Classifying n-types in o-minimal theories

Janak Ramakrishnan

Université Lyon I

February 13, 2009

Background

- We work in an o-minimal structure unless otherwise specified.
- An element, c, has a type which is principal over an ordered structure, M, iff there is an element, $a \in M \cup \{\pm\infty\}$, such that $(c, a) \cap M = \emptyset$ (or $(a, c) \cap M = \emptyset$). We may also refer to the element as principal.
- Note that in an o-minimal structure, there is only one 1-type principal above (below) any given element.

Background

- We work in an o-minimal structure unless otherwise specified.
- An element, c, has a type which is principal over an ordered structure, M, iff there is an element, $a \in M \cup \{\pm\infty\}$, such that $(c, a) \cap M = \emptyset$ (or $(a, c) \cap M = \emptyset$). We may also refer to the element as principal.
- Note that in an o-minimal structure, there is only one 1-type principal above (below) any given element.

Background

- We work in an o-minimal structure unless otherwise specified.
- An element, c, has a type which is principal over an ordered structure, M, iff there is an element, $a \in M \cup \{\pm\infty\}$, such that $(c, a) \cap M = \emptyset$ (or $(a, c) \cap M = \emptyset$). We may also refer to the element as principal.
- Note that in an o-minimal structure, there is only one 1-type principal above (below) any given element.

Getting on the scale

Definition (∼Marker-Steinhorn)

Let $M \preceq N$, and $p \in S_1(N)$, with p non-principal over N. Let c be any realization of p.

- If there is an N-definable k-ary function, f, such that $f(M^k)$ is both cofinal in N below c and coinitial in N above c, we say that p is k-in scale on M.
- Otherwise, if there is such an f with $f(M^k)$ cofinal or coinitial, but not both, we say that p is k-near scale on M.
- If no such f exists for a given k, we say that p is k-out of scale on M, and if no such f exists for any k, we say that p is all out of scale on M.

Getting on the scale

Definition ($\sim$Marker-Steinhorn)

Let $M \preceq N$, and $p \in S_1(N)$, with p non-principal over N. Let c be any realization of p.

- If there is an N-definable k-ary function, f, such that $f(M^k)$ is both cofinal in N below c and coinitial in N above c, we say that p is k-in scale on M.
- Otherwise, if there is such an f with $f(M^k)$ cofinal or coinitial, but not both, we say that p is k-near scale on M.
- If no such f exists for a given k, we say that p is k-out of scale on M, and if no such f exists for any k, we say that p is all out of scale on M.

Getting on the scale

Definition ($\sim$Marker-Steinhorn)

Let $M \preceq N$, and $p \in S_1(N)$, with p non-principal over N. Let c be any realization of p.

- If there is an N-definable k-ary function, f, such that $f(M^k)$ is both cofinal in N below c and coinitial in N above c, we say that p is k-in scale on M.
- Otherwise, if there is such an f with $f(M^k)$ cofinal or coinitial, but not both, we say that p is k-near scale on M.
- If no such f exists for a given k, we say that p is k-out of scale on M, and if no such f exists for any k, we say that p is all out of scale on M.

Getting on the scale

Definition ($\sim$Marker-Steinhorn)

Let $M \preceq N$, and $p \in S_1(N)$, with p non-principal over N. Let c be any realization of p.

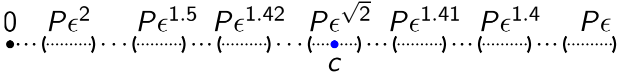
- If there is an N-definable k-ary function, f, such that $f(M^k)$ is both cofinal in N below c and coinitial in N above c, we say that p is k-in scale on M.
- Otherwise, if there is such an f with $f(M^k)$ cofinal or coinitial, but not both, we say that p is k-near scale on M.
- If no such f exists for a given k, we say that p is k-out of scale on M, and if no such f exists for any k, we say that p is all out of scale on M.

Practicing scales

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$. Let $N = M(\epsilon)$, where ϵ is infinitesimal. For compactness of notation, let $P = \mathbb{R}_+$.

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$. Let $N = M(\epsilon)$, where ϵ is infinitesimal. For compactness of notation, let $P = \mathbb{R}_+$.

1. If $c \models p = \mathrm{tp}(\epsilon^{\sqrt{2}}/N)$, then

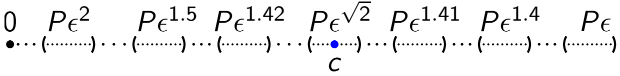


p is 1-out of scale on M.

Practicing scales

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$. Let $N = M(\epsilon)$, where ϵ is infinitesimal. For compactness of notation, let $P = \mathbb{R}_+$.

