Frames from first principles: Error correction, symmetry goals, and numerical efficiency

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1 Overview of the Field

In many areas of mathematics, orthonormal bases are a standard tool for representing vectors or operators. For certain problems, however, it is preferable to use an overcomplete, non-orthogonal set of vectors instead of an orthonormal basis, and thereby incorporate redundancy in the representation. Frames are overcomplete sets which are associated with redundant linear representations that have stable analysis and reconstruction maps. Driven by applications in engineering, the theory of frames has recently grown very rapidly.

This workshop focuses on optimality principles in frame design.

Particular topics of interest in the proposed workshop are the use of redundant representations for achieving:

(1) Sparsity and ℓ_1 Minimization,

(2) Sigma-Delta Quantization,

(3) Pure Frame Theory,

(4) Structured Decompositions.

2 Recent Developments and Open Problems

We first give a brief summary of the current status of research and mention some open questions which were central to our workshop. Later we will then delve deeper in to these four particular topics.

2.1 Sparsity and ℓ_1 Minimization

The efficiency of the reconstruction of a transmitted signal depends on the number of measurements and the computational effort required to determine the signal to a desired accuracy.

The rapid advance of digital data gathering mechanisms has dramatically increased the volume of data to be processed. The conventional approach to data processing is to first acquire the data - along with much redundant and unwanted information - and then compress it (throwing away the unwanted part). The new paradigm is to directly acquire, or ”sense”, the essential part of the data, using few probes of the data called ”measurements”. This relies on the natural assumption that the essential part of the data is usually small.

Obtaining the most efficient reconstruction for a typical class of signals is the central idea of sparse representation theory, which grew out of the groundbreaking work of Donoho, Hu and Stark. This work advocated that the reconstruction problem to compute a signal from few linear measurements can be turned into a linear program. This fact now has broad application in signal processing - recovering signals from highly incomplete measurements.

This technique for signal reconstruction, referred to as ”compressed sensing”, has attracted interest in pure and applied mathematics, computer science, statistics, and industry. At a recent international conference on ”Sparse Representation and High-Dimensional Geometry” in Snowbird, Utah, the one problem central to all of the presentations was the lack of examples of frames which can be used for sparse representation theory. Although there are various methods for showing the existence of frames for the theory, none of these results produces an actual example. Equally problematic, given a frame, none of these results could verify whether the frame works within the context of this theory. Advances made in frame design could bring significant progress to this important, extremely active and broadly applicable area of research.

In a large number of new applications, such as all types of sensor networks, the set-up can no longer be modeled naturally by one single frame system. Many of these applications share the common property of requiring distributed processing. In particular, distributed sensing and packet encoding can also be regarded as a means to study dimension-reduction processes for the analysis of high dimensional data. Fusion frames provide an extensive mathematical framework to model those applications. A fusion frame can be viewed as a frame-like collection of low-dimensional subspaces, thereby providing the possibility to study, for instance, design and robustness questions of those cutting-edge applications.

2.2 Sigma-Delta Quantization

Frame representations can be seen as linear codes over the real or complex numbers. The redundancy in the encoding of vectors provides robustness to signal degradation such as noise, quantization, and erasures.

Noise and quantization errors need to be controlled in the analog-to-digital (A/D) conversion process preceding the transmission of a signal on a network or its storage on digital media. Using an overcomplete representation, such as oversampling of a band-limited, noise-corrupted audio signal, allows one to achieve high accuracy even in the presence of physical limitations of the analog devices used for the conversion.

One important class of such strategies are the so-called sigma-delta quantizers, the very basic examples of which were invented by engineers in the 1960s. The practice has extensively matured since, and many of the high-resolution A/D converters in today’s technology employ sigma-delta quantization of one kind or another. Typical sigma-delta quantizers sample a bandlimited signal up to 100 times faster than the critical Nyquist rate, allocate as few as one bit for the ”rounding” of sample values, and incorporate memory when computing the quantized value of each sample. Mathematically, this amounts to calculating a frame expansion of the underlying signal and replacing each coefficient with values from a finite set that has only a few elements.

Despite the common use of sigma-delta schemes in practice, a general mathematical theory of such coarsely quantized representations did not experience a parallel development until the work of Daubechies and DeVore in the late 1990s established an approximation theoretical framework for the problem. Since then, there has been a rapid development in the theoretical analysis and applications of sigma-delta quantization. The deeper understanding of the sigma-delta schemes opened the door to the use of sigma-delta methods in the quantization of general frame expansions. In particular, it has been shown that for the case of finite frames, rate-distortion characteristics of sigma-delta schemes are superior to those of PCM schemes, which round each coefficient independently. Moreover, inspired by the relationship between the redundancy in an expansion and the implementation robustness of quantization, new A/D methods have emerged. For example, beta encoders, based on redundant beta expansions, were shown to improve the rate-distortion characteristics of sigma-delta schemes while being robust with respect to implementation imperfections in many aspects. Unlike sigma-delta methods, beta encoders are memoryless, and are therefore very easy to implement in conjunction with, for example, a compressed-sensing type sampling scheme.

Apart from quantization errors in the encoding, data loss is a major source of error in the transmission of analog signals. The reason for data loss may be an unreliable network or an error-correcting protocol that detects and ”purges” corrupted packets of data. In some situations, one may even intentionally erase coefficients as part of a data reduction strategy.

The central assumption of common network models is that a sequence of vectors is transmitted in the form of their frame coefficients. These coefficients are sent in parallel streams to the receiver according to a model used by Goyal, Kovacevic and Kelner in 2001. It is commonly assumed in such network models that losing one packet in the transmission process is rare, and that the occurrence of two lost packets is much less likely, and a similar hierarchy of probabilities holds for a higher number of lost packets. This motivated the design of frames following an inductive scheme, which successively selects optimal designs for each amount of losses.

