Amalgamation properties for types in stable theories and beyond

John Goodrick

University of Maryland, College Park

Banff
February 2009
We ask: when can certain systems of types be amalgamated, and when is the result unique?
This yields properties such as $n$-existence (or the $n$ amalgamation property) and $n$-uniqueness.
Many natural algebraic examples have these properties, and they have nice consequences, such as:

**Theorem**

*(De Piro, Kim, Young)* If $T$ is simple and has 5 complete amalgamation over models, then the existence of a hyperdefinable group configuration implies the existence of a hyperdefinable group.

More recently, Hrushovski showed that in stable $T$, 4-existence is equivalent to the eliminability of “generalized imaginary sorts” as well as the collapsing of certain definable groupoids.
The 3-amalgamation problem

3-amalgamation is about the following question:

**Question**

*Given complete types \( p_{12}(x_1, x_2) \), \( p_{23}(x_2, x_3) \), and \( p_{13}(x_1, x_3) \), when is \( p_{12} \cup p_{23} \cup p_{13} \) consistent?*

Equivalently: given any realization \((a_1, a_2)\) of \( p_{12} \), is there a common realization of the two types \( p_{23}(a_2, x_3) \) and \( p_{13}(a_1, x_3) \)?

A minimal necessary requirement is **coherence**: \( p_{12} \upharpoonright x_1 = p_{13} \upharpoonright x_1 \), \( p_{12} \upharpoonright x_2 = p_{23} \upharpoonright x_2 \), and \( p_{13} \upharpoonright x_3 = p_{23} \upharpoonright x_3 \).
Failures of 3-amalgamation

Question

Given complete types $p_{12}(x_1, x_2)$, $p_{23}(x_2, x_3)$, and $p_{13}(x_1, x_3)$, when is $p_{12} \cup p_{23} \cup p_{13}$ consistent?

But many coherent triples of types cannot be amalgamated, e.g.:

If the universe is linearly ordered by "<", $x_1 < x_2 \in p_{12}$, $x_2 < x_3 \in p_{23}$, and $x_3 < x_1 \in p_{13}$;

Or in a theory with an equivalence relation $E$ with exactly two classes, if $\neg E(x_i, x_j) \in p_{ij}$.
### Theorem

Suppose that $T$ is stable, $B = \text{acl}^{eq}(B)$, $a_1 \downarrow_B a_2$, and the types $p_1(a_1, x_3)$ and $p_2(a_2, x_3)$ are nonforking extensions of a common type $p(x_3) \in S(B)$. Then there is a realization $a_3$ of $p_1(a_1, x_3) \cup p_2(a_2, x_3)$ such that $a_3 \downarrow_B a_1 a_2$.

### Proof.

Pick any $a_3$ realizing $p_1(a_1, x_3)$ such that $a_3 \downarrow_{Ba_1} a_2$. By stationarity of $p$, $a_3 \models p_2(a_2, x_3)$. 

---

Amalgamation properties for types in stable theories and beyond
Kim and Pillay generalized this to simple theories:

**Theorem**

Suppose that $T$ is simple, $B = \text{bdd}^{heq}(B)$, $a_1 \downarrow_B a_2$, and the types $p_1(a_1, x_3)$ and $p_2(a_2, x_3)$ are nonforking extensions of a common type $p(x_3) \in S(B)$. Then there is a realization $a_3$ of $p_1(a_1, x_3) \cup p_2(a_2, x_3)$ such that $a_3 \downarrow_B a_1 a_2$.

If $T$ has elimination of hyperimaginaries (e.g. if $T$ is supersimple), then $B = \text{acl}^{eq}(B)$ is enough.
In the terminology we are about to define, we have shown that all stable theories have 3-existence (or the 3-amalgamation property). Now we will generalize this property from 3 to $n$. 
n-amalgamation problems

Notation: \( \mathcal{P}^- (n) = \{ s : s \subsetneq \{1, \ldots, n\} \} \).

**Definition**

1. An *n*-amalgamation problem is a functor \( A : \mathcal{P}^- (n) \to \mathcal{P}(\mathcal{C}) \), where the maps on the right are elementary.
2. A solution to an *n*-amalgamation problem \( A \) is an extension to a functor \( A' : \mathcal{P}(n) \to \mathcal{P}(\mathcal{C}) \) (again with elementary maps on the right).

