Drinfeld Modules and t-Modules
A Very Brief Introduction

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Outline

1 Classical Forebears

2 Analogues for Function Fields

Classical Forebears

Arithmetic objects from characteristic 0

- The multiplicative group and $\exp(z)$
- Elliptic curves and elliptic functions
- Abelian varieties

The multiplicative group

We have the usual exact sequence of abelian groups

$$0 \to 2\pi i\mathbb{Z} \to \mathbb{C} \stackrel{\exp}{\to} \mathbb{C}^\times \to 0,$$

where

$$\exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} \in \mathbb{Q}[[z]].$$

For any $n \in \mathbb{Z}$,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times \\ \downarrow{\scriptstyle z\mapsto nz} & & \downarrow{\scriptstyle x\mapsto x^n} \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times \end{array}$$

which is simply a restatement of the functional equation

$$\exp(nz) = \exp(z)^n.$$

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Roots of unity
Torsion in the multiplicative group

The n-th roots of unity are defined by

$$\mu_n := \{\zeta \in \mathbb{C}^\times \mid \zeta^n = 1\} = \{\exp(2\pi i a/n) \mid a \in \mathbb{Z}\}$$

- $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
- **Kronecker-Weber Theorem:** The cyclotomic fields $\mathbb{Q}(\mu_n)$ provide explicit class field theory for $\mathbb{Q}$.
- For $\zeta \in \mu_n$,
$$\log(\zeta) = \frac{2\pi i a}{n}, \quad 0 \le a < n.$$

Elliptic curves over $\mathbb{C}$

Smooth projective algebraic curve of genus 1.

$$E : y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C}$$

$E(\mathbb{C})$ has the structure of an abelian group through the usual chord-tangent construction.

Weierstrass uniformization

There exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, so that if we consider the lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

then the *Weierstrass $\wp$-function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

The function $\wp(z)$ has double poles at each point in Λ and no other poles.

We obtain an exact sequence of abelian groups,

$$0 \to \Lambda \to \mathbb{C} \overset{\exp_E}{\to} E(\mathbb{C}) \to 0,$$

where

$$\exp_E(z) = (\wp(z), \wp'(z)).$$

with commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\exp_E} & E(\mathbb{C}) \\ {\scriptstyle z \mapsto nz}\downarrow & & \downarrow{\scriptstyle P \mapsto [n]P} \\ \mathbb{C} & \xrightarrow{\exp_E} & E(\mathbb{C}) \end{array}$$

where $[n]P$ is the n-th multiple of a point P on the elliptic curve E.

Periods of E

How do we find ω_1 and ω_2?

An elliptic curve E,

$$E : y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C},$$

has the geometric structure of a torus in $\mathbb{P}^2(\mathbb{C})$. Let

$$\gamma_1, \gamma_2 \in H_1(E, \mathbb{Z})$$

be generators of the homology of E.

Then we can choose

$$\omega_1 = \int_{\gamma_1} \frac{dx}{\sqrt{4x^3 + ax + b}}, \qquad \omega_2 = \int_{\gamma_2} \frac{dx}{\sqrt{4x^3 + ax + b}}.$$

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Quasi-periods of E

- The differential dx/y on E generates the space of holomorphic 1-forms on E (differentials of the first kind).
- The differential $x\,dx/y$ generates the space of differentials of the second kind (differentials with poles but residues of 0).
- We set

$$\eta_1 = \int_{\gamma_1} \frac{x\,dx}{\sqrt{4x^3 + ax + b}}, \qquad \eta_2 = \int_{\gamma_2} \frac{x\,dx}{\sqrt{4x^3 + ax + b}},$$

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Quasi-Periods as Periods of Extensions

η_1, η_2 arise as special values of the Weierstrass ζ-function because of the way ζ is involved in the exponential functions of extensions of E by $\mathbb{G}_a$.

For $c \in \mathbb{C}$, the function of two variables

$$(z, t) \longmapsto (\wp(z), \wp'(z), t + c\zeta(z))$$

is the exponential function of a group extension G of E by $\mathbb{G}_a$:

$$0 \to \mathbb{G}_a \to G \to E \to 0.$$

Its periods are of the form $(\omega, -c\eta)$, since $\zeta(\omega/2) = \eta/2$.

When $c = 0$, the extension splits: $G = E \times \mathbb{G}_a$.

