Algebraic relations among periods and logarithms for Drinfeld modules

BIRS Workshop on t-motives

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(Joint work with Matt Papanikolas)

NCTS and National Central University

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Notation

1. $A := \mathbb{F}_q[\theta]$;
2. $k := \mathbb{F}_q(\theta)$, $|\theta|_\infty = q$;
3. $k_\infty := \mathbb{F}_q((1/\theta))$;
4. $\mathbb{C}_\infty := \widehat{\overline{k_\infty}}$;
5. t: independent variable of θ;
6. $\mathbb{T} := \{f \in \mathbb{C}_\infty[[t]];\ f$ converges on $|t|_\infty \leq 1\}$;
7. ρ: a rank r Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$;
8. Λ_ρ: the period lattice of ρ;
9. $H^1_{DR}(\rho)$: the DeRham cohomology of ρ;
10. F_δ: the quasi-periodic function of ρ associated to a given biderivation δ

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DeRham Isomorphism

Recall the well-defined pairing:

$$\begin{array}{ccc} H^1_{DR}(\rho) \times \Lambda_\rho & \to & \mathbb{C}_\infty \\ ([\delta], \lambda) & \mapsto & \int_\lambda \delta := F_\delta(\lambda). \end{array}$$

Anderson, Gekeler: The above map is a perfect pairing. So we have the isomorphism as comparison between the DeRham and Betti cohomologies of the Drinfeld module ρ:

$$H^1_{DR}(\rho) \to \mathrm{Hom}_A(\Lambda_\rho, \mathbb{C}_\infty) =: H^{Betti}(\rho).$$

For any basis $\{[\delta_1], \ldots, [\delta_r]\}$ of $H^1_{DR}(\rho)$ defined over $\bar{k}$, i.e., $\delta_i(\mathbb{F}_q[t]) \subseteq \bar{k}[\tau]\tau$, and any A-basis $\{\lambda_1, \ldots, \lambda_r\}$ of Λ_ρ, the $r \times r$ matrix

$$P_\rho = (\int_{\lambda_i} \delta_j)$$

is called period matrix of the Drinfeld module ρ.

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Natural Relations among Entries of Period Matrix

Each endomorphism f of ρ induces a homomorphism

$$f^* : (\delta \mapsto f^*\delta\ (t \mapsto \delta_t f)) : H_{DR}(\rho) \to H_{DR}(\rho).$$

The quasi-periodic function of $f^*\delta$ is given by $F_{f^*\delta}(z) = F_\delta(b_0 x)$ for $f = \sum_{i=0}^n b_0 \tau^i$. Write $f^*\delta_j = \sum_{\ell=1}^r c_\ell \delta_\ell$ and $b_0 \lambda_i = \sum_{\ell=1}^r d_\ell \lambda_\ell$, then evaluating $z = \lambda_i \in \Lambda_\rho$ we obtain

$$\sum_{\ell=1}^r c_\ell F_{\delta_\ell}(\lambda_i) = \sum_{\ell=1}^r d_\ell F_{\delta_j}(\lambda_\ell).$$

If $f \notin \rho(\mathbb{F}_q[t])$, then it is a nontrivial $\bar{k}$-linear relation among the values

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Period Conjecture for Drinfeld modules

Yu 1997, Brownawell 2001

All the $\bar{k}$-linearly relations among the entries of the period matrix P_ρ are those induced from the endomorphisms of ρ. In particular, $\dim_{\bar{k}} \bar{k}\text{-Span}\left\{\int_{\lambda_i} \delta_j;\ 1 \leq i, j \leq r\right\} = r^2/s$, where $s := [\mathrm{End}(\rho) : A]$.

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$$\mathrm{tr.deg}_{\bar{k}} \bar{k}\left(\int_{\lambda_i} \delta_j\right) = r^2/s.$$

Theorem 1 (Chang-Papanikolas 2009)

The period conjecture is true (also true for general A).

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Algebraic independence of Drinfeld logarithms

Yu 1997 (Analogue of Baker's Theorem)

Let $u_1, \ldots, u_n \in \mathbb{C}_\infty$ satisfy $\exp_\rho(u_i) \in \bar{k}$ for all i. If $u_1, \ldots, u_n$ are linear independent over $\mathrm{End}(\rho)$, then $1, u_1, \ldots, u_n$ are linearly independent over $\bar{k}$.