1. If $c \models p = \mathrm{tp}(\epsilon^{\sqrt{2}}/N)$, then

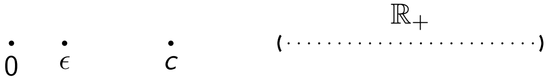


   p is 1-out of scale on M.

2. Let $M = (\mathbb{Q}^{\mathrm{rcl}}, +, \cdot, 0, 1, <)$, and let $N = M(\epsilon)$. If $c \models p = \mathrm{tp}(\pi\epsilon/N)$, then p is 1-in scale on M since, if $f(x) = x\epsilon$, $f(M)$ is both cofinal and coinitial at c in N.

More practicing scales

3. Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$ and let $N = M(\epsilon)$. Let c be smaller than every real, but larger than ϵ^d, for any rational $d > 0$.



tp(c/N) is 1-near scale on M since, if $f(x) = x$, $f(M)$ is coinitial at c in N. Note that, with $N' = M(c)$, then ϵ is principal over N'.

Scaling a peak

Theorem (Marker-Steinhorn)

Let p be an n-type over M, with $c = \langle c_1, \dots, c_n \rangle$ a realization. Then p is definable iff for each $i \leq n$, $\mathrm{tp}(c_i/Mc_{<i})$ is principal, or there is a k such that it is k-near scale on M, or all out of scale on M.

The proof does the hard work of showing that k-in scale implies 1-in scale over a definable extension, and 1-in scale easily shows that a non-principal element is definable from c. We can use this and an easy lemma to simplify the above statement.

Scaling a peak

Theorem (Marker-Steinhorn)

Let p be an n-type over M, with $c = \langle c_1, \ldots, c_n \rangle$ a realization. Then p is definable iff for each $i \leq n$, $\mathrm{tp}(c_i/Mc_{<i})$ is principal, or there is a k such that that it is k-near scale on M, or all out of scale on M.

The proof does the hard work of showing that k-in scale implies 1-in scale over a definable extension, and 1-in scale easily shows that a non-principal element is definable from c. We can use this and an easy lemma to simplify the above statement.

Many to one

Lemma

k-near scale implies 1-near scale over a definable extension.

Proof.

Since the full tuple, including the k-near scale element, is definable by the theorem, we can definably choose a cell whose image is cofinal at the element, and then consider fibers so that we drop in dimension. $\square$

Corollary

Let p be an n-type over M, with $c = \langle c_1, \ldots, c_n \rangle$ a realization. Then p is definable iff for each $i \leq n$, $\operatorname{tp}(c_i / Mc_{<i})$ is principal, or 1-near scale on M, or 1-out of scale on M.

Many to one

Lemma

k-near scale implies 1-near scale over a definable extension.

Proof.

Since the full tuple, including the k-near scale element, is definable by the theorem, we can definably choose a cell whose image is cofinal at the element, and then consider fibers so that we drop in dimension. $\square$

Corollary

Let p be an n-type over M, with $c = \langle c_1, \ldots, c_n \rangle$ a realization. Then p is definable iff for each $i \leq n$, $\mathrm{tp}(c_i / Mc_{<i})$ is principal, or 1-near scale on M, or 1-out of scale on M.

Many to one

Lemma

k-near scale implies 1-near scale over a definable extension.

Proof.

Since the full tuple, including the k-near scale element, is definable by the theorem, we can definably choose a cell whose image is cofinal at the element, and then consider fibers so that we drop in dimension. $\square$

Corollary

Let p be an n-type over M, with $c = \langle c_1, \ldots, c_n \rangle$ a realization. Then p is definable iff for each $i \leq n$, $\mathrm{tp}(c_i/Mc_{<i})$ is principal, or 1-near scale on M, or 1-out of scale on M.

Many to one?

Question

Let $M \prec N$, with N an arbitrary elementary extension. Let $p \in S_1(N)$ be k-in (-near) scale on M. Is p 1-in (-near) scale on M?

Value and scales

Many to one?

Question

Let $M \prec N$, with N an arbitrary elementary extension. Let $p \in S_1(N)$ be k-in (-near) scale on M. Is p 1-in (-near) scale on M?

▸ Value and scales

Making scales less slippery

We assume from now on that all principal types are interdefinable (this is true if M expands a field).

Definition

Let M be a structure. Define $a \prec_M b$ iff $\mathrm{tp}(a/Mb)$ is principal near an element of $M \cup \{\pm\infty\}$. Define $a \sim_M b$ if $a \not\prec_M b$ and $b \not\prec_M a$.

Lemma

$\sim_M$ is an equivalence relation, and $\prec_M$ totally orders the $\sim_M$-classes.

