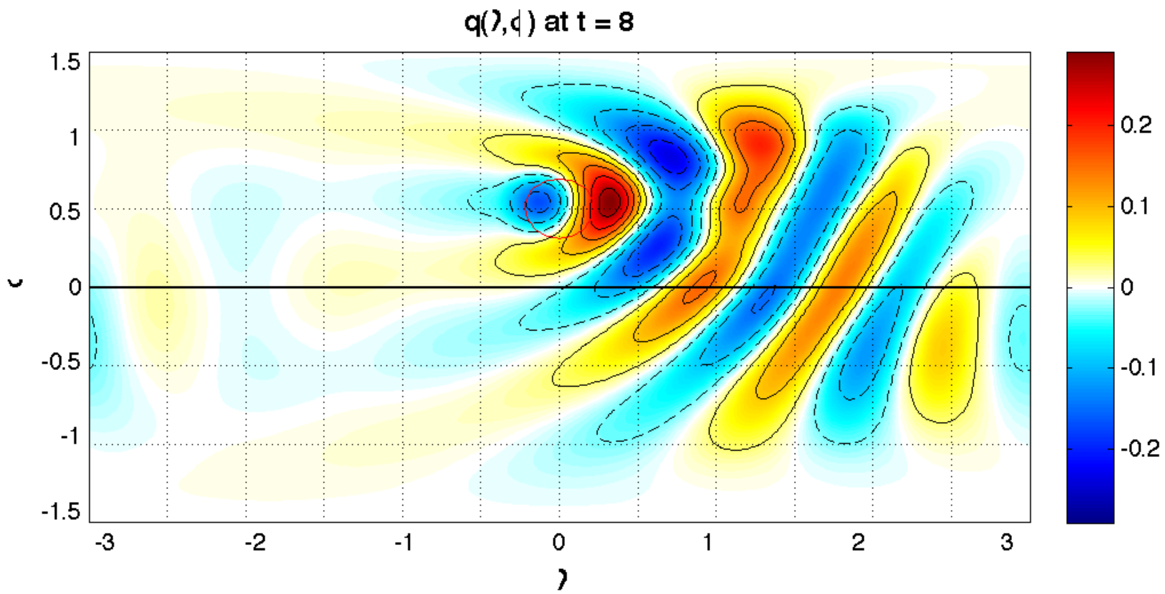
A PV Dynamics for Rotating Shallow Water on the Sphere

- ▷ search for a balance dynamics on the full sphere
- ▷ potential vorticity inversion & advection
- ▷ ray theory for short scale waves



- ▷ David J Muraki & Andrea Blazenko, Simon Fraser University
- ▷ Chris Snyder, NCAR

Midlatitude Balanced Dynamics

Rotating Shallow Water (rSW), β-Scaled

- winds, $\vec{u}$ & height field, $H = 1 + \epsilon\,\eta$
- longitude-latitude <u>local</u> coordinates, $(\lambda, \phi) \approx (\lambda_0, \phi_0)$

$$\epsilon^2 \frac{D\vec{u}}{Dt} \quad + \quad (1 + \beta(\phi - \phi_0))(\hat{r}_0 \times \vec{u}) \quad \sim \quad -\,\epsilon\,\nabla\eta$$

$$\epsilon\ \frac{D\eta}{Dt} \quad + \quad (1 + \epsilon\,\eta)\ \ (\nabla \cdot \vec{u}) \quad = \quad 0$$

- dimensionless parameters: Rossby number & Coriolis-β

$$\epsilon = \frac{\sqrt{gH_0}}{2\Omega \sin\phi_0\, r} \ll 1 \qquad ; \qquad \beta = \cot\phi_0$$

- restrict to short deformation scales: $\lambda - \lambda_0, \phi - \phi_0 \sim O(\epsilon)$

Quasigeostrophy (QG)

- *balanced* dynamics: slow Rossby waves & NO fast gravity waves
  - geostrophy: $\hat{r}_0 \times \vec{u} \sim -\,\epsilon\,\nabla\eta \quad \longrightarrow$ non-divergent winds
  - limit as $\epsilon \longrightarrow 0$, geostrophic degeneracy
  - advection & inversion of potential vorticity (PV)
- geometrical obstacle: <u>Rossby number singular at Equator, $\epsilon \longrightarrow \infty$</u>

rSW on the Full Sphere

Spherical Coordinates

- longitude-latitude global coordinates, (λ, ϕ)

$$\epsilon^2 \left\{ \frac{Du}{Dt} + v\hat{\lambda} \cdot \frac{D\hat{\phi}}{Dt} \right\} \quad - \; v \sin\phi \quad = \quad -\,\epsilon \frac{1}{\cos\phi}\eta_\lambda$$

$$\epsilon^2 \left\{ \frac{Dv}{Dt} + u\hat{\phi} \cdot \frac{D\hat{\lambda}}{Dt} \right\} \quad + \; u \sin\phi \quad = \quad -\,\epsilon \; \eta_\phi$$

$$\epsilon \quad \frac{D\eta}{Dt} \; + \; \{1 + \epsilon\,\eta\} \, \frac{u_\lambda + (v\cos\phi)_\phi}{\cos\phi} \quad = \quad 0$$