The characterization of optimality in this context started with results by Casazza and Kovacevic in 2001, and further investigated in works by Heath, Strohmer, Holmes, Paulsen, Bodmann and Kalra. The most recent results on optimality combine geometric aspects with combinatorics or even number theory, and may be of interest to a much wider community. In addition, other measures for optimality have been considered, such as probabilistic models for the input as well as the statistics of data losses. A major, remaining challenge is the optimal design when a combination of errors is present, such as quantization and data loss.

2.3 Pure Frame Theory

Although frame theory is basically an applied subject, there are a number of important problems in the *theory itself* which need to be resolved in order for the subject - as well as the applications - to go forward. These problems get to the very heart of the theory and require a deep understanding of the inner workings of frames and their construction.

One important area in pure frame theory involves the *construction* of frames with certain required properties. Over the last 10 years there has been a major advance on showing the existence of frames with added properties - initiated by Benedetto and Fickus with the introduction of *frame potentials*. This idea was heavily developed leading to many breakthroughs in the field. Unfortunately, these methods just show the *existence* of frames with prescribed properties. But what is really needed, are more construction techniques for building such frames. The construction problem is particularly important in the area of fusion frames which have the potential for broad application to problems in distributed processing, communication, sensing, coding and more. But each of these applications requires the construction of fusion frames with added properties. Gitta Kutyniok presented an overview of some of the applications of fusion frames as well as some of the directions for future research. A recent major breakthrough in the construction of tight frames and tight fusion frames has taken place (See Matt Fickus' talk below). But there are still many important open problems concerning the construction of frames with added properties which were presented at the meeting. We now have characterizations of the sequences of possible norms of tight frame vectors for finite dimensional Hilbert spaces. At the meeting, Marcin Bownik presented some deep results concerning the corresponding infinite dimensional form of these results.

It is now known that the famous Kadison-Singer Problem in C^*-algebras is equivalent to fundamental open problems in frame theory. Also, abstract frame theory was recently used to show that this problem is actually equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering. Each of these problems represents a fundamental notion for frames which we have little understanding of. Solving these problems will require bringing fundamental new tools into the area and bringing forward a fundamental new understanding of the behavior of abstract (tight) frames. Eric Weber presented some results concerning the Paving Conjecture and the Feichtinger Conjecture which are now known to be equivalent forms of the Kadison-Singer Problem. Related to the Kadison-Singer Problem is the 1989 Bourgain-Tzafriri Restricted Invertibility Theorem. It is an open problem whether there is an infinite dimensional version of the theorem. Recently, Casazza and Pfander used frame theory to give the correct form of this result and proved that it holds for ℓ_1-localized operators. But it is unknown if the localization assumption is really needed.

A new model for image fusion was introduced by Shidong Li in the form of one-sided frame perturbations. The idea here is different from standard perturbations in that it asks when an estimation of a vector f can be a stable approximation to the actual f.

There are a number of long standing questions in pure frame theory for which solutions would bring an important understanding of parts of the field. The 10 year old Paulsen Problem is now known to be equivalent to a deep problem in operator theory: How close is a nearly equal-norm, nearly Parseval frame to an equal-norm Parseval frame? Recently, Bodmann and Casazza gave the first partial solution to the problem. But much work has to be done to get at the general solution. Another direction of research in frame theory which has been ongoing for several years now is: What is the correct notion of redundancy for infinite dimensional Hilbert spaces? Recently, Balan, Casazza and Landau produced the first candidate which satisfies all of our *wish list* for redundancy in infinite dimensions. But these results require ℓ_1-localization and it is unknown if this assumption is necessary.

These problems and more were presented at the meeting (See the section on *pure frame theory*) and are helping to *frame* directions of research for the future.

2.4 Structured Decompositions

In many practical applications of frame design, symmetry constraints need to be satisfied. This could mean that each individual frame vector is required to be invariant under some group action, or that the entire set of frame vectors is invariant.

In wireless transmissions, typical errors are time delays or frequency shifts caused by motions of the handset. Specifically designed Gabor frames, sets obtained from discrete time-frequency shifts of one function, have proved very resilient against such distortions. The design by Strohmer, for example, uses Wexler-Raz relations to compute approximate dual Gabor frames used to measure signals, together with an associated approximation rate.

Image processing and analysis is another situation in which frames are used to achieve a desired transformation behavior under a group of symmetry transformations. The commonly used filter-bank algorithms for image analysis are not well-adapted to rotations. Recently, Papadakis developed a multiresolution theory, which has an equally efficient implementation as the standard wavelet algorithms, but allows a decomposition of images that commutes with rotations. Moreover, anisotropic structures, e.g. edges in images, curvilinear singularities in seismic data or shocks in hyperbolic PDEs, often require a representation which precisely detects orientation and provides sparse representations of typical signals. Recently, this has been achieved by the creation of group-adapted frame families such as Shearlets and Curvelets.

Finally, alpha-Modulation Spaces are a form of interpolating between Gabor and wavelet frames, i.e., the choice of the parameter alpha tunes the system to have more Gabor or more wavelet-like properties. This additional freedom may be useful to optimize the trade-off between the detection of singularities and of textures. Like non-exact Gabor frames, alpha-Modulation frames can have better time-frequency localization than allowed by the Balian-Low Theorem. The use of redundancy for obtaining frames with optimal localization properties remains an actively pursued problem with relevance in wireless communications.

3 Practical Aspects of the Topic of this Workshop

This workshop focused on optimality principles in the digital encoding of analog signals, a topic which poses challenges for mathematics, electrical engineering and computer science.