Making scales less slippery

We assume from now on that all principal types are interdefinable (this is true if M expands a field).

Definition

Let M be a structure. Define $a \prec_M b$ iff $\mathrm{tp}(a/Mb)$ is principal near an element of $M \cup \{\pm\infty\}$. Define $a \sim_M b$ if $a \not\prec_M b$ and $b \not\prec_M a$.

Lemma

$\sim_M$ is an equivalence relation, and $\prec_M$ totally orders the $\sim_M$-classes.

Making scales less slippery

We assume from now on that all principal types are interdefinable (this is true if M expands a field).

Definition

Let M be a structure. Define $a \prec_M b$ iff $\mathrm{tp}(a/Mb)$ is principal near an element of $M \cup \{\pm\infty\}$. Define $a \sim_M b$ if $a \not\prec_M b$ and $b \not\prec_M a$.

Lemma

$\sim_M$ is an equivalence relation, and $\prec_M$ totally orders the $\sim_M$-classes.

Decreasing types: definition

Definition

Assume that we have a fixed sequence $c = \langle c_i \rangle_{i \in I}$. Then the $\prec_i$-ordering is the $\prec_{c_{<i}}$-ordering. If we also have a fixed base, M, then it will be the $\prec_{Mc_{<i}}$-ordering.

Definition

A sequence, $c = \langle c_i \rangle_{i \in I}$, is decreasing if $c_j \precsim_i c_i$, for $j > i$. A type is decreasing if any realization of it is.

Lemma

Any n-type can have its coordinates reordered so that it is decreasing.

Decreasing types: definition

Definition

Assume that we have a fixed sequence $c = \langle c_i \rangle_{i \in I}$. Then the $\prec_i$-ordering is the $\prec_{c_{<i}}$-ordering. If we also have a fixed base, M, then it will be the $\prec_{Mc_{<i}}$-ordering.

Definition

A sequence, $c = \langle c_i \rangle_{i \in I}$, is decreasing if $c_j \precsim_i c_i$, for $j > i$. A type is decreasing if any realization of it is.

Lemma

Any n-type can have its coordinates reordered so that it is decreasing.

Decreasing types: definition

Definition

Assume that we have a fixed sequence $c = \langle c_i \rangle_{i \in I}$. Then the $\prec_i$-ordering is the $\prec_{c_{<i}}$-ordering. If we also have a fixed base, M, then it will be the $\prec_{Mc_{<i}}$-ordering.

Definition

A sequence, $c = \langle c_i \rangle_{i \in I}$, is decreasing if $c_j \precsim_i c_i$, for $j > i$. A type is decreasing if any realization of it is.

Lemma

Any n-type can have its coordinates reordered so that it is decreasing.

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta\rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta\rangle}/c_{<\langle j,\gamma_j\rangle})$ is principal.*
- *If i has a predecessor, $i-1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta\rangle}/Mc_{<\langle i,0\rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta\rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0\rangle})$.*

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta\rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta\rangle}/c_{<\langle j,\gamma_j\rangle})$ is principal.*
- *If i has a predecessor, $i-1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta\rangle}/Mc_{<\langle i,0\rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta\rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0\rangle})$.*

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta \rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta \rangle}/c_{<\langle j,\gamma_j \rangle})$ is principal.*
- *If i has a predecessor, $i-1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta \rangle}/Mc_{<\langle i,0 \rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta \rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0 \rangle})$.*

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta \rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta\rangle}/c_{<\langle j,\gamma_j\rangle})$ is principal.*
- *If i has a predecessor, $i-1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta\rangle}/Mc_{<\langle i,0\rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta \rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0\rangle})$.*

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta \rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta \rangle}/c_{<\langle j,\gamma_j \rangle})$ is principal.*
- *If i has a predecessor, $i - 1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta \rangle}/Mc_{<\langle i,0 \rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta \rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0 \rangle})$.*

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta\rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta\rangle}/c_{<\langle j,\gamma_j\rangle})$ is principal.*
- *If i has a predecessor, $i-1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta\rangle}/Mc_{<\langle i,0\rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta\rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0\rangle})$.*

Plummeting

Proposition

Let $M \prec N$. Then there is an ordered set, I, and sequence $c = \{c_\alpha \mid \alpha \in I\}$, with $N \setminus M = \{c_\alpha \mid \alpha \in I\}$, and the following properties for c and I:

- *c is decreasing.*
- *$I = \{\langle i, \beta\rangle : i \in I_0, \beta \in \gamma_i\}$, for some linear order I_0 and cardinals γ_i.*
- *If $i > j \in I_0$, then for any $\beta \in \gamma_i$, $\mathrm{tp}(c_{\langle i,\beta\rangle}/c_{<\langle j,\gamma_j\rangle})$ is principal.*
- *If i has a predecessor, $i-1$, in I_0, then $\mathrm{tp}(c_{\langle i,\beta\rangle}/Mc_{<\langle i,0\rangle})$ is principal above 0.*
- *If $\alpha = \langle i, \beta\rangle$, with $\beta > 0$ and i not the smallest element of I_0, and $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is not algebraic, then $\mathrm{tp}(c_\alpha/Mc_{<\alpha})$ is all out of scale on $M(c_{<\langle i,0\rangle})$.*

Obstructions

The case for finite-dimensional extensions is straightforward, but with infinite-dimensional extensions, the expected technique – given a sequence and an element, find a place in the sequence for the element to go – fails.

Example

Take the extension over $\mathbb{R}$ generated by $\{\epsilon_i\}_{i\in\mathbb{N}}$ and $\mu \approx \sum_{i\in\mathbb{N}} \prod_{0\leq j\leq i} \epsilon_j$ with $\epsilon_i \gg \epsilon_j$ for $i > j$. Then, if we have the sequence $\{\epsilon_i\}_{i\in\omega^*}$, there is no place for μ to go.

▸ Sketch of Proof

Obstructions

The case for finite-dimensional extensions is straightforward, but with infinite-dimensional extensions, the expected technique – given a sequence and an element, find a place in the sequence for the element to go – fails.

Example

Take the extension over $\mathbb{R}$ generated by $\{\epsilon_i\}_{i \in \mathbb{N}}$ and $\mu \approx \sum_{i \in \mathbb{N}} \prod_{0 \leq j \leq i} \epsilon_j$ with $\epsilon_i \gg \epsilon_j$ for $i > j$. Then, if we have the sequence $\{\epsilon_i\}_{i \in \omega^*}$, there is no place for μ to go.

Sketch of Proof

Adding in non-principal types

At every stage, we had a definable extension. We now consider all remaining elements, each of which must generate a non-principal element over M. It is clear that we can insert these elements at the start of our sequence. But we must ensure that each out of scale element remains out of scale. We do this by means of the following lemmas.

Lemma

Let $M \preceq N$. Let $\mathrm{tp}(c/N)$ be all out of scale on M. Let b be strictly $\prec_M$-maximal over $N \setminus M$. Then $\mathrm{tp}(c/Nb)$ is all out of scale on $M(b)$.

Lemma

Let $M \preceq N$, and let N be definable over M. Let b be non-principal over M. Then $N(b)$ is definable over $M(b)$.

Adding in non-principal types

At every stage, we had a definable extension. We now consider all remaining elements, each of which must generate a non-principal element over M. It is clear that we can insert these elements at the start of our sequence. But we must ensure that each out of scale element remains out of scale. We do this by means of the following lemmas.

Lemma

Let $M \preceq N$. Let $\mathrm{tp}(c/N)$ be all out of scale on M. Let b be strictly $\prec_M$-maximal over $N \setminus M$. Then $\mathrm{tp}(c/Nb)$ is all out of scale on $M(b)$.

Lemma

Let $M \preceq N$, and let N be definable over M. Let b be non-principal over M. Then $N(b)$ is definable over $M(b)$.

Adding in non-principal types

At every stage, we had a definable extension. We now consider all remaining elements, each of which must generate a non-principal element over M. It is clear that we can insert these elements at the start of our sequence. But we must ensure that each out of scale element remains out of scale. We do this by means of the following lemmas.

Lemma

Let $M \preceq N$. Let $\mathrm{tp}(c/N)$ be all out of scale on M. Let b be strictly $\prec_M$-maximal over $N \setminus M$. Then $\mathrm{tp}(c/Nb)$ is all out of scale on $M(b)$.

Lemma

Let $M \preceq N$, and let N be definable over M. Let b be non-principal over M. Then $N(b)$ is definable over $M(b)$.

Proof of first lemma

- We show that, if $f(b, M(b)^k)$ is cofinal (coinitial) at c in $N(b)$, then $f(M^{k+1})$ was cofinal (coinitial) at c in N.
- If not, we can find some interval, (a, c), with $a \in N$, $(a, c) \cap f(M^{k+1}) = \emptyset$, but $f(b, \alpha(b)) \in (a, c)$ for some tuple of M-definable functions α.
- It can be seen that there is some $a_1 \in (f(b, \alpha(b)), c) \cap N$. Then we consider the set

$$A = \{x_1 : f(x_1, \alpha(x_1)) \in (a, a_1)\}.$$

- Since $b \in A$ and A is N-definable, there is an interval in A around b.
- Since b is strictly $\prec_M$-maximal over $N \setminus M$, there must be an element of M in that interval – contradiction.