- Lamb parameter

$$\epsilon = \frac{\sqrt{gH_0}}{2\Omega r} \ll 1$$

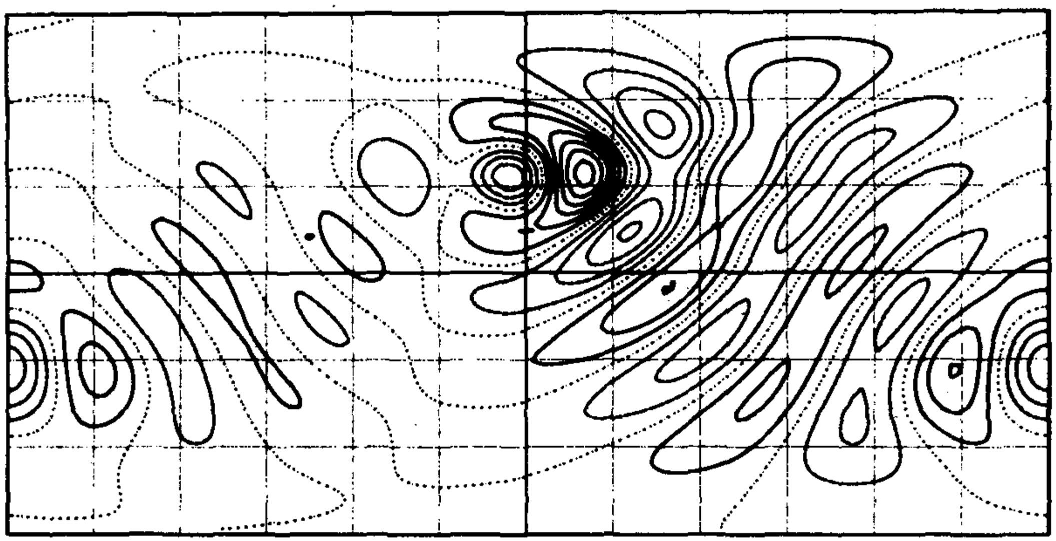
Balanced Dynamics

- *PV Inversion on a Hemisphere*, McIntyre/Norton 1999
  - landmark for PV as prognostic variable
  - non-divergent winds (at leading-order)
  - dynamical oddity: mirror symmetry across Equator
- on midlatitude scales, limit as $\epsilon \longrightarrow 0$ is now inconsistent!

A Case for Global Balanced Dynamics

Grose & Hoskins (1979)

- rSW flow past mountain at 30°N
  - steady, super-rotation zonal wind
- steady waves by rayleigh-damped perturbations
  - perturbation vorticity after 17 days (with damping) & $\epsilon \approx 0.33$

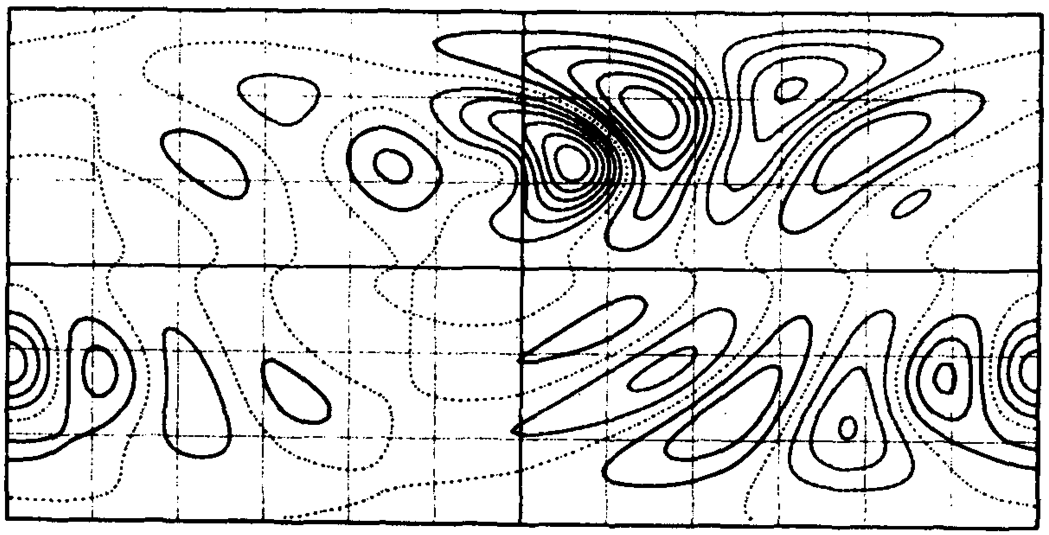


- waves propagate smoothly across Equator
- nearly non-divergent flow

A Case for Global Balanced Dynamics

Grose & Hoskins (1979)

- rSW flow past mountain at 30°N
  - steady, super-rotation zonal wind
- steady waves by rayleigh-damped perturbations
  - perturbation height after 17 days (with damping) & $\epsilon \approx 0.33$



- essentially zero height disturbance at Equator
- non-divergent balance relations, with streamfunction ψ

$$u = -\,\epsilon\,\psi_{\phi} \quad ; \quad v = \epsilon\,\frac{1}{\cos\phi}\,\psi_{\lambda} \quad ; \quad \eta = \psi \sin\phi$$

$$\epsilon^2 \frac{Du}{Dt} - v \sin\phi \sim -\epsilon \frac{1}{\cos\phi} \left(\frac{\eta}{\sin\phi} \right)_\lambda \sin\phi$$

$$\epsilon^2 \frac{Dv}{Dt} + u \sin\phi \sim -\epsilon \left(\frac{\eta}{\sin\phi} \sin\phi \right)_\phi$$

$$\epsilon \frac{D\eta}{Dt} + \{1 + \epsilon\,\eta\} \frac{u_\lambda + (v \cos\phi)_\phi}{\cos\phi} = 0$$

- non-divergent balance relations, with streamfunction ψ

$$u = -\epsilon\,\psi_\phi \quad ; \quad v = \epsilon \frac{1}{\cos\phi} \psi_\lambda \quad ; \quad \eta = \psi \sin\phi$$

rSW on the Sphere Redone

$$\epsilon^2 \frac{Du}{Dt} \quad - v \sin\phi \quad \sim \quad -\epsilon \frac{1}{\cos\phi} \psi_\lambda \sin\phi$$

$$\epsilon^2 \frac{Dv}{Dt} \quad + u \sin\phi \quad \sim \quad -\epsilon \psi_\phi \sin\phi \quad - \epsilon \psi \cos\phi$$

$$\epsilon \frac{D\eta}{Dt} + \{1 + \epsilon\eta\} \frac{u_\lambda + (v\cos\phi)_\phi}{\cos\phi} = 0$$

Geostrophic Degeneracy Restored

- on midlatitude scales, limit as $\epsilon \longrightarrow 0$ is consistently degenerate
- non-divergent balance relations
- β-effect displaced: meridional advection of planetary PV