Period matrix of E

- The period matrix of E is the matrix

$$P = \begin{bmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{bmatrix}.$$

It provides a natural isomorphism

$$H^1_{\mathrm{sing}}(E, \mathbb{C}) \cong H^1_{\mathrm{DR}}(E, \mathbb{C}).$$

- **Legendre Relation:** From properties of elliptic functions, the determinant of P is

$$\omega_1 \eta_2 - \omega_2 \eta_1 = \pm 2\pi i.$$

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Abelian varieties

Higher dimensional analogues of elliptic curves

- An *abelian variety A* over $\mathbb{C}$ is a smooth projective variety that is also a group variety.
- Elliptic curves are abelian varieties of dimension 1.
- Much as for $\mathbb{G}_m$ and elliptic curves, an abelian variety of dimension d has a uniformization,

$$\mathbb{C}^d \big/ \Lambda \cong A(\mathbb{C}),$$

where Λ is a discrete lattice of rank $2d$.

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The period matrix of an abelian variety

Let A be an abelian variety over $\mathbb{C}$ of dimension d.

- As in the case of elliptic curves, there is a natural isomorphism,

$$H^1_{\text{sing}}(A, \mathbb{C}) \cong H^1_{\mathrm{DR}}(A, \mathbb{C}),$$

  given by period integrals, whose defining matrix P is called the *period matrix of A*.
- We have

$$P = \left[\omega_{ij} \,\middle|\, \eta_{ij} \right] \in \mathrm{Mat}_{2d}(\mathbb{C}),$$

  where $1 \leq i \leq 2d$, $1 \leq j \leq d$.
- The ω_{ij}'s provide coordinates for the period lattice Λ.
- The η_{ij}'s occur in periods of extensions of A by $\mathbb{G}_a$.

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Analogues for Function Fields

- Function field notation
- Drinfeld modules
  - The Carlitz module
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- t-modules (higher dimensional Drinfeld modules) & t-motives

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Function fields

Let p be a fixed prime; q a fixed power of p.

$$A := \mathbb{F}_q[\theta] \quad \longleftrightarrow \quad \mathbb{Z}$$

$$k := \mathbb{F}_q(\theta) \quad \longleftrightarrow \quad \mathbb{Q}$$

$$\overline{k} \quad \longleftrightarrow \quad \overline{\mathbb{Q}}$$

$$k_\infty := \mathbb{F}_q((1/\theta)) \quad \longleftrightarrow \quad \mathbb{R}$$

$$\mathbb{C}_\infty := \widehat{\overline{k_\infty}} \quad \longleftrightarrow \quad \mathbb{C}$$

$$|f|_\infty = q^{\deg f} \quad \longleftrightarrow \quad |\cdot|$$

Twisted polynomials

- Let $F\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the q-th power Frobenius map: $F(x) = x^q$.
- For a subfield $\mathbb{F}_q \subseteq K \subseteq \mathbb{C}_\infty$, the ring of *twisted polynomials* over K is

$$K[F] = \text{polynomials in } F \text{ with coefficients in } K,$$

subject to the conditions

$$Fc = c^q F, \quad \forall\, c \in K.$$

- In this way,

$$K[F] \cong \{\mathbb{F}_q\text{-linear endomorphisms of } K^+\}.$$

For $x \in K$ and $\phi = a_0 + a_1 F + \cdots + a_r F^r \in K[F]$, we write

$$\phi(x) := a_0 x + a_1 x^q + \cdots + a_r x^{q^r}.$$

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Drinfeld modules

Function field analogues of $\mathbb{G}_m$ and elliptic curves

Let $\mathbb{F}_q[t]$ be a polynomial ring in t over $\mathbb{F}_q$.

Definition

A *Drinfeld module* over is an $\mathbb{F}_q$-algebra homomorphism,

$$\rho : \mathbb{F}_q[t] \to \mathbb{C}_\infty[F],$$

such that

$$\rho(t) = \theta + a_1 F + \cdots + a_r F^r.$$

- ρ makes $\mathbb{C}_\infty$ into a $\mathbb{F}_q[t]$-module in the following way:

$$f * x := \rho(f)(x), \quad \forall f \in \mathbb{F}_q[t], x \in \mathbb{C}_\infty.$$

- If $a_1, \ldots, a_r \in K \subseteq \mathbb{C}_\infty$, we say ρ is *defined over K*.
- When $a_r \neq 0$, r is called the *rank* of ρ.