With \( A \) as above and \( s \subseteq t \subseteq n \), let \( \tau^s_t : A(s) \to A(t) \) be the image of the inclusion \( s \subseteq t \).

Fuctoriality says: \( \tau^t_u \circ \tau^s_t = \tau^s_u \) whenever this makes sense.

[Draw picture of 3-amalgamation problem]
Bases of amalgamation

**Definition**

If $A$ is an $n$-amalgamation problem, then $A$ is over $B$ if $B = A(\emptyset)$ and for every $s \subseteq n$, $\tau^s_\emptyset$ fixes $B$ pointwise.

If we are looking at the solutions of $A$, clearly we may assume that $A$ is over $A(\emptyset)$ (just shift the $A(s)$’s by appropriate automorphisms).

From now on we always assume $A$ is over $A(\emptyset)$. 
Independent amalgamation problems

We write “$i$” for $\{i\}$ to simplify notation.

**Definition**

An $n$-amalgamation problem $A$ is independent if for every $s$ s.t. $\emptyset \neq s \subseteq n$,

1. $\{\tau_{s}^{i}(A(i)) : i \in s\}$ is an $A(\emptyset)$-independent set;
2. If $t \subseteq s$, then $\tau_{s}^{t}(A(t)) = \text{bdd}^{heq}(A(\emptyset) \cup \{\tau_{s}^{i}(A(i)) : i \in t\})$.

(If $T$ is stable, replace “bdd$^{heq}$” by “acl$^{eq}$.”)

So if the $\tau$-maps are all inclusions, then $A(t) \downarrow_{A(t \cap u)} A(u)$.

Independent solutions to $A$ are defined in a similar way.
Independent amalgamation problems

We write “i” for \( \{i\} \) to simplify notation.

**Definition**

An \( n \)-amalgamation problem \( A \) is independent if for every \( s \) s.t. \( \emptyset \neq s \subsetneq n \),

1. \( \{\tau_i^s(A(i)) : i \in s\} \) is an \( A(\emptyset) \)-independent set;
2. If \( t \subseteq s \), then \( \tau_t^s(A(t)) = \text{bdd}^{\text{heq}}(A(\emptyset) \cup \{\tau_i^s(A(i)) : i \in t\}) \).

(If \( T \) is stable, replace “\( \text{bdd}^{\text{heq}} \)” by “\( \text{acl}^{eq} \)”.)

So if the \( \tau \)-maps are all inclusions, then \( A(t) \downarrow_{A(t \cap u)} A(u) \).

Independent solutions to \( A \) are defined in a similar way.
Assume $T$ is simple.

**Definition**

1. $T$ has $n$-existence if every independent $n$-amalgamation problem has an independent solution.
2. $T$ has $n$-uniqueness if every independent $n$-amalgamation problem $A$ has at most one independent solution up to isomorphism over $A$.
3. $T$ has $n$-complete amalgamation if for every $k$ with $3 \leq k \leq n$, $T$ has $k$-existence.
So $n$-existence and $n$-uniqueness give two different ways to classify simple theories:

2-existence is true in any simple theory, by the existence of nonforking extensions;
3-existence is true in any stable theory, and all known examples of simple theories;
4-existence can fail even in stable theories (we’ll see an example).

2-uniqueness is true in any stable theory (by stationarity of strong types), but fails for unstable simple $T$;
3-uniqueness can fail even for stable $T$. 
The theory of a random graph is simple and has $n$-existence for all $n \geq 2$.

But if $A(i) = a_i$ (for $i = 1, 2$), then there are two solutions to the 2-amalgamation problem $A$: one with an edge between the points and one with no edge. So the random graph does not have 2-uniqueness.
Example: random hypergraphs

A **hypergraph** is a set with a symmetric ternary relation $R$.

The theory of a random “tetrahedron-free hypergraph” (where $R$ cannot hold of every 3-element subset of a 4-element set) turns out to be simple.

However, it fails 4-existence: consider a 4-amalgamation problem where $A(\{i, j, k\})$ is a triple of points on which $R$ holds.