Digital signal transmissions have revolutionized our daily lives, from cellular phones and Voice-over-Internet-Protocol telephony to High-Definition Television and other streaming media. The use of digitization helps suppress distortions typical for analog devices and allows seemingly faultless communication by incorporating redundancy, that is, repetitive information. However, at times the digital nature of error suppression leads to artifacts that do not resemble the graceful degradation we recall from analog technology. Such problems could range from blocky images, choppy satellite radio, or dropped cell-phone calls to possible instabilities in digital fly-by-wire control systems.

This undesirable behavior can be caused by imperfections in the digitization process or by transmissions using digital error-correction protocols which are not well-adapted to the type of analog signal under consideration. Understanding the best way to use redundancy in the digitization as well as in the transmission processes seems the key to significant progress in this field. In both cases, the quality of the reconstructed analog signal should be the guiding principle to the optimal design of systems for conversion and transmission. Engineers have already contributed many approaches to address this challenge, which give rise to interesting mathematical questions. Answering these questions as well as developing a systematic treatment is a primary motivation for this workshop.

The theory of frames is the mathematical formulation for incorporating redundancy in a linear representation of a signal. Oversampling of audio signals stored on Compact Discs is a simple example of such digital encoding. Recently, significant progress has been made to reduce the impact of typical errors in digital transmissions (round-off errors, noise and data loss) by the design of frames appropriate for signal encoding. However, these results have mostly addressed different sources of errors separately. For a more comprehensive treatment, the exchange of ideas between mathematics and engineering needs to be strengthened.

A frequent challenge in the design of frames is to meet a symmetry requirement. This could mean adapting the frame to resolve an image without directional preferences or encoding a cell phone signal in a way that makes it easy to adjust for delays or shifts in frequency due to motion of the handset. The detailed analysis of the group structure underlying such design constraints is essential for new results.

Numerical efficiency is often an important factor in the design of encoding. Recently, it has been shown that signals with sparse frame expansions can be recovered accurately by relatively few measurements. This is essential for ultra-wideband transmissions, where sampling at the Nyquist rate is unfeasible. Apart from minimizing the number of required measurements, the computational effort of signal reconstruction is important for numerical efficiency. Hierarchical structures mimicking our cognitive system seem to offer a good trade-off between performance and computational effort. Wavelets, which are used for the encoding of images for High-Definition Television, incorporate such an efficient hierarchical structure, but their redundancy-free design does not provide much flexibility to realize desirable symmetry properties, leading to commonly known block artifacts in images. Within the last few years, efficient image encoding techniques have been emerging, which avoid directional preferences with the help of frame representations.

Certain situations require a more refined notion of redundancy, for example, when sensors have been somewhat randomly scattered across a terrain or, in medical applications, a patient body. Assuming a fixed monitoring range for each sensor, they may overlap to varying degrees in different locations, which means they report with a varying amount of repetitive information. The flexible architecture of fusion frames offers a general setting to explore optimal designs in this context. In fact, the underlying concept may be a more realistic model for our cognitive process, and allow us to realize its versatility in many applications of signal analysis and communication.

Generally, a comprehensive mathematical answer to the typical combination of problems treated by engineers requires a better understanding of both practical and theoretical aspects of frame design. Therefore, this workshop brought together researchers from engineering and mathematics, in order to promote the exchange necessary for significant contributions to this field.

4 Outcome of the Meeting

In the sequel, we intend to now delve deeper into the particular topics of the talks which were given at the workshop and detail recent developments, the conjectures and problems, the points of controversy, relevant literature as well as new directions associated with those.

The topics this workshop is devoted to can be partitioned – as already mentioned in the previous section – into four major subtopics: Sparsity and ℓ_1 Minimization, Sigma-Delta Quantization, Pure Frame Theory, and Structured Decompositions. We now focus on each of those subtopics and devote one section to discuss the particular way it appeared in our workshop.

4.1 Sparsity and ℓ_1 Minimization: Recent Developments and Open Problems

The role of randomness was the main aspect of sparsity that was discussed during the workshop. The talk by Robert Calderbank concerned random signals and deterministic sensing vs. the usual formulation of compressed sensing with deterministic vectors and random sensing matrices.

4.1.1 Robert Calderbank: Fast Reconstruction Algorithms for Deterministic Sensing Matrices and Applications

Summary: Compressed sensing is a novel technique to acquire sparse signals with few measurements. Normally, compressed sensing uses random projections as measurements. Here we design deterministic measurements and an algorithm to accomplish signal recovery with computational efficiency. A measurement matrix is designed with chirp sequences forming the columns. Chirps are used since an efficient method using FFTs can recover the parameters of a small superposition. We show that this type of matrix is valid as compressed sensing measurements. Simulations show successful recovery of signals with sparsity levels similar to those possible by matching pursuit with random measurements. For sufficiently sparse signals, our algorithm recovers the signal with computational complexity O(KlogK) for K measurements. This is a significant improvement over existing algorithms. References: [2, 45].

Holger Rauhut addresses the need for reducing randomness in the sensing matrix. This can be achieved with matrices which are obtained from the orbits of random vectors under the cyclic group or the Heisenberg-Weyl group.

4.1.2 Holger Rauhut: Circulant and Toeplitz matrices in Compressed Sensing

Summary: Compressed sensing seeks to recover a sparse vector from a small number of linear and non-adaptive measurements. While most work so far focuses on Gaussian or Bernoulli random measurements we investigate the use of partial random circulant and Toeplitz matrices in connection with recovery by ℓ_1-minization. In contrast to recent work in this direction we allow the use of an arbitrary subset of rows of a circulant and Toeplitz matrix. Our recovery result predicts that the necessary number of measurements to ensure sparse reconstruction by ℓ_1-minimization with random partial circulant or Toeplitz matrices scales linearly in the sparsity up to a $\log$-factor in the ambient dimension. This represents a significant improvement over previous recovery results for such matrices. As a main tool for the proofs we use a new version of the non-commutative Khintchine inequality. Reference: [62].