Potential Vorticity

- total PV is advected quantity: $DQ/Dt = 0$

$$Q = \sin\phi + \epsilon q = \frac{\sin\phi + \epsilon^2 \hat{r} \cdot (\nabla \times \vec{u})}{1 + \epsilon\eta}$$

PV Dynamics on the Sphere (sPV)

Inversion, Streamfunction & Advection

- PV-streamfunction relation ($b(\lambda, \phi)$ = topography):

$$\epsilon^2 \nabla^2 \psi - (\sin^2 \phi)\, \psi = q - b \sin \phi$$

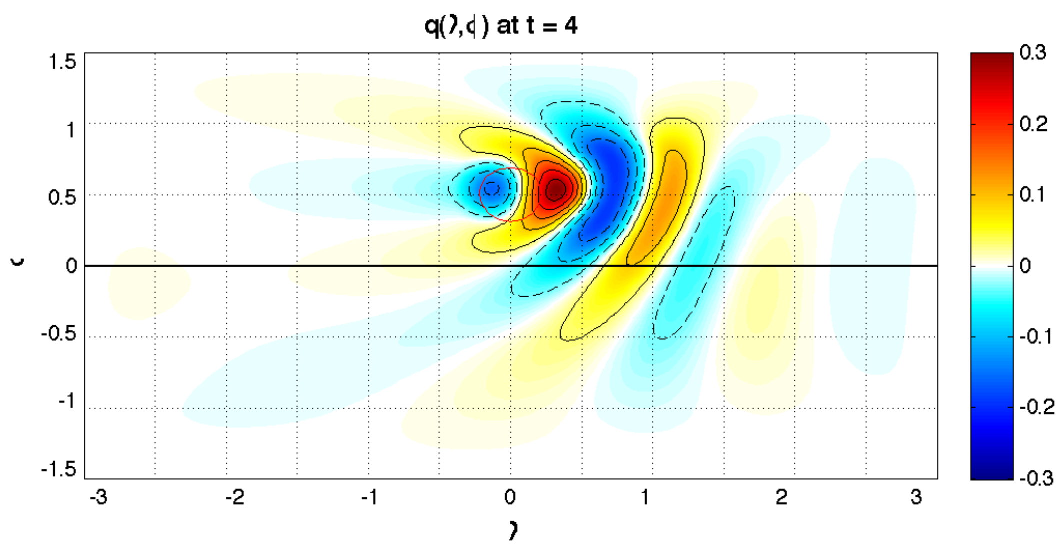
- balance relations for non-divergent winds:

$$u = -\epsilon\, \psi_\phi \quad ; \quad v = \epsilon \frac{1}{\cos \phi} \psi_\lambda \quad ; \quad \eta = \psi \sin \phi$$

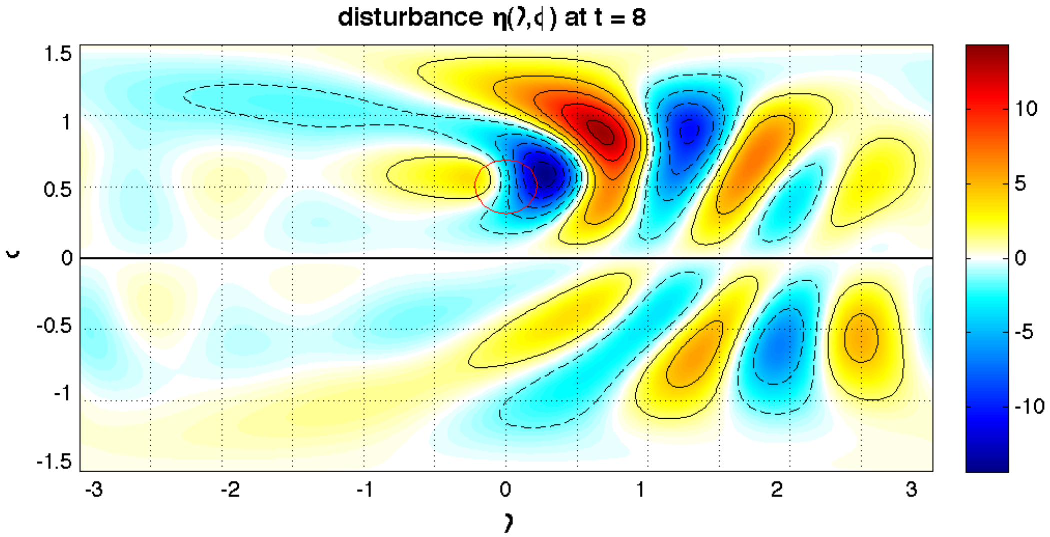
- distubance PV advection:

$$\epsilon \left\{ \frac{\partial q}{\partial t} + \frac{u}{\cos \phi} \frac{\partial q}{\partial \lambda} + v \frac{\partial q}{\partial \phi} \right\} + v \cos \phi = 0$$

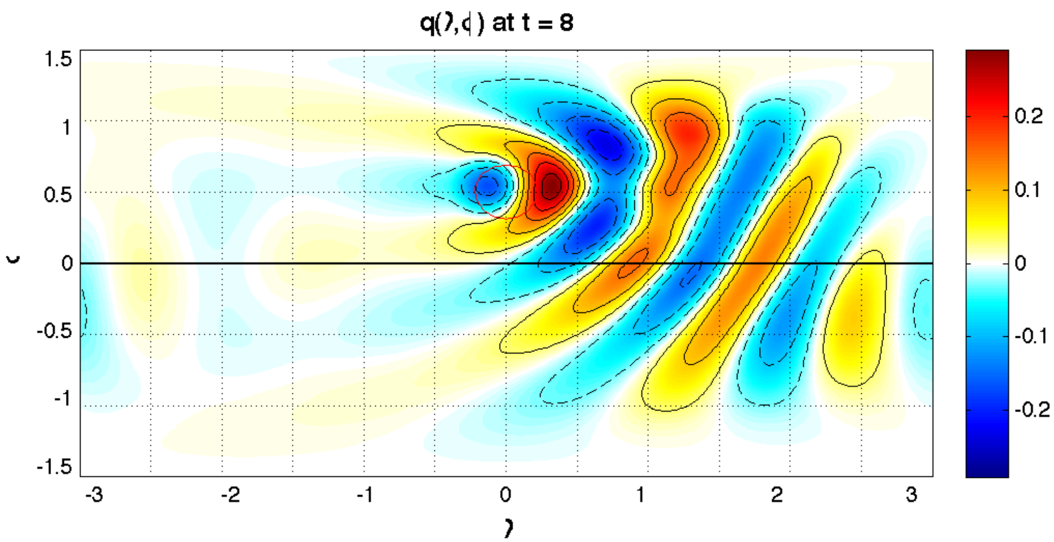
Mountain Waves, Disturbance PV ($t = 4$)