Theorem 2 (Chang-Papanikolas 2009)

Assumption as above. Then $u_1, \ldots, u_n$ are algebraically independent over $\bar{k}$ (also valid for general A).

Classical conjecture

Let $u_1, \ldots, u_n$ satisfy $e^{u_i} \in \overline{\mathbb{Q}}$ for all i. If $u_1, \ldots, u_n$ are linearly independent over $\mathbb{Q}$, then $u_1, \ldots, u_n$ are algebraically independent over $\overline{\mathbb{Q}}$.

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Logarithms and Quasi-Periodic Functions

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Fix a basis $\{[\delta_1], \ldots, [\delta_r]\}$ of $H^1_{DR}(\rho)$ defined over $\bar{k}$. Let $u_1, \ldots, u_n \in \mathbb{C}_\infty$ satisfy $\exp_\rho(u_i) \in \bar{k}$ for all i. Suppose that $u_1, \ldots, u_n$ are linearly independent over $\mathrm{End}(\rho)$, then the following rn values

$$\begin{array}{c} F_{\delta_1}(u_1), \ldots, F_{\delta_1}(u_n) \\ \vdots \\ F_{\delta_r}(u_1), \ldots, F_{\delta_r}(u_n) \end{array}$$

are linearly independent over $\bar{k}$.

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Assumption as above. Suppose that the fraction field of $\mathrm{End}(\rho)$ is separable over k. Then the above rn values are algebraically independent over $\bar{k}$.

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Sketch of the proof of Period Conjecture

Step I: Solving difference equations

W.L.O.G, we may assume that ρ is given by $\rho_t := \theta + \kappa_1\tau + \ldots + \kappa_{r-1}\tau^{r-1} + \tau^r$. Let

$$\Phi := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t-\theta) & -\kappa_1^{1/q} & -\kappa_2^{1/q^2} & \cdots & -\kappa_{r-1}^{1/q^{r-1}} \end{bmatrix} \in \mathrm{Mat}_r(\bar{k}[t]),$$

then following Pellarin we use Anderson generating functions to create $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that

$$\Psi^{(-1)} = \Phi\Psi, \text{ and } \bar{k}(\Psi(\theta)) = \bar{k}(\int_{\lambda_i} \delta_j).$$

By Papanikolas' theory, it suffices to prove $\dim \Gamma_\Psi = r^2/s$.

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Sketch of the proof of Period Conjecture

Let M be the rigid analytically trivial pre-t-motive defined by Φ. Anderson showed that there is a fully faithful functor

$$\left\{ \text{Drinfeld } \mathbb{F}_q[t]\text{-modules}/\bar{k} \text{ up to isogeny} \right\} \to \{\text{R.A.T. pre-}t\text{-motives}\},$$

we have

$$\mathrm{frac}(End(\rho)) \cong \mathrm{End}_{\bar{k}(t)[\sigma,\sigma^{-1}]}(M) =: \mathcal{K}$$

Note that $[\mathcal{K} : \mathbb{F}_q(t)] = s$.
Step II: Prove

$$\Gamma_\Psi = \mathrm{Cent}_{GL_{r/\mathbb{F}_q(t)}}(\mathcal{K}) \cong \mathrm{Res}_{\mathcal{K}/\mathbb{F}_q(t)}(\mathrm{GL}_{\frac{r}{s}/\mathcal{K}})$$

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$$\left\{ \text{Drinfeld } \mathbb{F}_q[t]\text{-modules}/\bar{k} \text{ up to isogeny} \right\} \to \{\text{R.A.T. pre-}t\text{-motives}\},$$

we have

$$\mathrm{frac}(End(\rho)) \cong \mathrm{End}_{\bar{k}(t)[\sigma,\sigma^{-1}]}(M) =: \mathcal{K}$$

Note that $[\mathcal{K} : \mathbb{F}_q(t)] = s$.
Step II: Prove

$$\Gamma_\Psi = \mathrm{Cent}_{GL_{r/\mathbb{F}_q(t)}}(\mathcal{K}) \cong \mathrm{Res}_{\mathcal{K}/\mathbb{F}_q(t)}(\mathrm{GL}_{\frac{r}{s}/\mathcal{K}})$$

and hence finish the proof of Period Conjecture.