Similarly, the $n$-simplex-free hyper$^{n-3}$ graph is simple and has $(n – 1)$-complete amalgamation but not $n$-existence.
We now give an example of a stable $T$ which fails 3-uniqueness.

Let $I$ be some infinite set, $[I]^2$ is all 2-element subsets of $I$, $E \subseteq I \times [I]^2$ is set membership, $P = \{0, 1\} \times [I]^2$, with projection map $\pi : P \to [I]^2$, And $Q \subseteq P \times P \times P$ be the set of all $((i, s), (j, t), (k, u))$ such that:

1. $s, t, u$ are all distinct sets,
2. $|s \cup t \cup u| = 3$, and
3. $i + j + k$ is even.

[Draw picture on blackboard]

$T = \text{Th}(I, [I]^2, E, P, \pi, Q)$. 
$T = \text{Th}(I, [I]^2, E, P, \pi, Q)$

Note that if $a, b \in I$, then $|\pi^{-1}({a, b})| = 2$, so $\pi^{-1}({a, b}) \subseteq \text{acl}(a, b)$.

It turns out that $T$ is totally categorical, hence stable.

Note that if $Q(x, y, z)$ holds, then $z \in \text{dcl}(x, y)$.

Therefore, for any three distinct elements $a_1, a_2, a_3 \in I$, note that

$$\pi^{-1}({a_1, a_2}) \subseteq \text{dcl}(\pi^{-1}({a_1, a_3}) \cup \pi^{-1}({a_2, a_3})).$$
Given three distinct elements $a_1, a_2, a_3 \in I$, let $A$ be the 3-amalgamation problem given by $A(\{i\}) = a_i$ and $A(\{i, j\}) = acl(a_i, a_j)$.

There are two solutions $A_1, A_2$ to $A$, defined by:

$A_1(\{1, 2, 3\}) = A_2(\{1, 2, 3, \}) = acl(\{a_1, a_2, a_3\})$;

All transition maps in $A_1$ are inclusion maps.

In $A_2$, the transition maps $A(\{1, 3\}) \to A_2(\{1, 2, 3\})$ and $A(\{2, 3\}) \to A_2(\{1, 2, 3\})$ are inclusions, but the transition map $A(\{1, 2\}) \to A_2(\{1, 2, 3\})$ fixes $a_1$ and $a_2$ but switches the two elements of $\pi^{-1}(\{a_1, a_2\})$.

$A_1 \not\cong A_2$ because of the relation $Q$ on the fibers.
Suppose $T$ is stable and $T$ has $k$-uniqueness for all $2 \leq k \leq n$ (where $n \geq 2$). Then $T$ has $(n + 1)$-existence.

Proof.

Suppose $A$ is an independent $(n + 1)$-amalgamation problem. Let $A'(\{1, \ldots, n + 1\})$ be the algebraic closure of independent copies of $A(\{1, \ldots, n\})$ and $A(\{n + 1\})$. Define maps $\tau_{1, \ldots, n+1}^i : A(\{i\}) \to A'(\{1, \ldots, n + 1\})$ in the natural way.

For any $i \leq n$, there is only one way to define the transition map $\tau_{1, \ldots, n+1}^{i, n+1} : A(\{i, n + 1\}) \to A'(\{1, \ldots, n + 1\})$ (by 2-uniqueness). If $n > 3$, 3-uniqueness implies there is a unique way to extend these transition maps to “faces.” Repeat using induction.
Characterizing 3-uniqueness in stable $T$

Theorem

(Hrushovski) If $T$ is stable, then TFAE:

1. $T$ has 3-uniqueness;
2. $T$ has 4-existence;
3. Every connected definable groupoid in $T$ with finite automorphism groups is “equivalent” to a group.
Retractable groupoids

Definition

1. A groupoid is a category $\mathcal{G}$ in which every morphism has a (unique, 2-sided) inverse.
2. A groupoid is connected if there is a morphism between any two objects.

In a connected groupoid, any two automorphism groups \( \text{Mor}_{\mathcal{G}}(a, a) \) and \( \text{Mor}_{\mathcal{G}}(b, b) \) are isomorphic. (Conjugate by \( f \in \text{Mor}_{\mathcal{G}}(a, b) \).)