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The Carlitz module

The analogue of $\mathbb{G}_m$

Define a particular Drinfeld module $C : \mathbb{F}_q[t] \to \mathbb{C}_\infty[F]$ by

$$C(t) := \theta + F.$$

Thus, for any $x \in \mathbb{C}_\infty$,

$$C(t)(x) = \theta x + x^q.$$

Carlitz exponential

Set

$$\exp_C(z) := z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})}.$$

- $exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is entire, surjective, and $\mathbb{F}_q$-linear.
- Functional equation:

$$\exp_C(\theta z) = \theta \exp_C(z) + \exp_C(z)^q,$$
$$\exp_C(f(\theta) z) = C(f)(\exp_C(z)), \quad \forall f(t) \in \mathbb{F}_q[t].$$

Carlitz uniformization and the Carlitz period

We have a commutative diagram of $\mathbb{F}_q[t]$-modules,

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\exp_C} & \mathbb{C}_\infty \\ {\scriptstyle z \mapsto \theta z} \downarrow & & \downarrow {\scriptstyle x \mapsto \theta x + x^q} \\ \mathbb{C}_\infty & \xrightarrow{\exp_C} & \mathbb{C}_\infty \end{array}$$

The kernel of $\exp_C(z)$ is

$$\ker(\exp_C(z)) = \mathbb{F}_q[\theta]\pi_q,$$

where

$$\pi_q = \theta \sqrt[q-1]{-\theta} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}.$$

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Wade's result

Thus we have an exact sequence of $\mathbb{F}_q[t]$-modules,

$$0 \to \mathbb{F}_q[\theta]\pi_q \to \mathbb{C}_\infty \stackrel{\exp_C}{\to} \mathbb{C}_\infty \to 0.$$

Theorem (Wade 1941)

The Carlitz period π_q is transcendental over $\bar{k}$.

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Torsion for the Carlitz module

Theorem (Carlitz-Hayes)

Torsion of the Carlitz module provides explicit class field theory over $\mathbb{F}_q(\theta)$.

Drinfeld modules of rank r

- Suppose $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ is a rank r Drinfeld module defined over $\overline{k}$ by
$$\rho(t) = \theta + a_1 F + \cdots + a_r F^r.$$
- Then there is an unique, entire, $\mathbb{F}_q$-linear function
$$\exp_\rho : \mathbb{C}_\infty \to \mathbb{C}_\infty,$$
so that
$$\exp_\rho(f(\theta)z) = \rho(f)(\exp_\rho(z)), \quad \forall f \in \mathbb{F}_q[t].$$

Periods of Drinfeld modules of rank r

- Furthermore, there are $\omega_1, \ldots, \omega_r \in \mathbb{C}_\infty$ so that

$$\ker(\exp_\rho(z)) = \mathbb{F}_q[\theta]\omega_1 + \cdots + \mathbb{F}_q[\theta]\omega_r =: \Lambda,$$

  is a discrete $\mathbb{F}_q[\theta]$-submodule of $\mathbb{C}_\infty$ of rank r.

- **Chicken vs. Egg:**

$$\exp_\rho(z) = z \prod_{0 \neq \omega \in \Lambda} \left(1 - \frac{z}{\omega}\right).$$

- Again we have a uniformizing exact sequence of $\mathbb{F}_q[t]$-modules

$$0 \to \Lambda \to \mathbb{C}_\infty \xrightarrow{\exp_\rho} \mathbb{C}_\infty \to 0.$$

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Riemann-Legendre Relations

Quasi-periods: Quasi-periods $\eta_1, \ldots, \eta_r \in \mathbb{C}_\infty$ for ρ arise in periods of extensions of ρ by $\mathbb{G}_a$.