▸ Proof of second lemma ▸ Value and decreasing types

Proof of first lemma

- We show that, if $f(b, M(b)^k)$ is cofinal (coinitial) at c in $N(b)$, then $f(M^{k+1})$ was cofinal (coinitial) at c in N.
- If not, we can find some interval, (a, c), with $a \in N$, $(a, c) \cap f(M^{k+1}) = \emptyset$, but $f(b, \alpha(b)) \in (a, c)$ for some tuple of M-definable functions α.
- It can be seen that there is some $a_1 \in (f(b, \alpha(b)), c) \cap N$. Then we consider the set

$$A = \{x_1 : f(x_1, \alpha(x_1)) \in (a, a_1)\}.$$

- Since $b \in A$ and A is N-definable, there is an interval in A around b.
- Since b is strictly $\prec_M$-maximal over $N \setminus M$, there must be an element of M in that interval – contradiction.

Proof of second lemma   Value and decreasing types

Proof of first lemma

- We show that, if $f(b, M(b)^k)$ is cofinal (coinitial) at c in $N(b)$, then $f(M^{k+1})$ was cofinal (coinitial) at c in N.
- If not, we can find some interval, (a, c), with $a \in N$, $(a, c) \cap f(M^{k+1}) = \emptyset$, but $f(b, \alpha(b)) \in (a, c)$ for some tuple of M-definable functions α.
- It can be seen that there is some $a_1 \in (f(b, \alpha(b)), c) \cap N$. Then we consider the set

$$A = \{x_1 : f(x_1, \alpha(x_1)) \in (a, a_1)\}.$$

- Since $b \in A$ and A is N-definable, there is an interval in A around b.
- Since b is strictly $\prec_M$-maximal over $N \setminus M$, there must be an element of M in that interval – contradiction.

Proof of second lemma   Value and decreasing types

Proof of first lemma

- We show that, if $f(b, M(b)^k)$ is cofinal (coinitial) at c in $N(b)$, then $f(M^{k+1})$ was cofinal (coinitial) at c in N.
- If not, we can find some interval, (a, c), with $a \in N$, $(a, c) \cap f(M^{k+1}) = \emptyset$, but $f(b, \alpha(b)) \in (a, c)$ for some tuple of M-definable functions α.
- It can be seen that there is some $a_1 \in (f(b, \alpha(b)), c) \cap N$. Then we consider the set

$$A = \{x_1 : f(x_1, \alpha(x_1)) \in (a, a_1)\}.$$

- Since $b \in A$ and A is N-definable, there is an interval in A around b.
- Since b is strictly $\prec_M$-maximal over $N \setminus M$, there must be an element of M in that interval – contradiction.

Proof of second lemma   Value and decreasing types

Proof of first lemma

- We show that, if $f(b, M(b)^k)$ is cofinal (coinitial) at c in $N(b)$, then $f(M^{k+1})$ was cofinal (coinitial) at c in N.
- If not, we can find some interval, (a, c), with $a \in N$, $(a, c) \cap f(M^{k+1}) = \emptyset$, but $f(b, \alpha(b)) \in (a, c)$ for some tuple of M-definable functions α.
- It can be seen that there is some $a_1 \in (f(b, \alpha(b)), c) \cap N$. Then we consider the set

$$A = \{x_1 : f(x_1, \alpha(x_1)) \in (a, a_1)\}.$$

- Since $b \in A$ and A is N-definable, there is an interval in A around b.
- Since b is strictly $\prec_M$-maximal over $N \setminus M$, there must be an element of M in that interval – contradiction.

Proof of second lemma   Value and decreasing types

Proof of first lemma

- We show that, if $f(b, M(b)^k)$ is cofinal (coinitial) at c in $N(b)$, then $f(M^{k+1})$ was cofinal (coinitial) at c in N.
- If not, we can find some interval, (a, c), with $a \in N$, $(a, c) \cap f(M^{k+1}) = \emptyset$, but $f(b, \alpha(b)) \in (a, c)$ for some tuple of M-definable functions α.
- It can be seen that there is some $a_1 \in (f(b, \alpha(b)), c) \cap N$. Then we consider the set

$$A = \{x_1 : f(x_1, \alpha(x_1)) \in (a, a_1)\}.$$

- Since $b \in A$ and A is N-definable, there is an interval in A around b.
- Since b is strictly $\prec_M$-maximal over $N \setminus M$, there must be an element of M in that interval – contradiction.