Mountain Waves, Disturbance Height ($t = 8$)

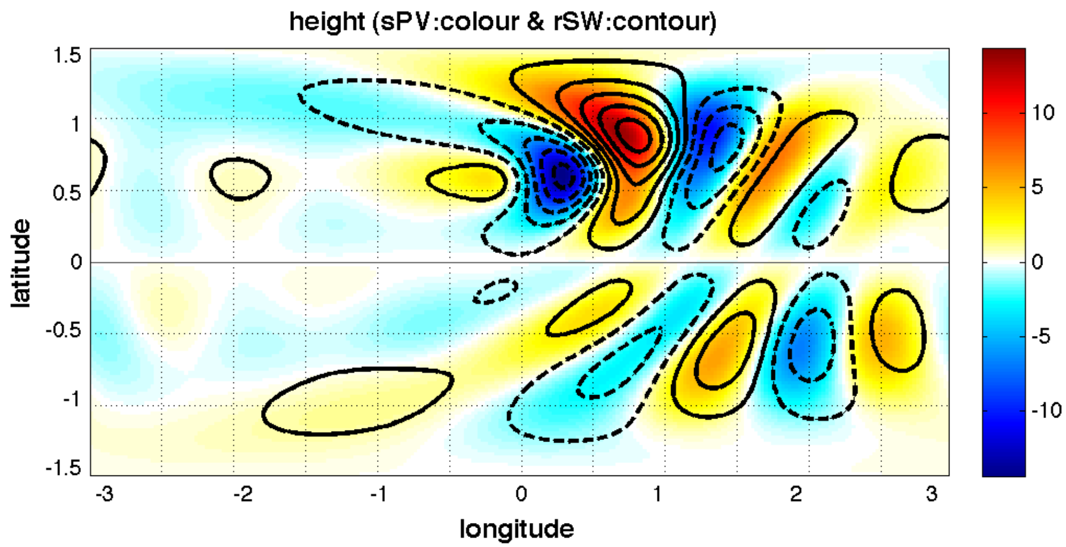


Mountain Waves, Disturbance PV ($t = 8$)

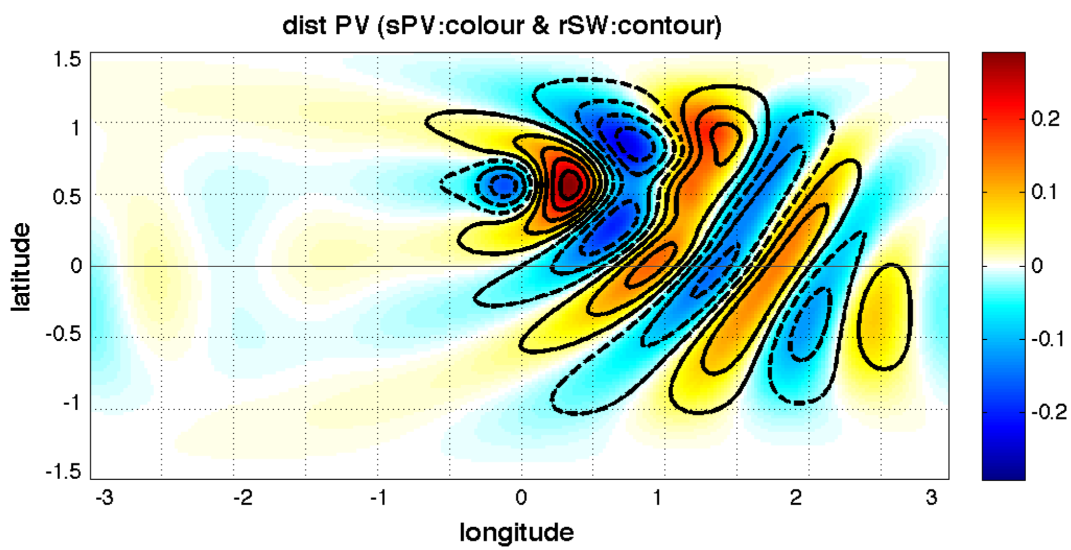


sPV on the Sphere: Comparison with rSW

Disturbance Height: sPV (colour) & rSW (contour)