Definition

A connected definable groupoid $\mathcal{G}$ is retractable if there is a definable family of commuting morphisms 
\( \{ f_{ab} \in \text{Mor}_{\mathcal{G}}(a, b) : a, b \in \text{Ob}_{\mathcal{G}} \} \).
Suppose that $T$ is stable. $T$ does not have 3-uniqueness if and only if there is a set $A$, elements $a_1, a_2, a_3$, and elements $f_{12}, f_{23}, f_{31}$ such that:

1. $a_1, a_2, a_3$ is a Morley sequence over $A$;
2. $f_{ij} \in \text{acl}(Aa_ia_j) \setminus \text{dcl}(Aa_ia_j)$;
3. $a_1a_2f_{12} \equiv_A a_2a_3f_{23} \equiv_A a_3a_1f_{31}$;
4. If $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$, then $f_{ij} \in \text{dcl}(Af_{jk}f_{ki})$.

$\{a_1, a_2, a_3, f_{12}, f_{23}, f_{31}\}$ as above is called a symmetric witness to non-3-uniqueness.
Non-retractable groupoids from failure of 3-uniqueness

Theorem

(G.-Kolesnikov) Suppose $T$ is stable and $\{a_1, a_2, a_3, f_{12}, f_{23}, f_{31}\}$ is a symmetric witness to non-3-uniqueness over $A$. Then $\text{tp}(\text{acl}(Aa_i)/\text{acl}(A))$ defines the object class of a connected $\star$-definable non-retractable groupoid $G$, with

$$\text{Mor}_G(a_1, a_2) = \{f' : f' \equiv_{Aa_1a_2} f_{12}\}.$$ 

Corollary

If $T$ is stable, then $T$ does not have 3-uniqueness if and only if there is a connected $\star$-definable groupoid with algebraically closed objects which is not retractable.
Generalizations?

**Question**

*Does failure of $n$-uniqueness in stable $T$ corresponded to the definability of a certain kind of “higher-dimensional groupoid” for $n \geq 4$?*

There are various different notions of “$n$-category” and “$n$-groupoid” in the literature, and it is not clear which one is appropriate here.

**Question**

*In stable $T$, is $(n+1)$-existence equivalent to $n$-uniqueness (for $n \geq 4$)?*
For stable $T$, Hrushovski proves there is an expansion $\mathcal{C}^*$ of the monster model $\mathcal{C}$ such that:

1. $\mathcal{C}^*$ is $\mathcal{C}$ plus a bounded collection of new sorts;
2. $\mathcal{C}$ is stably embedded in $\mathcal{C}^*$;
3. Each sort $S \in \mathcal{C}^*$ admits a definable map into $\mathcal{C}$ with finite fibers;
4. $\mathcal{C}^*$ has $n$-uniqueness and $n$-existence for all $n$.

However, we lack an “explicit” description of the new sorts in $\mathcal{C}^*$ – presumably they are related to higher groupoids definable in $\mathcal{C}$. 
Question

If $T$ is simple, is there an expansion $\mathcal{C}^* \supseteq \mathcal{C}$ with $n$-existence into which $\mathcal{C}$ is stably embedded?
What kinds of amalgamation can we expect in rosy theories?

O-minimal structures can’t have 3-existence (we can’t amalgamate \( x_1 < x_2 < x_3 \), and \( x_3 < x_1 \)). But they do have the following property:

**Definition**

(T rosy) \( T \) has **consistent** \( n \)-amalgamation if any thorn-independent \( n \)-amalgamation problem with a solution has a **thorn-independent** solution.
(Onshuus) There is a rosy theory which does not have consistent 3-amalgamation.

The example he constructs is a variation of Hrushovski’s ab initio construction, and has $U$-thorn-rank 1, but it is not dependent.

If $T$ is rosy and NIP, then $T$ has consistent 3-amalgamation.

What about consistent $n$-amalgamation?
Tristram De Piro, Byunghan Kim, and Jessica Millar, “Constructing the hyperdefinable group from the group configuration,” preprint.