Finally, the talk by Rayan Saab explores replacing ℓ^1-optimization with non-convex optimization with respect to an ℓ^p-function, $0 < p < 1$.

4.1.3 Rayan Saab: Sparse recovery via non-convex optimization: instance optimality

Summary: It has been recently shown that one can recover/decode estimates of sparse and compressible signals from an ”incomplete” set of noisy measurements via l1-norm minimization methods under certain conditions on the ”measurement matrix”. For example, these conditions are satisfied when the matrix is a random matrix whose entries are drawn i.i.d. from a Gaussian distribution.

In this talk, we present the theoretical recovery guarantees obtained when decoding by p-quasinorm minimization with $0 < p < 1$ in the setting described above, and we prove that the corresponding guarantees can be better than those one can obtain in the case of one-norm minimization. In particular, we show that decoders based on p-quasinorm minimization are (l^2,l^p) instance optimal. Moreover, these decoders are (l^2,l^2) instance optimal in probability (this latter relates to a result on distances of p-convex bodies to their convex hulls). Finally, we comment on algorithmic issues.

The following problems remain challenging:

Problem 4.1

1. *Find compressive sensing matrices for specific applications.*
2. *Find structured sensing matrices which admit a low-complexity check for the RIP property.*
3. *Optimize the computational complexity of decoding.*

4.2 Sigma-Delta Quantization: Recent Developments and Open Problems

4.2.1 John J. Benedetto: Nonlinear frame-theoretic problems and some solutions

Here is a brief summary of the talk by Benedetto: “Frame theoretic modeling has emerged as an effective means of addressing certain problems where numerical stability and robust signal representation are desired goals. There is also a new level of applicability where frames are intrinsic to realistic modeling of some physical phenomena. We presented three examples from current and central research areas. These areas are the following: classification problems for hyper- and multi-spectral imaging data; coding or quantization in low bit environments; and the formulation of ambiguity functions in the setting of vector-valued codes arising in multi-sensor or MIMO settings. Besides frame theoretic ideas and harmonic analysis, other necessary mathematical tools involved finite group representations and algebraic number theory. Some of the relevant references for this presentation are [5, 6, 7, 8, 9].

4.2.2 Bernhard Bodmann: Correcting erasures of quantized frame coecients

Here is a brief summary of the talk by Bodmann: "In this talk we investigate an algorithm for the suppression of errors caused by quantization of frame coecients and by erasures in their subsequent transmission. The erasures are assumed to happen independently, modeled by a Bernoulli experiment. The algorithm for error correction in this study embeds check bits in the quantization of frame coecients, causing a possible, but controlled quantizer overload. If a single-bit quantizer is used in conjunction with codes which satisfy the Gilbert Varshamov bound, then the contributions from averaging over erasures and from the quantization error are shown to have bounds with the same asymptotics in the limit of large numbers of frame vectors. Joint work with Pete Casazza, Gitta Kutyniok and Steven Senger."

4.2.3 Sinan Güntürk: Recent advances in sigma-delta modulation

Here is a brief summary of this talk by Güntürk: "Sigma-Delta modulation is a popular method for analog-to-digital conversion of bandlimited signals that employs coarse quantization coupled with oversampling. The standard mathematical model for the error analysis of the method measures the performance of a given scheme by the rate at which the associated reconstruction error decays as a function of the oversampling ratio λ. It was recently shown that exponential accuracy of the form $O(2^{-r\lambda})$ can be achieved by appropriate one-bit Sigma-Delta modulation schemes. By general information-entropy arguments r must be less than 1. The current best known value for r is approximately 0.076. The schemes that were designed to achieve this accuracy employ the "greedy" quantization rule coupled with feedback filters that fall into a class we call "minimally supported". In this talk, we present the minimization problem that corresponds to optimizing the error decay rate for this class of feedback filters. We solve a relaxed version of this problem exactly and provide explicit asymptotics of the solutions. From these relaxed solutions, we find asymptotically optimal solutions of the original problem, which improve the best known exponential error decay rate to $r \approx 0.102$. Our method draws from the theory of orthogonal polynomials; in particular, it relates the optimal filters to the zero sets of Chebyshev polynomials of the second kind."

4.2.4 Mark Lammers: Uncertainty in finite frames with application to quantization

Mark Lammers started his talk by motivating a characterization of localization/uncertainty in the nite dimensional setting using some recent work in Sigma Delta quantization [10], cf, [55]. This joint work with Blum/Powell/Yılmaz exploits an alternate dual, i.e., a non-canonical dual, to reduce the error of the quantization process. A good alternate dual is found by minimizing the ℓ^2 norms of the dual frame vectors after applying a nite dierence matrix – this difference matrix depends on the order of the underlying sigma delta quantization scheme; the dual that is obtained for an rth order scheme is the so-called rth-order Sobolev dual.

In the second part of his talk, Mark Lammers presented his recent joint work with Fickus and Powell: "Using a nite dierence Matrix D and the Discrete Fourier Transform matrix F leads to a natural representation of the Heisenberg product, $\|Dv\|\|DFv\|$, in the nite setting. A number of authors have used these matrices to develop nite versions of the Gauss and Hermite functions as eigenvectors of the Discrete Fourier Transform, as well as some nite versions of the classical uncertainty principle." Lammers presented some initial ndings, inspired by the Balian-Low theorem, for both general nite frames and nite Gabor systems.

4.2.5 Alex Powell: Error bounds for consistent reconstruction by soft thresholding

Here is a brief summary of the talk by Powell: "We considered the problem of signal reconstruction from quantized frame coecients under Bennetts white noise model. The Rangan-Goyal (RG) algorithm addresses this problem with a recursive soft thresholding procedure based on consistent reconstruction; the RG algorithm may be viewed as a generalization of the Kaczmarz algorithm that is specically adapted to bounded noise. We derived rened mean squared error bounds for the Rangan-Goyal algorithm in the settings of random and deterministic frame measurements. In particular, he showed that the RG algorithm achieves MSE of order $1/N^2$, where N is the number of measurements. We also showed that frame ordering issues play an important role in the analysis."