Legendre relation: When $r = 2$, $\omega_1\eta_2 - \omega_2\eta_1 = \zeta\pi_q$ for some $\zeta \in \mathbb{F}_q^\times$.

t-modules (Anderson)
Higher dimensional Drinfeld modules

- A *t-module A* of dimension d is a pair $(A, \mathbb{G}_a^d)$ consisting of an $\mathbb{F}_q$-linear homomorphism,

$$A : \mathbb{F}_q[t] \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty^d) \cong \mathrm{Mat}_d(\mathbb{C}_\infty[F]),$$

  such that

$$A(t) = \theta \mathrm{Id} + N + a_0 F + \cdots a_r F^r,$$

  where $N \in \mathrm{Mat}_d(\mathbb{C}_\infty)$ is nilpotent.
- Thus $\mathbb{C}_\infty^d$ is given the structure of an $\mathbb{F}_q[t]$-module via

$$f * x := A(f)(x), \quad \forall f \in \mathbb{F}_q[t],\ x \in \mathbb{C}_\infty^d.$$

Exponential functions of t-modules

- There is a unique entire $\exp_A : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$ so that

$$\exp_A((\theta \mathrm{Id} + N)z) = A(t)(\exp_A(z)).$$

- If $\exp_A$ is surjective, we have an exact sequence

$$0 \to \Lambda \to \mathbb{C}_\infty^d \xrightarrow{\exp_A} \mathbb{C}_\infty^d \to 0,$$

  where Λ is a discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{C}_\infty^d$.
- Λ is called the *period lattice* of A.
- Quasi-periods are defined via periods of extensions by copies of the additive group.

Remarks on t-modules

- When $A(t) \in \overline{k}$, we say that the t-module is *defined over* $\overline{k}$.
- In that case, $\exp_A$ has coefficients from $\overline{k}$.

Subtleties

- Surjectivity of exponential function not assured, but here *posited*.
- We do not have a product expansion for $\exp_A$ or indeed any series expansion in terms of Λ.
- Exponential function does not always completely determine t-module

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Easiest examples of t-modules

- Direct sums of t-modules, in particular Drinfeld modules
- Extensions of t-modules by $\mathbb{G}_a$ (De Rham cohomology controls how much new stuff can be obtained this way.)
- Tensor products of t-modules

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A *morphism* Θ between two t-modules $(A_1, \mathbb{G}_a^{d_1})$ and $(A_2, \mathbb{G}_a^{d_2})$ is a matrix of twisted polynomials $\Theta \in \mathrm{Mat}_{d_2 \times d_1}(\mathbb{C}_\infty[F])$ such that

$$\Theta A_1(t) = A_2(t)\Theta.$$

An *isogeny* is a morphism when $d_1 = d_2$ and the kernel of Θ is finite.

t-Motives (Anderson)

Let $\mathbb{C}_\infty[t, F] := \mathbb{C}_\infty[F][t]$, the ring of polynomials in the commuting variable t over the non-commuting ring $\mathbb{C}_\infty[F]$. A *t-motive M* is a left $\mathbb{C}_\infty[t, F]$-module which is free and finitely generated as a $\mathbb{C}_\infty[F]$-module and for which there is an $\ell \in \mathbb{N}$ with

$$(t - \theta)^\ell (M/FM) = \{0\},$$

Morphisms are morphisms of left $\mathbb{C}_\infty[t, F]$-modules.

Motives from Modules

Every t-module $(A, \mathbb{G}_a^d)$ gives rise to a unique t-motive over $\mathbb{C}_\infty$, viz.

$$M := \mathrm{Hom}^q_{\mathbb{C}_\infty}(\mathbb{G}_a^d, \mathbb{G}_a),$$

the module of $\mathbb{F}_q$-linear morphisms of algebraic groups. The action of $\mathbb{C}_\infty[t, F]$ is given by

$$(ct^i, m) \mapsto c \circ m \circ A(t^i).$$

Projections on the d coordinates give a $\mathbb{C}_\infty[F]$-basis for M, $d = \mathrm{rank}_{\mathbb{C}_\infty[F]} M$, and ℓ need not be taken greater than d.

Modules from Motives

A t-motive M has a $\mathbb{C}_\infty[F]$-basis $m_1, \ldots, m_d$ which we can use to express the t-action via a matrix $A(t) \in \mathrm{Mat}_d(\mathbb{C}_\infty[F])$.
This is compatible with the above because, if we represent arbitrary $m \in M$ as

$$m = (k_1, \ldots, k_d) \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = \mathbf{k} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix},$$

gives according to the commutativity of t with elements of $\mathbb{C}_\infty[F]$, that, with $a \in L[F]$,

$$at \cdot \mathbf{k} \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = a\mathbf{k} \cdot t \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix} = a\mathbf{k}A(t) \begin{pmatrix} m_1 \\ \vdots \\ m_d \end{pmatrix}$$

Theorem (Anderson)

The above correspondence between t-modules and t-motives gives an anti-equivalence of categories.