Examples in Dependent theories

"You can fool some of the people all the time, and those are the ones you want to concentrate on." (George W. Bush)

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Introduction

We will give some peculiar examples of dependent theories, in which things that once thought to be impossible happen.

- First we discuss existence of indiscernibles (as in [Sheb]) and prove (sorry) that not much can be said of general dependent theories.
- Then we say a few words on directionality of a theory.
- In the end, we show that a Generic pair may not be dependent even if the theory is stable.

Existence of indiscernibles

Definition

$\lambda \to (\mu)_{T,n}$ means: for every sequence $\langle a_\alpha \,|\alpha \in \lambda \rangle \in^\lambda (\mathfrak{C}^n)$, there is a subset $u \subseteq \lambda$ of size μ such that $\langle a_\alpha \,|\alpha \in u \rangle$ is an indiscernible sequence.

Some history: Morley, in [Mor65], proved that for ω-stable T, and for λ regular big enough, $\lambda \to (\lambda)_{T,1}$. In fact, for stable theories, and for $\lambda = \lambda^{|T|}$, $\lambda^+ \to (\lambda^+)_{T,n}$ for all $n < \omega$ (or even $n \leq |T|$) (for example by local character of non-forking and Fodor's lemma - see [She90, III]). In the dependent context we have the following theorem (from [Sheb]):

Theorem

If T is strongly dependent then $\beth_{|T|^+}(\lambda) \to (\lambda^+)_{T,n}$ for all $n < \omega$.

Existence of indiscernibles

However,

Theorem

There exists a dependent T, such that $\lambda \nrightarrow (\mu)_{T,1}$ for any $\lambda \geq \mu$ such that in $[\mu, \lambda]$ there are no strongly inaccessible cardinals.

For each $I \subset \mathbb{Z}$, a finite subset, let $L_I = \{P_n, <_n, F_n \,|n \in I\} \cup \left\{H_n^1, H_n^2 \,|n, n+1 \in I\right\}$. Let T_I' be the following theory:

- P_n are disjoint unary predicates.
- $<_n$ is a partial order on P_n, and $(P_n, <_n)$ is a tree (i.e. $\{b \,|b <_n a\}$ is linearly ordered).
- H_n^1, H_n^2 are two unary functions from P_n to P_{n+1}.
- F_n is a binary function taking $a, b \in P_n$ to $a \wedge b = max\,(c\,|c \leq a, b)$.

Existence of indiscernibles

Claim

1. T_I' *is universal, it has JEP and AP.*
2. *If* $A \neq \emptyset$ *is a finite subset of a model of* T_I'*, then* $|\langle A \rangle| \leq f(n)$ *for some polynomial* f *(*$\langle A \rangle$ *is the generated substructure).*

Hence T_I' has a model completion T_I which eliminate quantifiers (and is ω categorical).

Existence of indiscernibles

Claim

T_I is dependent.

Proof.

By e.g. [She90], it is enough to show that given a finite set A, there is a polynomial f such that $|S_1(A)| \leq f(|A|)$. It is enough to check that $S^n = \{p \in S_1(A) \,|\, P_n(x) \in p\}$ is such.

Consider Tr = the model completion of the theory of trees. For all finite $B \subseteq M \models Tr$, and $n < \omega$, $|S_n(B)| \leq f_n(|B|)$ for some polynomial f_n (because it is dependent).

By QE, a type is determined by atomic formulas. Hence $|S^n| \leq f_1(|A \cap P_n|) \cdot f_2(|A \cap P_{n+1}|) \cdot \ldots \cdot f_{2^{|I|-1}}\left(A \cap P_{|I|-1}\right)$. □

Existence of indiscernibles

Definition

Let $L = \bigcup_{I \subseteq \mathbb{Z}, |I| < \infty} L_I$, and let $T = \bigcup_{I \subseteq \mathbb{Z}, |I| < \infty} T_I$.

- Note that for $J \subseteq I \subseteq \mathbb{Z}$ finite, $T_I|_{L_J} = T_J$, so this definition makes sense.
- T_I is strongly dependent. However T is not.

The main theorem is:

Theorem

For all $n \in \mathbb{Z}$, and $\aleph_0 \leq \mu \leq \lambda$ such that in $[\mu, \lambda]$ there is no strongly inaccessible, there is a set $U \subseteq P_n$ that witnesses $\lambda \nrightarrow (\mu)_{T,1}$.

Existence of Indiscernibles

The proof is by induction on μ and then on λ.