Disturbance PV: $\epsilon \approx 0.337$ — thanks to Joe Klemp, NCAR



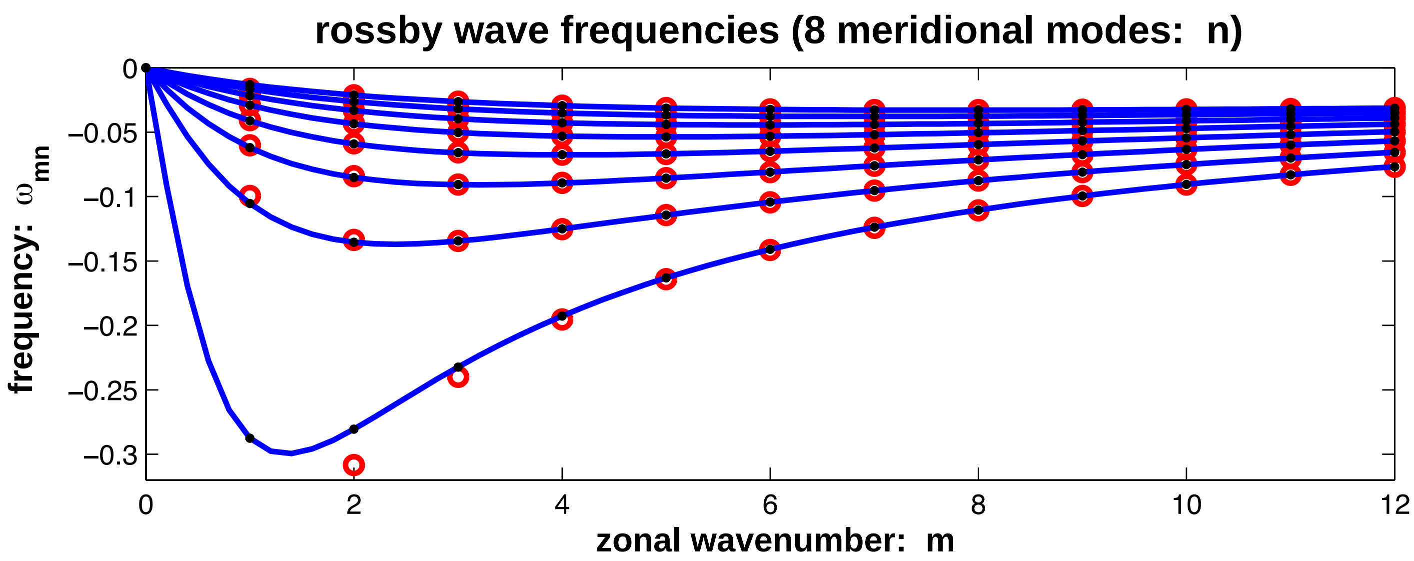
sPV is not a Global Asymptotic Theory?

$$
\begin{aligned}
\epsilon^2 \frac{Du}{Dt} \qquad\qquad - v \sin\phi &\sim -\epsilon \frac{1}{\cos\phi}\psi_\lambda \sin\phi \\
\epsilon^2 \frac{Dv}{Dt} \qquad\qquad + u \sin\phi &\sim -\epsilon\, \psi_\phi \sin\phi \qquad - \epsilon\, \psi \cos\phi \\
\epsilon \frac{D\eta}{Dt} + \{1 + \epsilon\, \eta\} \frac{u_\lambda + (v\cos\phi)_\phi}{\cos\phi} &= 0
\end{aligned}
$$

Midlatitude Synoptic-Scale Truncation

- leading-order balance assumptions
  - on deformation scales $\Rightarrow$ $\partial/\partial\lambda, \partial/\partial\phi = O(\epsilon)$
  - at midlatitudes $\Rightarrow$ $\sin\phi \neq 0$
- there is NO expectation of asymptotic validity at the Equator
  - sPV is not *globally* accurate, but is well-posed
  - PV inversion & velocity-streamfunction relationship are non-singular

Q: So, Why are Equatorial Crossings Faithfully Represented?



Rotating Waves, $\psi(\lambda - ct, \phi)$

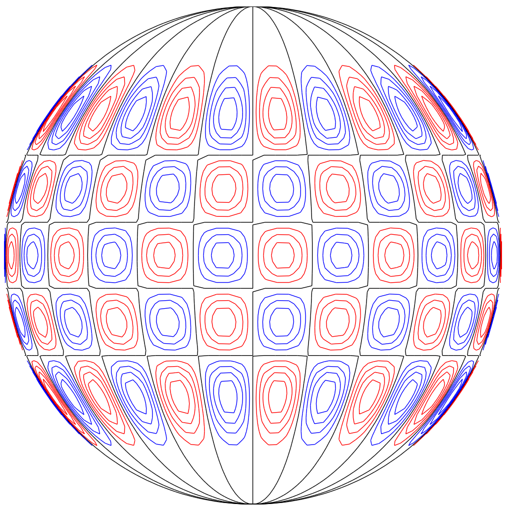
- ▷ ODE eigenfunction, Schubert (2008) $\longrightarrow$ exact nonlinear sPV solutions ($\epsilon \approx 0.337$)

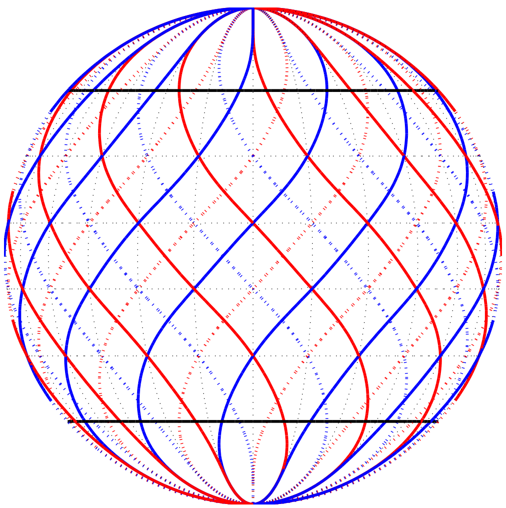
$$\epsilon^2 \nabla^2 \psi - (\sin^2 \phi)\, \psi + (1/c)\, \psi = 0 \quad ; \quad \psi(\pm\pi/2) = 0$$

- ▷ compare to modes for rSW on sphere (Margules, Hough, Longuet-Higgins, . . .)