▸ Proof of second lemma ▸ Value and decreasing types

Refining intermediates

The following is due to Baisalov and Poizat:

Lemma

Si $M \prec N$ est une extension élémentaire de modèles de T, on peut trouver un modèle intermédiaire N', $M \prec N' \prec N$, tel que tout a de $N' \setminus M$ ait un type non-définissable sur M, et tout b de $N \setminus N'$ ait un type definissable sur N'; on peut également trouver un modèle intermédiaire N'', $M \prec N'' \prec N$, tel que tout a de $N'' \setminus M$ ait un type définissables sur M, et tout b de $N \setminus N''$ ait un type non-définissables sur N''.

Refining intermediates

The following is due to Baisalov and Poizat:

Lemma

Let $M \prec N$ be an elementary extension. We can find an intermediate structure, N', such that every $a \in N' \setminus M$ is non-principal over M, and every $b \in N \setminus N'$ is principal over N'. We can also find N'' such that every $a \in N'' \setminus M$ is principal over M, and every $b \in N \setminus N''$ is non-principal over N''.

Refining intermediates

The following is due to Baisalov and Poizat:

Lemma

Let $M \prec N$ be an elementary extension. We can find an intermediate structure, N', such that every $a \in N' \setminus M$ is non-principal over M, and every $b \in N \setminus N'$ is principal over N'. We can also find N'' such that every $a \in N'' \setminus M$ is principal over M, and every $b \in N \setminus N''$ is non-principal over N''.

Note the asymmetry of the lemma. In the first case, not only is every $b \in N \setminus N'$ principal over N', it is interdefinable with a principal element over M. Not so in the second.

A sharper result

Lemma

Let $M \prec N$ be an elementary extension. We can find N'' such that every $a \in N'' \setminus M$ is principal over M, and every $b \in N \setminus N''$ is N''-interdefinable with a non-principal element over M.

Proof.

Put N in the form guaranteed by our proposition. Then if we remove the initial segment of the sequence consisting of non-principal elements, the remaining sequence satisfies the conditions on N''. $\square$

A sharper result

Lemma

Let $M \prec N$ be an elementary extension. We can find N'' such that every $a \in N'' \setminus M$ is principal over M, and every $b \in N \setminus N''$ is N''-interdefinable with a non-principal element over M.

Proof.

Put N in the form guaranteed by our proposition. Then if we remove the initial segment of the sequence consisting of non-principal elements, the remaining sequence satisfies the conditions on N''. □

Value and scales

Analyzing pairs of structures, we find another way to describe scale, assuming that our o-minimal structure expands a field.

Definition

Let N be an elementary extension of M. Let $\Gamma(N) = N^\times / M^\times$. $v : N^\times \to \Gamma(N)$ is the induced valuation.

Lemma

$\mathrm{tp}(c/N)$ is principal iff $v(c')$ is principal over $\Gamma(N)$ near ∞ for some $c' \in \mathrm{dcl}(Nc)$. $\mathrm{tp}(c/N)$ is 1-near scale on M iff $v(c')$ is principal over $\Gamma(N)$ near a finite element of $\Gamma(N)$. $\mathrm{tp}(c/N)$ is 1-in scale on M iff $v(c') \in \Gamma(N)$. $\mathrm{tp}(c/N)$ is 1-out of scale on M iff $v(c')$ is non-principal over $\Gamma(N)$ and not in $\Gamma(N)$.

Back

Value and scales

Analyzing pairs of structures, we find another way to describe scale, assuming that our o-minimal structure expands a field.

Definition

Let N be an elementary extension of M. Let $\Gamma(N) = N^\times / M^\times$. $v : N^\times \to \Gamma(N)$ is the induced valuation.

Lemma

$\mathrm{tp}(c/N)$ *is principal iff* $v(c')$ *is principal over* $\Gamma(N)$ *near* ∞ *for some* $c' \in \mathrm{dcl}(Nc)$. $\mathrm{tp}(c/N)$ *is* 1*-near scale on* M *iff* $v(c')$ *is principal over* $\Gamma(N)$ *near a finite element of* $\Gamma(N)$. $\mathrm{tp}(c/N)$ *is* 1*-in scale on* M *iff* $v(c') \in \Gamma(N)$. $\mathrm{tp}(c/N)$ *is* 1*-out of scale on* M *iff* $v(c')$ *is non-principal over* $\Gamma(N)$ *and not in* $\Gamma(N)$.

Back

Value and scales

Analyzing pairs of structures, we find another way to describe scale, assuming that our o-minimal structure expands a field.