4.3 Pure Frame Theory: Recent Developments and Open Problems

4.3.1 Matt Fickus: Tight Frame Constructions

Matt Fickus presented a recent groundbreaking piece of work on the construction of tight frames due to Casazza, Fickus, Mixon, Wang and Zhou [23]. For the last 20 years, frame theory has relied on a sequence of existence proofs to tell when tight frames exist. Often, we had little idea how to actually find these frames. This new work gives an exact and explicit construction of tight frames. As a consequence, the authors of [23] completely resolve a much worked on and important problem in fusion frame theory by giving a complete characterization of those tuples (K, L, N) so that a Hilbert space of dimension N has a family of subspaces $\{W_k\}_{k=1}^K$ with $dim\ W_k = L$, for all $k = 1, 2, \ldots, K$ and

$$\sum_{k=1}^{K} P_{W_k} = M \cdot I,$$

where P_{W_k} is the orthogonal projection onto W_k.

There are still a number of important open problems in fusion frame theory, such as:

Problem 4.2 *Classify the (K, L, N) and weights $\{v_k\}_{k=1}^K$ so that*

$$\sum_{k=1}^{K} v_k P_{W_k} = M \cdot I.$$

And preferably, give a construction for the projections.

Problem 4.3 *Given $\lambda_1, \lambda_2, \ldots, \lambda_N$, construct the projections and weights in Problem 4.2 so that the frame operator associated with the fusion frame $\{v_k, W_k\}_{k=1}^K$ has $\{\lambda_j\}_{j=1}^N$ as its eigenvalues (instead of constant eigenvalues M).*

4.3.2 Gitta Kutyniok: From frames to fusion frames

Gitta Kutyniok gave a talk about the new rapidly emerging theory of fusion frames. Frames have been a focus of study in the last two decades in applications where redundancy plays a vital and useful role. However, recently, a number of new applications have emerged which cannot be modeled naturally by one single frame system. They typically generally share a common property that requires distributed processing such as sensor networks.

4.4 Beyond Frame Theory

In the first part of her talk, Gitta Kutyniok gave an overview of the basic theory of fusion frames and of first results in this new area. Fusion frames, which were first introduced in [26], where they were still coined *frame of subspaces*, and then further developed in [28], are a notion which precisely satisfies the aforementioned required properties. Given a Hilbert space $\mathcal{H}$ and a family of closed subspaces $\{\mathcal{W}_i\}_{i\in I}$ with associated positive weights v_i, $i \in I$, a *fusion frame* for $\mathcal{H}$ is a collection of weighted subspaces $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|P_i f\|^2 \leq B\|f\|^2 \qquad \text{for any } f \in \mathcal{H},$$

where P_i is the orthogonal projection onto $\mathcal{W}_i$. Therefore a fusion frame can be regarded as a frame-like collection of subspaces in some Hilbert space. A fusion frame can indeed be regarded as a generalization of frame theory, therefore as *going beyond frame theory*.

This area poses many new challenging problems, due to the delicateness of the subspaces of different dimensions (in frame theory you basically only deal with 1-dimensional subspaces), and even more with weights being involved. As a source for further information the recently designed webpage on fusion frames [17] was referred to.

4.5 Further Directions in this Area

Then Gitta Kutyniok focussed on three very recent results in this area, each one being linked to a particular application where a necessity for the introduction and careful analysis of fusion frames occurs. These links indeed indicate the power of fusion frame theory and shows that this will be one direction the area of frame theory will develop towards.

- Communication: Noise and Erasure Resilience (with Calderbank, Liu, Pezeshki)
- Sensing: Sparse Reconstruction (with Boufounos)
- Coding: Erasure-Proof Coding (with Bodmann and Pezeshki)

4.5.1 Noise and Erasure Resilience

The fundamental question underlying this complex is, how robust fusion frame processing is with respect to noise and erasures (see earlier work in [27]). Gitta Kutyniok presented work joint with Calderbank, Liu, and Pezeshki [54, 61], in which a random signal $x \in \mathbb{R}^M$ with zero mean and $E[xx^T] = R_{xx}$ is considered. Interestingly, it could be shown that tight fusion frames are optimally robust with respect to noise, and fusion frames forming a Grassmannian packing are optimally robust with respect to erasures of subspaces.

4.5.2 Sparse Reconstruction

Sparsity has recently gained tremendous attention due to the fact that it allows for unique solutions of underdetermined systems. In a work joint with Boufounos [18], which Gitta Kutyniok reported upon, combinations of fusion frame measurements are considered which are underdetermined. In fact, it was very enlightning to see that in this case the angles of subspaces – as already for resilience analysis – become apparent, for instance, in the coherence property the combination coefficients need to satisfy.

4.5.3 Erasure-Proof Coding

At last, Gitta Kutyniok presented a work joint with Bodmann [16] in which the question was asked whether fusion frames can provide erasure-proof coding. She showed that when considering linear transmission of vectors through a memoryless analog erasure channel with erasures modeled by a family of binary random variables, fusion frames provide a way to achieve high error decay rates in the sense that the mean-square error decays faster than any inverse power of the number of transmitted coefficients.

4.6 Conclusion

We expect that eventually frame theory will be considered within the more flexible framework of fusion frame theory, thereby along the way opening possibilities for applications yet still to be imagined.