Claim

The theorem is true when $\mu = \lambda = \aleph_0$.

Proof.

Find a sequence of different elements $\left\langle a_i^{j+n} \,|\, i, j < \omega \right\rangle$ such that $\left\{ a_i^{j+n} \,|\, i < \omega \right\} \subseteq P_{j+n}$ and $H_j^1\left(a_i^{j+n}\right) = a_i^{j+n+1}$ for $i \geq j$ and a_0^{j+n+1} otherwise. So if $\langle a_i^n \,|\, i \in U \rangle$ is indiscernible, then let $i_0 < i_1$ be the first elements in U. For large enough j,
$H_{j+n}^1 \circ \ldots \circ H_n^1\left(a_{i_0}^n\right) = H_{j+n}^1 \circ \ldots \circ H_n^1\left(a_{i_1}^n\right)$, but this is not true for $i_2 \in U$ larger than j. $\square$

Existence of indiscernibles

Claim

The theorem is true when $\mu = \lambda$ is singular.

Proof.

$\lambda = \bigcup_{i<\kappa} \lambda_i$ where $\kappa = cf(\lambda) < \lambda_i < \lambda$. Find some sequence of different elements $\langle b_i \,|\, i < \kappa \rangle \subseteq P_{n+1}$. Now find some sequence $\langle a_\alpha \,|\, \alpha < \lambda \rangle \subseteq P_n$ of different elements such that $H_n^1(a_\alpha) = b_i$ where i is the unique ordinal such that $\bigcup_{j<i} \lambda_j \leq \alpha < \lambda_i$. If there was some $U \subseteq \lambda$, $|U| = \lambda$, such that $\langle a_\alpha \,|\, \alpha \in U \rangle$ is indiscernible, then there is some $\alpha < \beta \in U$ such that $H_n^1(a_\alpha) = H_n^1(a_\beta)$, H_n^1 is constant on U. But that is a contradiction to the fact that $|U| = \lambda$. $\square$

Existence of indiscernibles

Claim

If $\lambda \nrightarrow (\mu)_{T,1}$ (and there is a witness for this in P_{n+1}) then $2^\lambda \nrightarrow (\mu)_{T,1}$ (and there is a witness in P_n).

Proof.

There is a witness $\langle b_i \,|i < \lambda \rangle \subseteq P_{n+1}$ for our assumption. Let $\left\langle a_{\eta_\alpha} \left| \alpha < 2^\lambda \right. \right\rangle$ enumerate $2^{\leq\lambda}$. Find $\left\langle a_\eta \left| \eta \in 2^{\leq\lambda} \right. \right\rangle \subseteq P_n$ such that: $a_\nu <_n a_\eta$ iff $\nu \triangleright \eta$, $F_n(a_\eta, a_\nu) = a_{\eta\wedge\nu}$, $H_n^1(a_\eta) = b_{lg(\eta)}$.
Suppose $U \subseteq 2^\lambda$ of size μ, such that $\langle a_{\eta_\alpha} \,|\alpha \in U \rangle$ is indiscernible. For convenience assume that $\alpha \in U \Rightarrow \alpha + 1 \in U$. Then $lg(\eta_\alpha)$ is constant.
Given $\alpha < \beta < \gamma \in U$, $\eta_\alpha \wedge \eta_\beta = \eta_\alpha \wedge \eta_\gamma$ and $\eta_\alpha \wedge \eta_\beta = \eta_\beta \wedge \eta_\gamma$ (otherwise, by indiscernibility, we'll have an increasing sequence of length μ). Let $\delta := lg(\eta_\alpha \wedge \eta_\beta)$.
So $\eta_\gamma(\delta) \neq \eta_\alpha(\delta) \neq \eta_\beta(\delta)$ and $\eta_\beta(\delta) \neq \eta_\gamma(\delta)$ - contradiction. $\square$

Existence of indiscernibles

Claim

The theorem is true when $\mu < \lambda$ is singular, and in $[\mu, \lambda]$ there is no strongly inaccessible.

Proof.

