaserstein Problem: Does every holomorphic map $\mathbb{C}^n \to SL_m(\mathbb{C})$ decompose into a finite product of holomorphic maps sending $\mathbb{C}^n$ into unipotent subgroups in $SL_m(\mathbb{C})$?* In the talk we explain a complete solution to GROMOV'S *Vaserstein Problem*. This is joint work with B. Ivarsson.

László Lempert (Purdue University)
The uniqueness of geometric quantization
This is joint work with Szöke, and in progress. In geometric quantization (as in most other schemes of quantization) one associates with a Riemannian manifold a Hilbert space. The manifold represents the classical configurations of a mechanical system, and the Hilbert space is to represent its quantum states. Often the Hilbert space depends on additional choices, and these choices form a smooth or complex manifold S. The uniqueness problem asks whether there is a natural isomorphism between the Hilbert spaces H_s and H_t corresponding to different choices $s, t \in S$.

In the 1990s Axelrod, Della Pietra, and Witten suggested to view the H_s as fibers of a Hilbert bundle H over S, define a connection on H, and use parallel transport to identify its fibers. In the talk I will briefly explain what is unsatisfactory, from the mathematical point of view, in their work. Then I will discuss the mathematical structures to which their idea leads, and properties of these structures. Finally I will tackle the issue of uniqueness when geometric quantization is based on so called adapted Kähler structures.

Vakhid Masagutov (Purdue University)
Homomorphisms of infinitely generated analytic sheaves
We prove that every homomorphism $\mathcal{O}_\zeta^E \to \mathcal{O}_\zeta^F$, with E and F Banach spaces and $\zeta \in \mathbb{C}^m$, is induced by a $\mathrm{Hom}(E, F)$-valued holomorphic germ, provided that $1 \leq m < \infty$. A similar structure theorem is obtained

for the homomorphisms of type $\mathcal{O}_\zeta^E \to \mathcal{S}_\zeta$, where $\mathcal{S}_\zeta$ is a stalk of a coherent sheaf of positive depth. We later extend these results to sheaf homomorphisms, obtaining a condition on coherent sheaves which guarantees the sheaf to be equipped with a unique analytic structure in the sense of Lempert-Patyi.

Laurent Meersseman (PIMS/Université de Bourgogne)
Uniformization of deformation families of compact complex manifolds
Consider the following uniformization problem. Take two holomorphic (parametrized by the unit disk) or differentiable (parametrized by an interval containing 0) deformation families of compact complex manifolds. Assume they are pointwise isomorphic, that is for each point t of the parameter space, the fiber over t of the first family is biholomorphic to the fiber over t of the second family. Then, under which conditions are the two families locally isomorphic at 0?

After recalling some known results (positive and negative) on this problem, I will give a sufficient condition in the case of holomorphic families. I will then show that, surprisingly, this condition is not sufficient in the case of differentiable families. These results rely on a geometric study of the Kuranishi space of a compact complex manifold.

Joël Merker (École Normale Supérieure)
Effective algebraic degeneracy
In 1979, Green and Griffiths conjectured that in every projective algebraic variety X of general type, there exists a certain *proper* subvariety Y with the property that every *nonconstant* entire holomorphic curve $f\colon \mathbb{C} \to X$ landing in X must in fact lie inside Y. For projective hypersurfaces X, Siu showed in 2004 that there is an integer d_n such that every generic hypersurface X in $\mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geqslant d_n$, such an Y exists. The talk, based on Demailly's bundle of invariant jet differentials and on a new construction of explicit slanted vector fields tangent to the space of vertical jets to the universal hypersurface (realizing an idea of Siu), will present a recent complete detailed proof (joint with Diverio and Rousseau) of such a kind of algebraic degeneracy statement, with the *effective* degree bound :

$$d \geqslant n^{(n+1)^{n+5}}.$$

In the early 1980's, Lang conjectured a deep correspondence between degeneracy of entire holomorphic curves and finiteness/non-denseness of rational points on projective algebraic varieties that the whole subject is, unfortunately, still unable to put in concrete form.