4.6.1 Marcin Bownik (Joint with John Jasper) Characterization of sequences of frame norms

We show that frames with frame bounds A and B are images of orthonormal bases under positive operators with spectrum contained in $\{0\} \cup [\sqrt{A}, \sqrt{B}]$. Then, we give an explicit characterization of the diagonals of such operators, which in turn gives a characterization of the sequences which are the norms of a frame. Our result extends the tight case result of Kadison [47, 48], which characterizes diagonals of orthogonal projections, to a non-tight case. We illustrate our main theorem by studying the set of possible lower bounds of positive operators with prescribed diagonal.

The outstanding problem in this area is an infinite dimensional analogue of the Schur-Horn theorem.

Problem. Suppose that E is a self-adjoint operator on a Hilbert space $\mathcal{H}$. Characterize the set

$$\mathcal{D} = \{\{\langle Ee_i, e_i\rangle\}_{i\in I} \,\Big|\, \{e_i\} \text{ is an orthonormal basis of } \mathcal{H}\}.$$

General background on this problem can be found in [50]. Here, we make only a few historical observations. If $\mathcal{H}$ is finite dimensional, then the answer gives the classical Schur-Horn theorem. If E is an orthogonal projection, then the answer was given by Kadison in [47, 48]. If E is a trace class operator, then the answer was given by Arveson and Kadison [4]. Finally, Kaftal and Weiss [51] extended this characterization to compact operators E. Moreover, Neumann [60] gave an approximate answer to this problem in terms of the ℓ^∞ closure of the convexity condition.

Notable special cases of this problem include characterization of operators with 3 point spectrum, or in general, with n point spectrum. In this case the necessary condition (true for normal operators as well) was given by Arveson [3].

4.6.2 Eric Weber: Some Algebraic Aspects of the Paving and Feichtinger Conjectures

The set of symbols $\mathcal{P}_\mathcal{L}$ from $L^\infty[0,1]$ which satisfies the paving conjecture forms a closed ideal under convolution. The set of symbols $\mathcal{R}_\epsilon$ which satisfies the $\mathcal{R}_\epsilon$ conjecture is nearly a closed ideal, lacking possibly closure under addition. The set of symbols $\mathcal{F}$ which satisfies the Feichtinger Conjecture additionally may not be closed in norm. Which, if any, of these sets are all of $L^\infty[0,1]$ is an open problem.

Theorem 4.4 *The following are equivalent:*

1. $\mathcal{P}_\mathcal{L} = \mathcal{R}_\epsilon = \mathcal{F} = L^\infty[0,1]$*;*
2. $\mathcal{P}_\mathcal{L} = \mathcal{R}_\epsilon$*;*
3. $\mathcal{R}_\epsilon$ *is a subspace;*
4. $\mathcal{R}_\epsilon$ *is convex.*

Problem 4.5 *Is* $\mathcal{P}_\mathcal{L}$ *a maximal ideal in* $L^\infty[0,1]$*? (Not a proper one.)*

Theorem 4.6 *The set* $L^\infty[0,1]$ *is an abstract Segal algebra in* $L^1[0,1]$*, and* $\mathcal{P}_\mathcal{L}$ *is an ideal in* $L^\infty[0,1]$.

Question: What is the structure of ideals in an abstract Segal algebra without approximate units?
For background and related material see [1, 12, 13, 30, 41, 49]

4.6.3 Peter Casazza: Five deep problems in frame theory: A progress report

Pete Casazza gave a progress report on several deep problems in frame theory. The first was skipped since it was decided to have M. Fickus present this topic by itself. The first problem presented was *The Paulsen Problem* from Bodmann/Casazza [15]: Given $\epsilon > 0$ and N, find the optimal $\delta > 0$ so that for every δ-nearly equal-norm, δ-nearly Parseval frame $\{f_i\}_{i=1}^M$ for an N-dimensional Hilbert space $\mathbb{H}_N$, there exists an equal-norm Parseval frame $\{g_i\}_{i=1}^M$ for $\mathbb{H}_N$ satisfying:

$$\sum_{i=1}^M \|f_i - g_i\|^2 < \epsilon.$$

In [15] it is shown that when N, M are relatively prime, we can get

$$\sum_{i=1}^M \|f_i - g_i\|^2 \le \frac{27}{8} N^2 M (M-1)^8 \delta.$$

It was also shown in [15] that the Paulsen problem is equivalent to a fundamental open problem in operator theory: Given an orthogonal projection on $\mathbb{H}_N$ with nearly constant diagonal, what is the closest constant diagonal projection? We still have the problem,

Problem 4.7 *What is the best bound for the Paulsen problem?*

The second problem presented in this talk involves the search for the correct notion of redundancy for infinite dimensional Hilbert spaces? A recent paper of Balan, Casazza and Landau [14] has given the first answer to this problem by giving a notion of redundancy which satisfies our basic *wish list* for redundancy:

1. The redundancy of any frame for the whole space would be greater than or equal to one.
2. The redundancy of a Riesz basis would be exactly one.
3. The redundancy would be additive on unions of frames.
4. Any frame with redundancy bigger than one would contain in it a frame with redundancy arbitrarily close to one.

Also in [14], the old finite dimensional version of this question is also answered. In [14], they rely heavily on ℓ_1-localization.

Problem 4.8 *Can the results of [14] be proved with something weaker than ℓ_1-localization? In particular, is ℓ_2-localization enough?*

The third problem presented involves one of the most celebrated theorems in analysis, the *Bourgain-Tzafriri restricted invertibility theorem* [19]. In 1987 when they proved this theorem, the authors asked: *Is there an infinite dimensional restricted invertibility theorem?* Recently, Casazza and Pfander [24], using the notion of "density" from frame theory answered this question for ℓ_1-localized frames. The next step is to remove this assumption.