- ▷ only longest planetary waves have $O(1)$ wavespeed errors

sPV Rossby Waves





Rotating Waves, $\psi(\lambda - ct, \phi)$

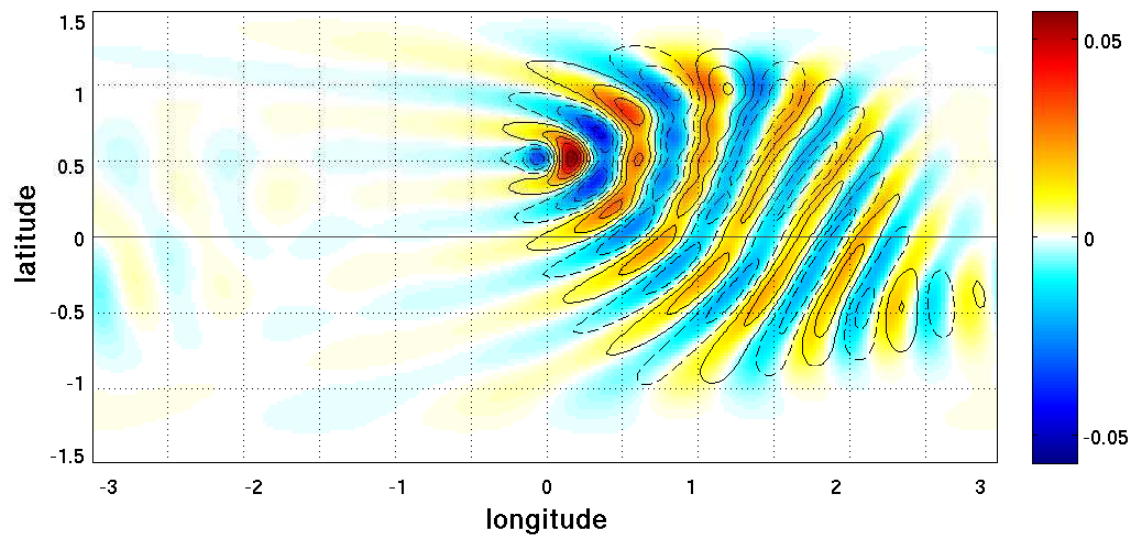
- ODE eigenfunction, Schubert (2008) $\longrightarrow$ exact nonlinear sPV solutions ($\epsilon \approx 0.337$)

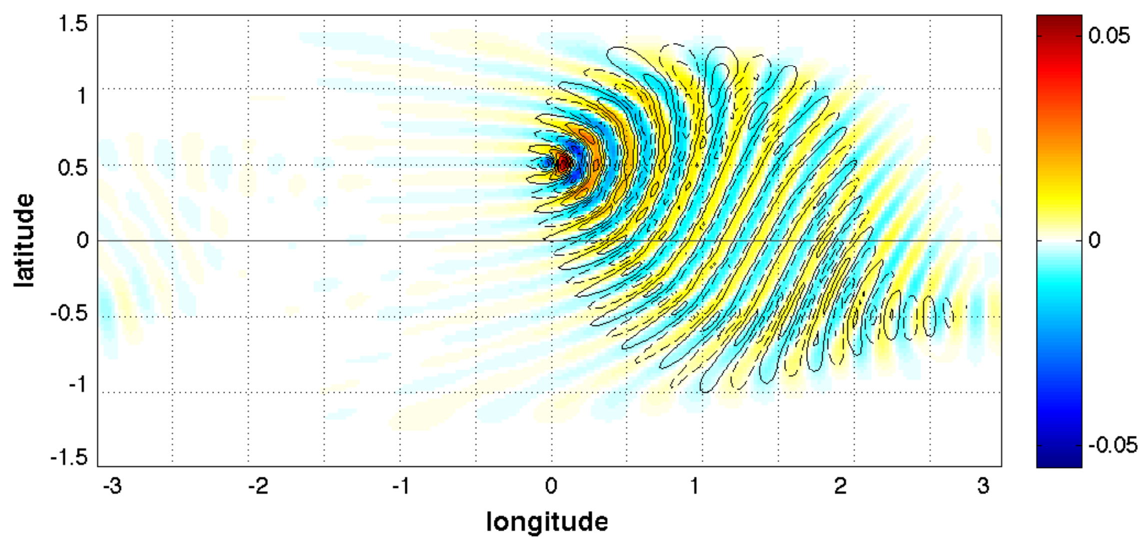
$$\epsilon^2 \, \nabla^2 \, \psi - (\sin^2 \phi) \, \psi + (1/c) \, \psi = 0 \quad ; \quad \psi(\pm\pi/2) = 0$$

- compare to modes for rSW on sphere (Margules, Hough, Longuet-Higgins, . . .)

- only longest planetary waves have $O(1)$ wavespeed errors

Wavepacket Limit of Short-Scale Waves: $\epsilon = \frac{1}{2}\,\epsilon_{\text{gh}}$ & $\frac{1}{4}\,\epsilon_{\text{gh}}$





Geometrical Optics Limit as $\epsilon \longrightarrow 0$

- (flow) = (steady, zonal $\bar{U}(\phi)$) + (wave amplitude) $e^{iS(\lambda,\phi,t)/\epsilon}$
- phase $S(\lambda, \phi, t)$ satisfies ϵ-independent Hamilton-Jacobi PDE

$$\left(S_t + \frac{\bar{U}(\phi)}{\cos\phi} S_\lambda\right)\left(\frac{S_\lambda^2}{\cos^2\phi} + S_\phi^2 + \sin^2\phi\right) - S_\lambda = 0$$

- midlatitude replacements

$$S_t \longrightarrow -\omega \quad ; \quad \frac{S_\lambda}{\cos\phi} \longrightarrow k \quad ; \quad S_\phi \longrightarrow l \quad ; \quad \sin\phi \longrightarrow \tilde{f} \quad ; \quad \cos\phi \longrightarrow \tilde{\beta}$$

- midlatitude Rossby wave dispersion

$$\omega = \bar{U}k - \frac{\tilde{\beta}k}{k^2 + l^2 + \tilde{f}^2}$$

Q: So, Why are Equatorial Crossings Faithfully Represented?