Definition

Let N be an elementary extension of M. Let $\Gamma(N) = N^\times / M^\times$.
$v : N^\times \to \Gamma(N)$ is the induced valuation.

Lemma

$\mathrm{tp}(c/N)$ *is principal iff* $v(c')$ *is principal over* $\Gamma(N)$ *near* ∞ *for some* $c' \in \mathrm{dcl}(Nc)$. $\mathrm{tp}(c/N)$ *is* 1*-near scale on* M *iff* $v(c')$ *is principal over* $\Gamma(N)$ *near a finite element of* $\Gamma(N)$. $\mathrm{tp}(c/N)$ *is* 1*-in scale on* M *iff* $v(c') \in \Gamma(N)$. $\mathrm{tp}(c/N)$ *is* 1*-out of scale on* M *iff* $v(c')$ *is non-principal over* $\Gamma(N)$ *and not in* $\Gamma(N)$.

Back

Decreasing means increasing value

If an n-type, p, realized by a sequence, c, conforms to the conditions of our proposition (non-principal in the beginning, principal above 0, always out of scale), then we have the following.

Lemma

1. *The T-convex closure of $Mc_{<i}$ is contained in the T-convex closure of $Mc_{<j}$, for $i \leq j$.*
2. $v(c_i) \in v(M(c_{<i}))$ *iff* $v(c_i) \in v(M)$ *iff* $\mathrm{tp}(c_i/Mc_{<i})$ *is non-principal iff* $\mathrm{tp}(c_i/M)$ *is non-principal.*
3. $v(c_i)$ *is non-principal over* $\Gamma(Mc_{<i})$ *iff* $\mathrm{tp}(c_i/Mc_{<i})$ *is all out of scale on M. Otherwise, $v(c_i)$ is principal near $\pm\infty$.*

Back

Decreasing means increasing value

If an n-type, p, realized by a sequence, c, conforms to the conditions of our proposition (non-principal in the beginning, principal above 0, always out of scale), then we have the following.

Lemma

1. *The T-convex closure of $Mc_{<i}$ is contained in the T-convex closure of $Mc_{<j}$, for $i \leq j$.*
2. *$v(c_i) \in v(M(c_{<i}))$ iff $v(c_i) \in v(M)$ iff $\mathrm{tp}(c_i/Mc_{<i})$ is non-principal iff $\mathrm{tp}(c_i/M)$ is non-principal.*
3. *$v(c_i)$ is non-principal over $\Gamma(Mc_{<i})$ iff $\mathrm{tp}(c_i/Mc_{<i})$ is all out of scale on M. Otherwise, $v(c_i)$ is principal near $\pm\infty$.*

Back

Decreasing means increasing value

If an n-type, p, realized by a sequence, c, conforms to the conditions of our proposition (non-principal in the beginning, principal above 0, always out of scale), then we have the following.

Lemma

1. *The T-convex closure of $Mc_{<i}$ is contained in the T-convex closure of $Mc_{<j}$, for $i \leq j$.*
2. $v(c_i) \in v(M(c_{<i}))$ *iff* $v(c_i) \in v(M)$ *iff* $\mathrm{tp}(c_i/Mc_{<i})$ *is non-principal iff* $\mathrm{tp}(c_i/M)$ *is non-principal.*
3. $v(c_i)$ *is non-principal over* $\Gamma(Mc_{<i})$ *iff* $\mathrm{tp}(c_i/Mc_{<i})$ *is all out of scale on* M. *Otherwise,* $v(c_i)$ *is principal near* $\pm\infty$.

Back

Decreasing means increasing value

If an n-type, p, realized by a sequence, c, conforms to the conditions of our proposition (non-principal in the beginning, principal above 0, always out of scale), then we have the following.

Lemma

1. *The T-convex closure of $Mc_{<i}$ is contained in the T-convex closure of $Mc_{<j}$, for $i \leq j$.*
2. $v(c_i) \in v(M(c_{<i}))$ *iff* $v(c_i) \in v(M)$ *iff* $\mathrm{tp}(c_i/Mc_{<i})$ *is non-principal iff* $\mathrm{tp}(c_i/M)$ *is non-principal.*
3. $v(c_i)$ *is non-principal over* $\Gamma(Mc_{<i})$ *iff* $\mathrm{tp}(c_i/Mc_{<i})$ *is all out of scale on M. Otherwise,* $v(c_i)$ *is principal near* $\pm\infty$.

Back

Sketch of proof

- Using inclusion of T-convex subrings of N, find a maximal "spine" of elements whose T-convex closures generate the maximal sequence of T-convex subrings – each element will be principal over the preceding ones.
- Go transfinitely, choosing b, an element of N not in the definable closure of c^i – what we have so far – such that $c^i b$ remains definable over M.
- If b (or an element interdefinable with b over $c^i M$) has a unique position in the sequence c^i, insert it.
- If not, we insert infinitely many elements, all interdefinable with each other over c^i, thus preventing any of them from "backsliding" up a level.