Sketch of the proof of $\Gamma_\Psi \cong \mathrm{Cent}_{GL_{r/\mathbb{F}_q(t)}}(\mathcal{K})$

Let $\mathcal{R}_M$ be the Tannakian subcategory generated by M. As $\mathcal{R}_M$ is functorial in M, we have a natural upper bound for Γ_Ψ:

$$\Gamma_\Psi \subseteq \mathrm{Cent}_{GL_{r/\mathbb{F}_q(t)}}(\mathcal{K}).$$

Question: How to obtain a lower bound for Γ_Ψ?
Answer: Connection to Galois representations.
Let K be a finite extension of k so that $End(\rho) \subseteq K[\tau]$. Given a prime v in $\mathbb{F}_q[t]$, we let

$$T_v(\rho) := \varprojlim \rho[v^n].$$

Let $\mathbf{A}_v := \mathbb{F}_q[t]_v$ and $\mathbf{k}_v := \mathbb{F}_q(t)_v$, then we have the v-adic Galois representation

$$\phi_v : G_K := \mathrm{Gal}(K^{sep}/K) \to \mathrm{Aut}(\mathbf{k}_v \otimes_{\mathbf{A}_v} T_v(\rho)) = \mathrm{GL}_r(\mathbf{k}_v).$$

Sketch of the proof of $\Gamma_\psi \cong \mathrm{Cent}_{GL_{r/\mathbb{F}_q(t)}}(\mathcal{K})$

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Pink 1997: $\phi_v(G_K) \subseteq \mathrm{Cent}_{GL_r(\mathbf{k}_v)}(\mathcal{K})$ is Zariski dense.

Key Lemma (Lower bound for Γ_Ψ): For $v = t$, enlarge K so that Spec $\bar{k}(t)[\Psi_{ij}, 1/\det\Psi]$ is defined over $K(t)$, then one has

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Pink's theorem implies $\Gamma_\Psi(\mathbf{k}_v) = \mathrm{Cent}_{GL_r(\mathbf{k}_v)}(\mathcal{K})$ for $v = t$ and hence

$$\Gamma_\Psi = \mathrm{Cent}_{GL_{r/\mathbb{F}_q(t)}}(\mathcal{K}).$$

Corollary: For each prime v, we have the analogue of Mumford-Tate conjecture

$$\phi_v(G_K) \subseteq \Gamma_\Psi(\mathbf{k}_v) \text{ is Zariski dense.}$$

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Drinfeld modular forms

For any $z \in \mathbf{H} := \mathbb{C}_\infty \setminus k_\infty$, we let $\Lambda_z := Az + A$. Its corresponding rank 2 Drinfeld $\mathbb{F}_q[t]$-module is given by

$$\phi^{\Lambda_z} : t \mapsto \theta + g(z)\tau + \Delta(z)\tau^2.$$

Regarding g and Δ as functions on $\mathbf{H}$,then

1. g is a Drinfeld modular form of weight $q-1$, type 0;
2. Δ is a Drinfeld modular form of weight q^2-1, type 0.

Goss, Gekeler: Put $g_{new} := g/\tilde{\pi}^{q-1}$ and $\Delta_{new} := \Delta/\tilde{\pi}^{q^2-1}$, then

$$g_{new}, \Delta_{new} \in \bar{k}[[q_\infty(z)]], \text{ where } q_\infty(z) := 1/\exp_C(\tilde{\pi}z).$$

There is a modular form $h \in \bar{k}[[q_\infty]]$ (Poincaré series) of weight $q+1$, type 1 for which $h^{q-1} = -\Delta_{new}$. Then graded ring generated by modular forms (graded by weights) is given by

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Drinfeld quasi-modular forms

Gekeler: Set $E := \frac{1}{\tilde{\pi}} \frac{\frac{d}{dz}\Delta(z)}{\Delta(z)} \in \bar{k}[[q_\infty]]$. Then E is called false Eisenstein series of weight 2 since for $\gamma \in GL_2(A)$,

$$E(\gamma z) = (cz+d)^2 (\det\gamma)^{-1} \left(E(z) - \frac{c}{\tilde{\pi}(cz+d)} \right).$$