Geometrical Optics Limit as $\epsilon \longrightarrow 0$

- (flow) = (steady, zonal) + (wave amplitude) $e^{iS(\lambda,\phi,t)/\epsilon}$
- phase $S(\lambda, \phi, t)$ satisfies Hamilton-Jacobi PDE

$$\left(S_t + \frac{\bar{U}(\phi)}{\cos\phi} S_\lambda\right)\left(\frac{S_\lambda^2}{\cos^2\phi} + S_\phi^2 + \sin^2\phi\right) - S_\lambda = 0$$

- midlatitude replacements, $\bar{U}$ = zonal flow

$$S_t \longrightarrow -\omega \;\; ; \;\; \frac{S_\lambda}{\cos\phi} \longrightarrow k \;\; ; \;\; S_\phi \longrightarrow l \;\; ; \;\; \sin\phi \longrightarrow \tilde{f} \;\; ; \;\; \cos\phi \longrightarrow \tilde{\beta}$$

- midlatitude Rossby wave dispersion

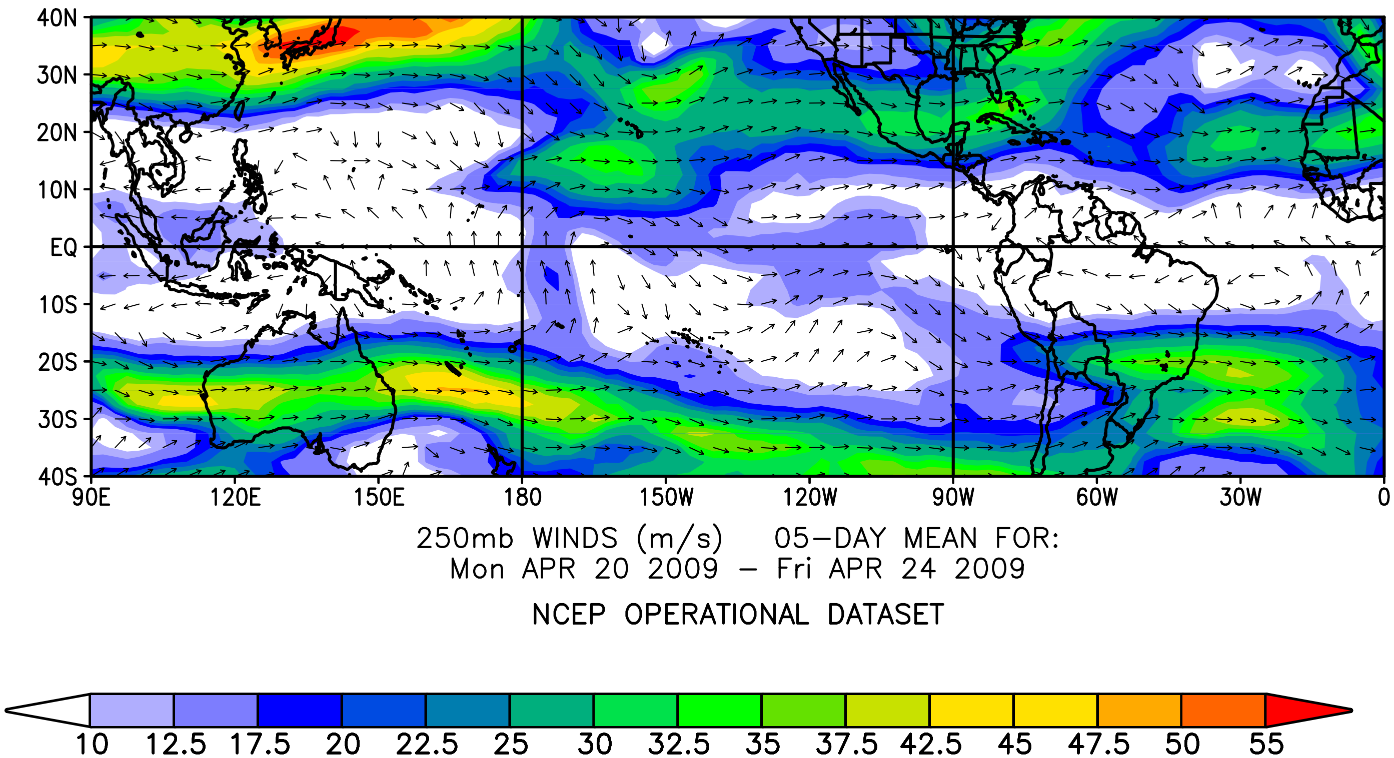
$$\omega = \bar{U}k - \frac{\tilde{\beta}k}{k^2 + l^2 + \tilde{f}^2}$$

A: Both sPV & rSW have the same global ray theory!