$\lambda = \bigcup_{i<\kappa} \lambda_i$ where $\mu \leq \kappa < \lambda_i < \lambda$. By the induction hypothesis, for every suitable $i < \kappa$, there is a sequence $I_i = \left\langle a_\alpha \middle| \bigcup_{j<i} \lambda_j \leq \alpha < \lambda_i \right\rangle \subseteq P_{n+1}$ that witnesses $\lambda_i \nrightarrow (\mu)_{T,1}$, and a sequence $\langle b_i \,|i < \kappa \rangle$ witnessing $\kappa \nrightarrow (\mu)_{T,1}$. Now find $\langle c_\alpha \,|\alpha < \lambda \rangle \subseteq P_n$ such that $H_n^1 (c_\alpha) = a_\alpha$ and $H_n^2 (c_\alpha) = b_i$ for the unique i such that $\bigcup_{j<i} \lambda_j \leq \alpha < \lambda_i$. If $U \subseteq \lambda$, $|U| = \mu$, $\{c_\alpha \,|\alpha \in U\}$ is indiscernible, then $H_n^2 (U)$ is constant, so $H_n^1 (U) \subseteq I_i$ - a contradiction. □

Existence of indiscernibles

Claim

The theorem is true.

Proof.

Take the first λ that this is not true for it. So λ is regular, so, as λ is not strongly limit, there is some $\kappa < \lambda$ such that $\lambda < 2^\kappa$. But by the induction hypothesis, $\kappa \nrightarrow (\mu)_{T,1}$ so by a claim above also $2^\kappa \nrightarrow (\mu)_{T,1}$, hence also $\lambda \nrightarrow (\mu)_{T,1}$. $\square$

Remark

1. The case of the inaccessible is currently under construction and will most probably appear in the paper, proving that unless there are some good set theoretical reasons for it, $\lambda \nrightarrow (\mu)_{T,1}$ for all μ, λ.
2. For strongly dependent theories, there is a similar result for ω-tuples.
3. Another example which is currently work in progress, will show the same for o-minimal theories.
4. Not all is lost: wait for the end!

Directionality

Definition

For a type $p \in S(A)$, let $uf(p) = \{q \in S(\mathfrak{C}) \,|\, q \text{ is f.s. on } A \text{ and } q \supseteq p\}$.
For a type $p(x) \in S(A)$, and Δ a set of formulas of the form $\varphi(x, \bar{y})$, $uf_\Delta(p) = \{q \in S_\Delta(\mathfrak{C}) \,|\, q \cup p \text{ is f.s. on } A\}$.

Definition

1. T is said to be of bounded directionality (or just, T is bounded) if for $p \in S^\alpha(M)$, $|uf(p)| \leq 2^{|T|+|\alpha|}$.
2. T is said to be of medium directionality (or just, T is medium) if for $p \in S^\alpha(M)$, $|uf(p)| \leq |M|^{|T|+|\alpha|}$ and T is not bounded.
3. T is said to be of large directionality (or just, T is large) T is not bounded nor medium.

Directionality

Claim

T is bounded iff for all finite Δ, and $p \in S(M)$, $uf_{\Delta}(p)$ is finite.

Claim

T is medium iff for every cardinality $\lambda \geq |T|$,
$\lambda = sup\left(|uf_{\Delta}(p)|\,|p \in S(M), \Delta \text{ finite}, |M| = \lambda\right)$.

Claim

T is large iff for every cardinality $\lambda \geq |T|$,
$ded^{*}(\lambda) = sup\left(|uf_{\Delta}(p)|\,|p \in S(M), \Delta \text{ finite}, |M| = \lambda\right)$.

Directionality

It has been known for a long time that not all dependent theories are bounded (see e.g. [Del84]).

Fact

The theory T defined above has large directionality, and every T_I.

Proof.

E.g. let $I = \{0, 1\}$. Let $M \models T_I$ countable,with branch B in P_0 that is not realized, and a dense branch C in P_1 with $2^{\aleph_0}$ cuts. $p \in S(M)$ says that $B <_0 x$, and that $H_0^1(x) = c$ for some $c \in P_1$.
Note that this implies a complete type.
Let $d \models p$, and for all cut $I \subseteq C$, the type $p \cup \left(I < H_0^1(F_0(x, d)) < C \backslash I\right)$ is f.s. in M. $\square$

Directionality

In fact, even o-minimal theories are not immune, and not even RCF. This next example was inspired by a conversation with Marcus Tressle.

Definition

Let $K \models RCF$. A cut p is called dense if it is not definable and the differences $b - a$ with $a, b \in K$ and $a < p < b$, are arbitrary (w.r.t. K) close to 0.