Definition/Theorem (Bosser-Pellarin 2008): Any such function

$$f = \sum_{(q-1)i+(q+1)j+2e=\ell} a_{ije} g_{new}^i h^j E^e \in \mathbb{C}_\infty[g_{new}, h, E]$$

is called a Drinfeld quasi-modular form of weight ℓ.
Defintion: A quasi-modular form f is called arithmetic if

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The algebraic points on $\mathrm{GL}_2(A)\backslash\mathbf{H}$

Recall that the set of isomorphism classes of rank 2 Drinfeld $\mathbb{F}_q[t]$-modules can be identified with $GL_2(A)\backslash\mathbf{H}$ and $GL_2(A)\backslash\mathbf{H}$ is analytically isomorphic to $\mathbb{C}_\infty$ via the j-invariant function

$$j(:= g^{q+1}/\Delta) : \begin{array}{ccc} GL_2(A)\backslash\mathbf{H} & \to & \mathbb{C}_\infty \\ z & \mapsto & j(z). \end{array}$$

Set

$$S := \{\alpha \in \mathbf{H};\ j(\alpha) \in \bar{k}\}$$

Then for each $\alpha \in S$, there exists $\omega_\alpha \in \mathbb{C}_\infty$ so that the rank 2 Drinfeld $\mathbb{F}_q[t]$-module ϕ^Λ is defined over $\bar{k}$, where $\Lambda := A\alpha\omega_\alpha + A\omega_\alpha$ (period lattice of ϕ^Λ). Note that

$$S = \mathrm{CM} \sqcup \mathrm{NCM},$$

1. $\mathrm{CM} := \{\alpha \in \mathbf{H};\ \alpha \text{ is quadratic over } k\}$ (set of CM points)
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Transcendence results

Theorem 4 (Chang 2009)

Let f be an arithmetic quasi-modular form of nonzero weight. Given any $\alpha \in S := \{\alpha \in \mathbf{H};\ j(\alpha) \in \bar{k}\}$ so that $f(\alpha) \neq 0$, then $f(\alpha)$ is transcendental over k.

Remark

1. Algebraic independence of $f(\alpha)$, $\alpha \in S$ (work in progress).
2. Similar question to $f(\alpha)$ in the classical case. The transcendence of $f(\alpha)$ is only known for CM point α.

Question: Why is $f(\alpha)$ interesting?

Answer:

1. It has connection to periods and quasi-periods of rank 2 Drinfeld $\mathbb{F}_q[t]$-modules defined over $\bar{k}$;
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Question: Why is $f(\alpha)$ interesting?

Answer:

1. It has connection to periods and quasi-periods of rank 2 Drinfeld $\mathbb{F}_q[t]$-modules defined over $\bar{k}$;
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Transcendence results

Theorem 4 (Chang 2009)

Let f be an arithmetic quasi-modular form of nonzero weight. Given any $\alpha \in S := \{\alpha \in \mathbf{H};\ j(\alpha) \in \bar{k}\}$ so that $f(\alpha) \neq 0$, then $f(\alpha)$ is transcendental over k.

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Special values of modular forms I

Given any $\alpha \in S$, consider $\Lambda_\alpha = A\alpha + A$. Then $\phi_t^{\Lambda_\alpha} = \theta + g(\alpha)\tau + \Delta(\alpha)\tau^2$. Choose any $\epsilon \in \mathbb{C}^\times$ so that $\Delta(\alpha)\epsilon^{q^2-1} = 1$. Set $\Lambda := \epsilon^{-1}\Lambda_\alpha$, then we have

$$\phi_t^\Lambda = \epsilon^{-1}\phi_t^\Lambda \epsilon = \theta + \sqrt[q+1]{j(\alpha)}\tau + \tau^2,$$

where $j(\alpha) := g(\alpha)^{q+1}/\Delta(\alpha) \in \bar{k}$. Note that the period lattice of ϕ^Λ is $\Lambda = A\frac{\alpha}{\epsilon} + A\frac{1}{\epsilon}$. Set $\omega_\alpha = \frac{1}{\epsilon}$, then