20-24 April 2009

NCEP Operational Analysis

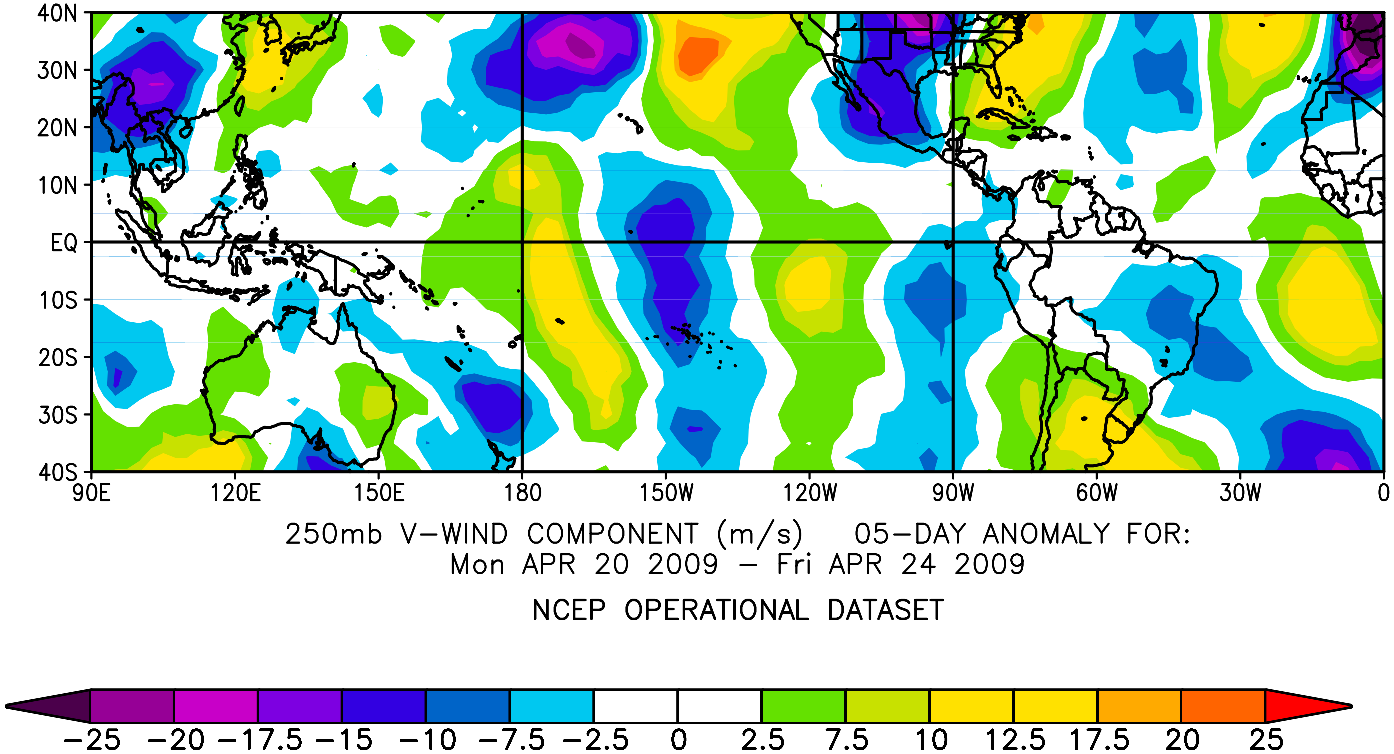
- ▷ 250mB wind vectors (5-day average) — thanks to Mel Shapiro, NCAR
- ▷ Rossby wave flow in Equatorial Pacific



20-24 April 2009

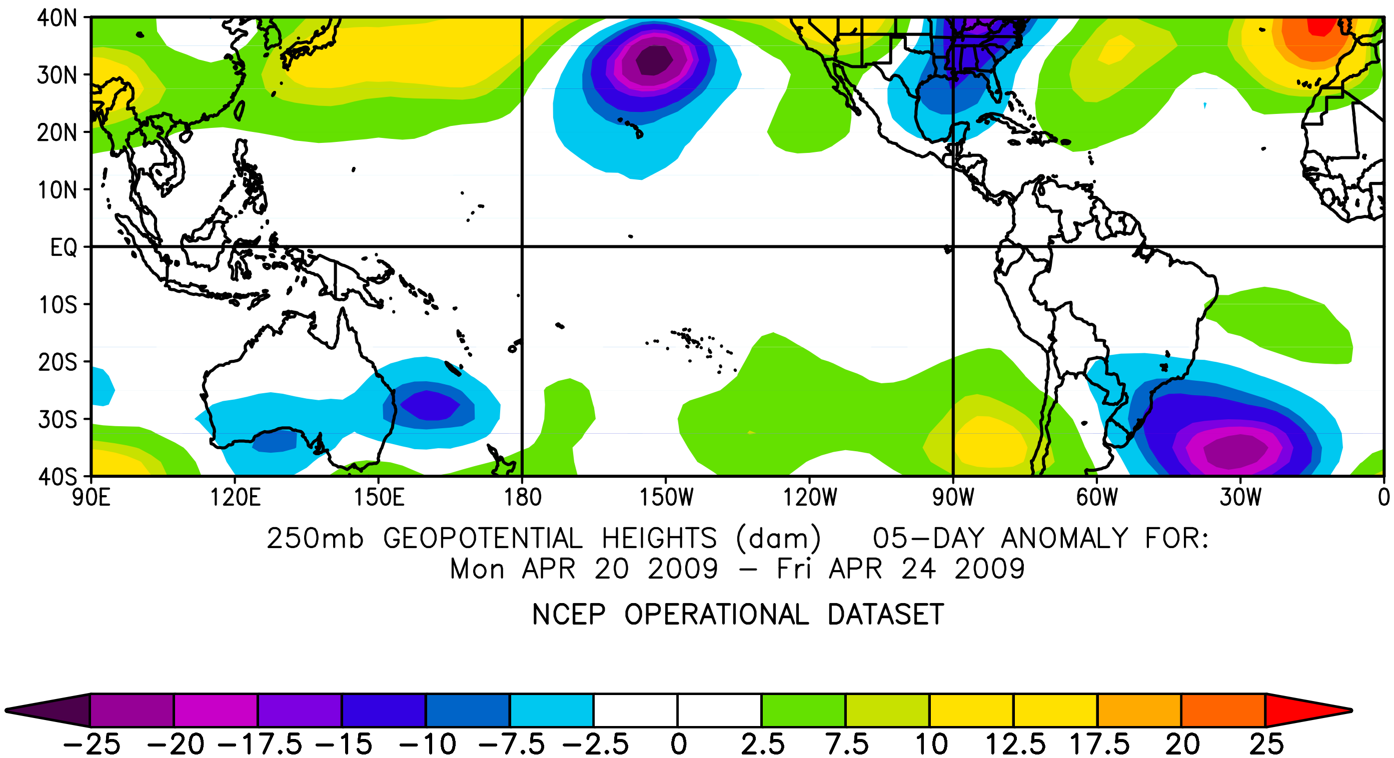
NCEP Operational Analysis

- 250mB meridional wind anomaly (5-day average)
- Rossby wave flow in Equatorial Pacific



NCEP Operational Analysis

- ▷ 250mB geopotential height anomaly (5-day average)
- ▷ Rossby wave flow in Equatorial Pacific



In Closing

PV Dynamics for the rSW Sphere

- ▷ local asymptotic validity in midlatitudes
- ▷ accurate for global Rossby waves at scales smaller than planetary
- ▷ recreates Grose & Hoskins (1979) equatorial wave crossing
- ▷ shares identical (linear) ray theory with rSW
  - ▷ rSW ray theory is degenerate theory, but globally valid
- ▷ sPV includes <u>nonlinear</u> PV advection