Fact

1. *There are real closed fields with arbitrary size with dense cuts.*
2. *If $q = tp(\omega / K)$ where $K < \omega$, and p is dense, then p and q are weakly orthogonal, i.e. $p \cup q$ implies a complete type.*

Directionality

Let $K \models RCF$ be countable with a dense type (for example, K could be the real algebraic numbers, and the type is π).
Let α realize some dense type over K. Let $p(x_\omega, x_\alpha) = tp(\omega, \alpha/K)$. Now, for every bounded first segment of K, $I \subseteq K$, let p_I be

$$p_I = p \cup \{\alpha + a/x_\omega < x_\alpha < \alpha + b/x_\omega : a \in I, b \notin I\}$$

then, this type is f.s. in K because of the weak orthogonality. For I, J 2 different first segments, p_I and p_J contradict each other.
So there is a finite Δ such that $|K| < uf_\Delta(p)$.

Generic pair

Here we give an example of a pair of structures $M \prec M_1$ of a dependent theory (even $\aleph_0$ stable) such that the pair (M, M_1) is independent. The pair is also generic:

Definition

A pair as above is generic if it comes from the generic pair conjuncture.

Fact

In a generic pair, for all formula $\varphi(x)$ with parameters from M, if φ has infinitely many solutions in M, then it has a solution in $M_1 \setminus M$.

Generic pair

Let $L = \{P_1, P_2, R, Q_1, Q_2\}$ where R, P_1, P_2 are unary predicates and Q_1, Q_2 are binary relations. Let M be the following structure for L:
$P_2^M = \{u \subseteq \omega \,|\, |u| < \aleph_0\}$, $P_1^M = \{u \subseteq \omega \,|\, |u| = 1\}$, R^M is the rest.
The universe is

$$M = P_2^M \cup \{(u, v, i) \,|\, u, v \subseteq \omega, |u| = 1, |v| < \aleph_0, i \in \omega, u \subseteq v \Rightarrow i < |v|\}$$

$Q_1^M = \{(u, (u, v, i)) \,|\, P_1(u)\}$, $Q_2^M = \{(v, (u, v, i)) \,|\, P_2(v)\}$.
Let $T = Th(M)$. So T is $\aleph_0$ stable.
(why? Add the relations $E_1((u, v, i), (u', v', i')) \Leftrightarrow u' = u$ and $E_2((u, v, i), (u', v', i')) \Leftrightarrow v' = v$. With them, T eliminates quantifiers, and the conclusion follows).

Generic pair

Now let (M, M_1) be a non-algebraic pair for T. In the language $L \cup \{P\}$ (P is a unary predicate), consider the formula

$$\varphi(u, v) = P_1(u) \wedge P_2(v) \wedge (\forall z (Rz \wedge Q_1 uz \wedge Q_2 vz \rightarrow z \in P))$$

For all $n < \omega$, we can find $u_0, \ldots, u_{n-1}$ in M such that for any subset $s \subseteq n$, there is some v_s such that: $P_1(u_i)$ for all $i < n$, $P_2(v_s)$ and most importantly, $|R(u_i, v_s, M)| < n$ if $i \in s$ and if not, $R(u_i, v_s, M)$ is infinite (they exist in the original model). Hence $\varphi(u_i, v_s) \Leftrightarrow i \in s$, so we have the independence property.

Not all is lost

In [Shea] many things are proved, despite the above examples:

- If T has bounded or medium directionality, then there exists indiscernibles.
- Smart counting of types: if M is saturated, then T is dependent iff the number of types over M up to isomorphism of M is bounded by $|M|$.
- A strong criteria for saturation is proved:

Criteria for saturation

Theorem

Assume $\sigma > \mu = (2^{|T|})^+ + \beth_\omega^+$.
Then M *is* σ*-saturated iff*

1. M *is* μ*-saturated*
2. *if* $\kappa \in [\mu, \sigma)$ *and* $\langle a_\alpha : \alpha < \kappa \rangle$ *is an indiscernible sequence in* M *then for some* $a \in M$ *the sequence* $\langle a_\alpha : \alpha < \kappa \rangle \hat{} \langle a \rangle$ *is indiscernible*
3. *if* $\kappa \in [\mu, \sigma)$ *is regular,* $\langle a_s : s \in I_1 + I_2 \rangle$ *is an indiscernible sequence in* M *where* $I_1 \cong (\kappa, <), I_2 \cong (\alpha, <)$ *for some* $\alpha \leq \kappa + 1$ *then for some* $a \in N$ *the sequence* $\langle a_s : s \in I_1 \rangle \hat{} \langle a \rangle \hat{} \langle a_t : t \in I_2 \rangle$ *is an indiscernible sequence.*

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