Imre Patyi (Georgia State University)
On holomorphic domination
We discuss the question of flexible exhaustions of pseudoconvex open sets in a Banach space by sublevel sets of the norm of holomorphic vector valued functions; this has applications to sheaf and Dolbeault cohomology of complex Banach manifolds. We show that if X is a Banach space with a Schauder basis (e.g., $X = C[0,1]$), $D \subset X$ is pseudoconvex open, $u : D \to (-\infty, \infty)$ is continuous, then there are a Banach space Z and a holomorphic function $h : D \to Z$ such that $u(x) < \|h(x)\|$ for $x \in D$; in this case we say that holomorphic domination is possible in D. On a different note we also show that many complex Banach submanifolds of the Banach space ℓ_1 of summable sequences admit many nowhere critical numerical holomorphic functions.

Evgeny Poletsky (Syracuse University)
Functions holomorphic along holomorphic vector fields
We will discuss the following generalization of Forelli's theorem: Suppose F is a holomorphic vector field with singular point at p, such that F is linearizable at p and the matrix is diagonalizable with eigenvalues whose ratios are positive reals. Then any function that has an asymptotic Taylor expansion at p and is holomorphic along the complex integral curves of F is holomorphic in a neighborhood of p. We also present an example to show that the requirement for ratios of the eigenvalues to be positive reals is necessary.

Jean-Pierre Rosay (University of Wisconsin)
Pluripolar sets in almost complex manifolds

The notion of plurisubharmonicity makes sense for functions defined on almost complex manifolds. Pluripolar sets are sets on which a plurisubharmonic function is $-\infty$. I shall discuss the notion of pluripolarity and the important question of logarithmic singularities versus weaker singularities.

The Chirka function with pole at a point has been efficiently used for localization of the Kobayashi metric (Gaussier-Sukhov and Ivashkovich-Rosay). I shall discuss more recent results on pluripolarity with applications to uniqueness results (Ivashkovich-Rosay).

Alexandre Sukhov (Université de Lille-1)
Constructions of pseudoholomorphic discs
We establish an existence and study the properties of J-complex curves with prescribed boundary conditions in almost complex Stein manifolds.

Sophia Vassiliadou (Georgetown University)
Hartogs extension theorems on complex spaces with singularities
I will discuss some generalizations of the classical Hartogs extension theorem to complex spaces with singularities and present an analytic proof using $\overline{\partial}$-techniques (joint work with Nils Øvrelid).

Jörg Winkelmann (Universität Bayreuth)
On Brody curves
We discuss a number of properties of Brody curves which underline that the class of Brody curves is rather "delicate", for example, the property of a variety of admitting a non-degenerate Brody curve does not behave well in families.

Aaron Zerhusen (Illinois Wesleyan University)
Local solvability of the $\bar{\partial}$-equation in certain Banach spaces.
In a sharp contrast to the situation in finite dimensions, Imre Patyi has shown that the $\bar{\partial}$-equation is not always solvable, even locally, in an infinite dimensional Banach space. On the other hand, László Lempert has shown that in ℓ_1, the Banach space of 1-summable sequences, the $\bar{\partial}$-equation is solvable for (0,1)-forms on pseudoconvex domains. I will discuss how Lempert's result leads to a proof of local solvability of the $\bar{\partial}$-equation in a large class of Banach spaces which includes any L_1 space and the dual space of any L_∞ space.

4 Scientific Progress Made

Almost all the talks generated many interesting questions from the audience, related to the results presented in the talks. Some of the questions were about new possible directions of research, while some pointed to possible connections of the exposed results to other fields of mathematics.

Aside from such questions, there were quite a few longer discussions between groups of participants regarding not only the topics presented in the lectures but also other important open questions. We note here a few such discussions.

A very recent result of Berman and Boucksom [5] shows that in higher dimensions the arrays of Fekete points in a compact set equidistribute to the equilibrium measure of that compact set. This was known in dimension one (Fekete's Theorem), and had been conjectured to be true in any dimension once the right analogues of equilibrium measures were introduced in pluripotential theory. This result provides important applications of pluripotential theory in approximation theory. Discussions about it went on throughout the duration of the workshop between Boucksom, Levenberg, Bloom, Bos and others.