Problem 4.9 *Are the results of [24] true with ℓ_2-localization? Do we need any assumption at all to get an infinite dimensional restricted invertibility theorem?*

The last problem presented was the Kadison-Singer Problem [49, 25, 30, 24], which has the following equivalent form in frame theory [24]:

Problem 4.10 *Do there exist universal constants $0 < c < 1$ and $r \in \mathbb{N}$ so that for all equal-norm Parseval frames $\{f_i\}_{i=1}^{2N}$ in $\mathbb{H}_N$, there is a partition $\{A_j\}_{j=1}^{r}$ of $\{1, 2, \ldots, 2N\}$ satisfying for all $j = 1, 2, \ldots, r$ and all scalars $\{a_i\}_{i \in A_j}$:*

$$\|\sum_{i\in A_j} a_i f_i\|^2 \geq c \sum_{i \in A_j} |a_i|^2?$$

In [24] it was shown that $r = 2$ fails in Problem 4.10, but this was an existence proof. Recently, Casazza, Fickus, Mixon, Tremain [22] gave a concrete construction of the frames which fail for $r = 2$ in problem 4.10. Most people believe that Problem 4.10 has a negative answer and this concrete construction (and a generalization of it in [22]) should give us a better chance at the general case.

4.6.4 Shidong Li: Image fusion, one-side frame perturbation and a dimension invariance principle

Three connected topics were reported, all resulting from frame fundamental image fusion applications.

1. From the modeling of imaging devices, a *one-side frame perturbation* (OSFP) [57] naturally arises.

   Let $\{h_n\}$ be the sensory frame of the image devices. Then the actual sensing is the process of $\{\langle f, h_n\rangle\}$. But $\{h_n\}$ is never known precisely. Consequently, the image reconstruction will have to be through an dual sensory frame $\{\tilde{h}_n^a\}$ of an approximation $\{h_n^a\}$ of $\{h_n\}$, namely,

   $$\forall f \in \mathcal{X}, \quad f^a = \sum_n \langle f, h_n\rangle \tilde{h}_n^a$$

   Stability of such an one-side frame perturbation is studied and established.

Points: (a) The notion of one-sided frame perturbation (OSPF) and its stability are new and exist in all applications. Potentially useful also in theoretical develop of frame approximations.

(b) The stability of OFSP is different from that of the traditional frame perturbation studies. The later is about how a frame remains a frame when elements are marginally altered. Here, the stability issue of OSFP is about how an estimation f^a can be a stable approximation to the actual f.

2. Image fusion through *sensory frame formulation* is also presented [58].

   The general observation model can always be written as $Hf = g$ where H consists of all sensory frames $\{h_n^{(k)}\}_{k,n}$, $(k = 1, \cdots, K)$, f is the original image and g is the observation.

   Points: (a) The frame image fusion formulation is new. It uses the spatial reversal of sensors' actual impulse response functions $\{r^{(k)}\}$ as sensory frames. $\{r^{(k)}\}$ is measurable, making this formulation accurate.

   (b) A new iterative reconstruction procedure is proposed whose convergence is also established. Let $\tilde{H}$ be a "low-pass operator" such that $0 < \tilde{H}H < Id$, and

$$\tilde{H}H + R = I. \tag{1}$$

   Let

$$f_{n+1} = \tilde{H}Hf + Rf_n = \tilde{H}g + Rf_n.$$

   Then the sequence of images $\{f_n\}$ converges to the original image f in Euclidean norm for any f_0.

   (c) New regularization operator. One can show that this iteration algorithm is equivalent to the classical Tikhonov regularization when $\tilde{H} = H'$. But $\tilde{H}$ can be a lot more general. Numerical studies show that $\tilde{H} = (H')^m$ for some integer $m > 1$ is more stable than that of $\tilde{H} = H'$.

3. Finally, *a dimension invariance principle* [20] is established for the inversion of a general class of circulant matrices, which certainly includes the evaluation of dual sensory frames.

   The principle states that if the sensory frame and its dual are compactly supported, compact duals can be evaluated from a subspace $\mathcal{X}$ with a fraction of the actual dimension (of the image space $\mathcal{H}$), and stay valid while naturally embedded to $\mathcal{H}$. This dimension invariance principle coupled with FFT based method make the implementation a lot more feasible. Error bounds are also established when approximate duals are utilized.

   Points: (a) The dimension invariance principle (DIP) applies to a variety of uniform and non-uniform multi-frames of translates and multi-Gabor frames.

   (b) DIP is indispensable in non-separable image fusion applications (instead of column-by-column processing), where image fusion tasks amounts to matrices of the order $10^{18} \times 10^{18}$ for (merely) images of the size 256×256. Without DIP, this is astronomically difficult if not impossible.

4.7 Structured Decompositions: Recent Developments and Open Problems

4.7.1 Ole Christensen: Dual pairs of Gabor frames

Ole Christensen's talk described ways of constructing explicitly given pairs of dual Gabor frames. The constructed pairs appear to be very attractive from the applied point of view: they are generated by compactly supported functions and fast decay in the Fourier domain can be achieved.

Another new aspect is an analysis of the necessary conditions for a compactly supported window to have a compactly supported dual window. It was shown that there is a clear relationship between the redundancy of the frame and the size of the support for the dual window.

Problem 4.11 *In order for the construction to work for smooth functions, a quite high redundancy is required. It is conjectured that this redundancy can not be avoided.*

4.7.2 Karlheinz Gröchenig: Gabor Frames

Gabor frames possess a rich and deep structure theory. These explain the density of Gabor frames and yield many characterizations of Gabor frames [36, 38].Sometimes these characterizations lead to checkable sufficient conditions for Gabor frames.

Yet despite the beautiful theory the following problem is largely unsolved: Fix a window function g. Determine all lattices Λ in the time-frequency plane, such that the corresponding Gabor system $\{e^{2\pi i\lambda_2 t}g(t-\lambda_1) : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ generates a frame.

Currently there are only three (three!) window functions for which a complete classification of all Gabor frames is known. These are the Gaussian, the hyperbolic cosine, and the one-sided exponential function. The case of the Gaussian was settled with complex methods by Lyubarski and Seip[59, 66], the two other functions can be reduced to the Gaussian [39].