Back

Sketch of proof

- Using inclusion of T-convex subrings of N, find a maximal "spine" of elements whose T-convex closures generate the maximal sequence of T-convex subrings – each element will be principal over the preceding ones.
- Go transfinitely, choosing b, an element of N not in the definable closure of c^i – what we have so far – such that $c^i b$ remains definable over M.
- If b (or an element interdefinable with b over $c^i M$) has a unique position in the sequence c^i, insert it.
- If not, we insert infinitely many elements, all interdefinable with each other over c^i, thus preventing any of them from "backsliding" up a level.

Back

Sketch of proof

- Using inclusion of T-convex subrings of N, find a maximal "spine" of elements whose T-convex closures generate the maximal sequence of T-convex subrings – each element will be principal over the preceding ones.
- Go transfinitely, choosing b, an element of N not in the definable closure of c^i – what we have so far – such that $c^i b$ remains definable over M.
- If b (or an element interdefinable with b over $c^i M$) has a unique position in the sequence c^i, insert it.
- If not, we insert infinitely many elements, all interdefinable with each other over c^i, thus preventing any of them from "backsliding" up a level.

Back

Sketch of proof

- Using inclusion of T-convex subrings of N, find a maximal "spine" of elements whose T-convex closures generate the maximal sequence of T-convex subrings – each element will be principal over the preceding ones.
- Go transfinitely, choosing b, an element of N not in the definable closure of c^i – what we have so far – such that $c^i b$ remains definable over M.
- If b (or an element interdefinable with b over $c^i M$) has a unique position in the sequence c^i, insert it.
- If not, we insert infinitely many elements, all interdefinable with each other over c^i, thus preventing any of them from "backsliding" up a level.

Back

Proof of second lemma

- Assume we fail, so $f(e, b)$ has non-principal type over Mb, for some M-definable f and tuple from N, e. We can choose f and e to minimize $k = \mathrm{lh}(e)$ and assume that it satisfies the proposition.
- If $\mathrm{tp}(e_k/Me_{<k})$ is principal, we can show that $\mathrm{tp}(e_k/Me_{<k}b)$ is also principal, above 0. This easily gives a contradiction, by minimality of k.
- If $\mathrm{tp}(e_k/Me_{<k})$ is non-principal, then it is all out of scale on M, but we can see that $M(b)$ is cofinal and coinitial at $f(e, b)$ in $M(e_{<k}b)$ by minimality of k, so $f^{-1}_{e_{<k}}(b, M(b))$ is cofinal and coinitial at e_k in $M(e_{<k}b)$. By the previous lemma, contradiction.

Back

Proof of second lemma

- Assume we fail, so $f(e, b)$ has non-principal type over Mb, for some M-definable f and tuple from N, e. We can choose f and e to minimize $k = \mathrm{lh}(e)$ and assume that it satisfies the proposition.
- If $\mathrm{tp}(e_k/Me_{<k})$ is principal, we can show that $\mathrm{tp}(e_k/Me_{<k}b)$ is also principal, above 0. This easily gives a contradiction, by minimality of k.
- If $\mathrm{tp}(e_k/Me_{<k})$ is non-principal, then it is all out of scale on M, but we can see that $M(b)$ is cofinal and coinitial at $f(e, b)$ in $M(e_{<k}b)$ by minimality of k, so $f^{-1}_{e_{<k}}(b, M(b))$ is cofinal and coinitial at e_k in $M(e_{<k}b)$. By the previous lemma, contradiction.

Back

Proof of second lemma

- Assume we fail, so $f(e, b)$ has non-principal type over Mb, for some M-definable f and tuple from N, e. We can choose f and e to minimize $k = \mathrm{lh}(e)$ and assume that it satisfies the proposition.
- If $\mathrm{tp}(e_k/Me_{<k})$ is principal, we can show that $\mathrm{tp}(e_k/Me_{<k}b)$ is also principal, above 0. This easily gives a contradiction, by minimality of k.
- If $\mathrm{tp}(e_k/Me_{<k})$ is non-principal, then it is all out of scale on M, but we can see that $M(b)$ is cofinal and coinitial at $f(e, b)$ in $M(e_{<k}b)$ by minimality of k, so $f^{-1}_{e_{<k}}(b, M(b))$ is cofinal and coinitial at e_k in $M(e_{<k}b)$. By the previous lemma, contradiction.

Back