The new variational method of solving Monge-Ampère type equations on compact Kähler manifolds presented in the talk of V. Guedj also generated much discussion during the workshop. The notion of foliated shells and their applications presented by S. Ivashkovich, and the recent characterization of envelopes of holomorphy of domains in Stein manifolds using analytic discs presented by B. Jöricke [22], were considered very interesting and discussed by some participants.

An interesting open question in complex geometry is to study whether in a complex manifold, analytic discs with boundary have a Stein neighborhood. The existence of such neighborhoods is useful in applications. The question was discussed a number of participants, including Ivashkovich, Poletsky, Rosay, Shcherbina.

One usually cannot expect major theorems to be proved during a five-day workshop. However, the organizers are confident that the ideas generated by the talks presented and by the many discussions that took place during the week will lead to important progress at least in some of the many topics covered by the conference.

5 Outcome of the Meeting

Although a single workshop cannot do justice to the breadth and depth of contemporary complex analysis and complex geometry, the organizers believe it was beneficial to bring together a group of experts from diverse subfields to discuss recent results and work in progress, and to share ideas on open questions. We chose a coherent collection of interrelated topics for the workshop, representing some of the most vibrant developments in the subject today.

There were 41 participants, ranging from leading experts to graduate students (5 in total) and recent PhDs (another 5). Among the participants were 6 female mathematicians.

The program consisted of 24 talks, each of 45 minutes, with a break of at least 15 minutes in between talks. Five of the talks were by recent PhDs or graduate students. There were three full days, in which the presentations ended by 5:00 pm, while the remaining two days consisted of just a morning session. This allowed ample time for questions and discussions.

There is general agreement among the participants that the workshop was both inspiring and stimulating. All very much appreciated the excellent facilities and the hospitality at BIRS. The beautiful scenery and the unseasonably warm weather during the week helped make the workshop a success.

6 List of Participants

Barrett, David (University of Michigan)
Blocki, Zbigniew (Jagiellonian University)
Bloom, Thomas (University of Toronto)
Boucksom, Sebastien (Institut Mathématique de Jussieu)
Chakrabarti, Debraj (University of Notre Dame)
Coman, Dan (Syracuse University)
de Oliveira, Bruno (University of Miami)
Dharmasena, Dayal (Syracuse University)
Gaussier, Hervé (Université Aix-Marseille)
Gong, Xianghong (University of Wisconsin)
Guedj, Vincent (Université Aix-Marseille)
Halfpap, Jennifer (University of Montana)
Heier, Gordon (University of California, Riverside)
Ho, Pak Tung (Purdue University)
Isaev, Alexander (Australian National University)
Ivashkovich, Sergey (University of Lille-1)
Jöricke, Burglind (Institut des Hautes Études Scientifiques)
Kruzhilin, Nikolay (Steklov Mathematical Institute)
Kutzschebauch, Frank (Universität Bern)
Lárusson, Finnur (University of Adelaide)
Lempert, László (Purdue University)
Levenberg, Norm (Indiana University)
Masagutov, Vakhid (Purdue University)
Meersseman, Laurent (PIMS/Université de Bourgogne)
Merker, Joel (École Normale Supérieure)
Patyi, Imre (Georgia State University)
Perkins, Tony (Syracuse University)

Pinchuk, Sergey (Indiana University)
Poletsky, Evgeny (Syracuse University)
Porten, Egmont (Mid Sweden University)
Rosay, Jean-Pierre (University of Wisconsin)
Shafikov, Rasul (University of Western Ontario)
Shcherbina, Nikolay (University of Wuppertal)
Stensones, Berit (University of Michigan)
Sukhov, Alexandre (University of Lille-1)
Taylor, Al (University of Michigan)
Vassiliadou, Sophia (Georgetown University)
Vivas, Liz (University of Michigan)
Winkelmann, Jörg (Mathematisches Institut, Bayreuth)
Zeager, Crystal (University of Michigan)
Zerhusen, Aaron (Illinois Wesleyan University)

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