A similar result seems to hold for Hermite functions, but currently only an explicit sufficient density condition is known [37]. This condition is surprising because the density depends on the order of the Hermite function. Some counter-examples indicate that this result might be optimal. For Hermite functions the methods used are again from complex analysis, but the Wexler-Raz conditions lead to a new type of interpolation problem that leaves experts clueless at the moment.

Problem 4.12 *It remains a challenge to find other classes of window functions for which a complete characterization of all lattices that generate Gabor frames is possible.*

1. *Is such a characterization possible for B-splines?*
2. *Which properties of a window function are relevant?*
3. *What other methods besides complex analysis are suitable to investigate this question?*

4.7.3 Bin Han: Matrix Extension with Symmetry and Symmetric Orthonormal Complex M-wavelets

Bin Han's talk concerned the matrix extension problem for wavelets. The matrix extension problem with symmetry is to find a unitary square matrix P of 2π-periodic trigonometric polynomials with symmetry such that the first row of P is a given row unit vector of 2π-periodic trigonometric polynomials with symmetry. Matrix extension plays a fundamental role in many areas such electronic engineering, wavelet analysis, and applied mathematics, for example, in the construction of symmetric orthonormal complex wavelets and symmetric tight wavelet frames. Though several exciting recent developments on the matrix extension with symmetry and symmetric orthonormal complex wavelets have been reported in [42, 43, 44], there are still many unresolved problems in this area.

Problem 4.13 *Of particular importance are the following questions:*

1. *Matrix extension problem with symmetry in high dimensions and its applications to multivariate symmetric wavelets. This problem is closely related to multivariate polynomials in algebraic geometry and is of importance for wavelet applications in image processing and high-dimensional problems.*
2. *Matrix extension problem with symmetry for biorthogonal wavelets and biorthogonal multiwavelets. Even in dimension one, this problem remains unsolved satisfactorily and it greatly hinders the applications of biorthogonal (multi)wavelets.*
3. *Directional complex wavelets. Directional representations are of fundamental importance in image processing. There are many interesting approaches on directional wavelets such as curvelets, shearlets, contourlets, and bandlets, etc. Another approach is to study directional complex wavelets. Though some initial encouraging research has been done in this direction, it is still in its early stage.*

4.7.4 Götz E. Pfander: Gabor frames for $\mathbb{C}^d$ and some applications

Götz Pfander's talk reviewed recent results on the geometry of Gabor systems in finite dimensions. For example, he discussed the coherence of Gabor systems, the linear independence of subsets of Gabor systems[52, 56], and the condition number of matrices formed by a small number of vectors from a Gabor system[64, 65]. He stated a result on the recovery of signals that have a sparse representation in certain Gabor systems. Below, on open questions on the linear independence, namely, whether there exists Gabor systems of d^2 vectors in general linear position in $\mathbb{C}^d$, will be posed.

Let G denote a finite Abelian group and $\widehat{G}$ its dual group. Recall $\widehat{G} \subseteq \mathbb{C}^G = \{f : G \longrightarrow \mathbb{C}\}$, $G \simeq \mathbb{G}$, and the Fourier transform of $f \in \mathbb{C}^G$ is in $\widehat{f}(\xi) = \sum_{x \in G} f(x)\,\overline{\xi(x)}$, $\xi \in \widehat{G}$. Translation operators T_x, $x \in G$, and modulation operators M_ξ, $\xi \in \widehat{G}$, on $\mathbb{C}^G$ are unitary and given by $(T_x f)(t) = f(t-x)$ and $(M_\xi f)(t) = f(t) \cdot \xi(t)$. Time-frequency shift operators are $\pi(\lambda) f = T_x \circ M_\xi f$, $\lambda = (x, \xi) \in G \times \widehat{G}$. The system $\{\pi(\lambda) g : \lambda \in G \times \widehat{G}\} \subseteq \mathbb{C}^G$ is called (full) Gabor system with window $g \in \mathbb{C}^G$, it consists of $|G|^2$ vectors in a $|G|$ dimensional space.

Theorem 4.14 *[52, 56]*

1. *If $G = \mathbb{Z}_p$, p prime, then exists $g \in \mathbb{C}^G$ such that the vectors in $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ are in general linear position.*
2. *If $G = \mathbb{Z}^2 \times \mathbb{Z}^2$, then exists no $g \in \mathbb{C}^G$ such that the vectors in $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ are in general linear position.*

Rudimentary numerical experiments encouraged the following question. Note that a positive answer would lead to a central generalization in the sampling theory for operators as discussed in [63].

Problem 4.15 *[56] For G cyclic, that is, $G = \mathbb{Z}_n$, $n \in \mathbb{N}$, does there exist some $g \in \mathbb{C}^G$ such that the vectors in $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ are in general linear position?*

4.7.5 Gabriele Steidl: The Continuous Shearlet Transform in Arbitrary Space Dimensions

Gabriele Steidl's talk concerned the definition of a continuous shearlet transform in higher dimensions, which in fact was even association to a square-integrable representation of the full n-variate shearlet group, thereby providing an exceptionally 'nice' mathematical structure. It was then shown that a shearlet coorbit theory could be established, and canonical scales of smoothness spaces could be derived with associated Banach frames. This shearlet transform was then applied to characterize certain singularities in signals (without numerical experiments).

Problem 4.16 *Some related problems can be stated as follows:*

1. *How do the shearlet coorbit spaces relate to classical smoothness spaces like Besov spaces and curvelet decomposition spaces?*
2. *Can (shearlet) interpolation spaces be established?*
3. *How to construct shearlets which are compactly supported in spatial domain?*
4. *What is the exact relation/connection with classical methods in image processing like structure tensors?*

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