$$\Delta(\alpha) = (\frac{1}{\epsilon})^{q^2-1} = \omega_\alpha^{q^2-1}.$$

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For any arithmetic modular form f of weight ℓ, consider $f^{q^2-1}/\Delta_{new}^{\ell}$ which has weight zero. Since f^{q^2-1} and Δ_{new}^{ℓ} are arithmetic, $f^{q^2-1}/\Delta_{new}^{\ell}$ belongs to the function field $\bar{k}(GL_2(A)\backslash\mathbf{H}) = \bar{k}(j)$. For $x, y \in \mathbb{C}_\infty^\times$, we denote by $x \sim y$ if $x/y \in \bar{k}$. Since $j(\alpha) \in \bar{k}$, $f^{q^2-1}(\alpha)/\Delta_{new}^{\ell}(\alpha) \in \bar{k}$ and hence

$$f(\alpha) \sim \left(\frac{\omega_\alpha}{\tilde{\pi}}\right)^{\ell}.$$

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1. The above formula is still valid for any arithmetic modular forms for a congruence subgroup of $GL_2(A)$.
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Special values of $E(\alpha)$ I

Recall that the quasi-modular forms in question are lying in $\bar{k}[g_{new}, h, E]$, and g_{new}, h are modular forms. So it suffices to investigate the value $E(\alpha)$. We claim that

$$E(\alpha) \sim \frac{\omega_\alpha F_{\phi^\Lambda,\tau}(\omega_\alpha)}{\tilde{\pi}^2}.$$

Classical case: Recall $G_2(z) = \sum_m \sum_n' \frac{1}{(mz+n)^2}$ and

$$E_2(z) = \frac{6}{\pi^2} G_2(z).$$

For $\tau \in \mathbb{H}$, let $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$. Let E_τ be the elliptic curve associated to Λ_τ and set

$$\eta_2 := \int_0^1 \wp_{\Lambda_\tau}(z)dz.$$

Katz: $\eta_2 = G_2(\tau)$.

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Gekeler: For any $z \in \mathbf{H}$, let $\Lambda_z = Az + A$. Then

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Note that ϕ^Λ is defined over $\bar{k}$ and so our Theorem 1 implies $\omega_\alpha/\tilde{\pi}$ and $F_{\phi^\Lambda,\tau}(\omega_\alpha)/\tilde{\pi}$ are algebraically independent over $\bar{k}$. Therefore we obtain the transcendence of $f(\alpha)$ for nonzero weight quasi-modular form $f \in \bar{k}[g_{new}, h, E]$, since $f(\alpha)$ is homogeneous over $\bar{k}$ in $(\omega_\alpha/\tilde{\pi})^{q-1}$, $(\omega_\alpha/\tilde{\pi})^{q+1}$ and $\frac{\omega_\alpha F_{\phi^\Lambda,\tau}(\omega_\alpha)}{\tilde{\pi}^2}$.

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Motivic interpretation of $E(\alpha)$

Given $\alpha \in S$, let $\kappa := \sqrt[q+1]{j(\alpha)} \in \bar{k}$. Then $\phi_t^\Lambda = \theta + \kappa\tau + \tau^2$. Define

$$\Phi_\alpha := \begin{pmatrix} -\kappa^{1/q}(t-\theta) & (t-\theta) \\ 1 & 0 \end{pmatrix}$$

define a pre-t-motive M_α. Then we have:

1. M_α is rigid analytically trivial and the solution matrix for $\Psi_\alpha^{(-1)} = \Phi_\alpha \Psi_\alpha$ is given by certain generating functions in terms of E and α (based on functions defined by Pellarin);
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Results for Drinfeld modules

1. Prove a period conjecture;
2. Establish an analogue of Mumford-Tate conjecture;
3. Algebraic independence of Drinfeld logarithms;
4. Tools: Papanikolas' theory + Pink's theorem on the size of v-adic Galois image.

Result for arithmetic quasi-modular forms

1. Transcendence of values of positive weight at $\alpha \in S$;
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No surprising algebraic relations among periods! All algebraic relations should be explained motivically.

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2. Establish an analogue of Mumford-Tate conjecture;
3. Algebraic independence of Drinfeld logarithms;
4. Tools: Papanikolas' theory + Pink's theorem on the size of v-adic Galois image.

Result for arithmetic quasi-modular forms

1. Transcendence of values of positive weight at $\alpha \in S$;
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Transcendence Philosophy

No surprising algebraic relations among periods! All algebraic relations should be explained motivically.

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