Report on 10FRG155 — Discrete Probability

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1 summary

This is a report on activities and results of the focused research group meeting held at BIRS on 2010-06-13 to 26. The participants were Omer Angel, Ander Holroyd, Gady Kozma, James Martin, Jim Propp (first week), Dan Romik, Johan Wastlund, David Wilson (second week), and Peter Winkler.

Tho format of the meeting was as follows. Each morning, one or two of the participants would discuss a few open problems in the broad area of discrete probability. These led to an open discussion, which continued in the afternoons, either at the BIRS facilities or occasionally during excursions. While the problems were broadly spread, some unexpected connections were discovered. The problems discussed ranged from very specific questions to more vague ideas and suggested attacks on other problems.

In the second week, the problem exposition component was reduced, and most of the time was spent pursuing the some promising approaches to solving the problems. However, some questions also arose later in the meeting, as a result of discussions.

Over all, the meeting was very successful, over 30 problems were posed and discussed. Several were solved completely, and significant progress was made on others. A number of papers are being currently being written as a direct result of the meeting (two are already complete), and much current research is still being done. A number of questions are of a very fundamental nature, and the resulting work is likely to pave the way to additional research in the future.

2 Diadic tilings

A direct outcome of the workshop [2]:

A dyadic tile of order n is any rectangle obtained from the unit square by n successive bisections by horizontal or vertical cuts. Let each dyadic tile of order n be available with probability p, independently of the others. We prove that for p sufficiently close to 1, there exists a set of pairwise disjoint available tiles whose union is the unit square, with probability tending to 1 as $n \to \infty$, as conjectured by Joel Spencer in 1999. In particular we prove that if $p = 7/8$, such a tiling exists with probability at least $1 - (3/4)^n$. The proof involves a surprisingly delicate counting argument for sets of unavailable tiles that prevent tiling. This problem is also related to bootstrap percolation on lamplighter groups.

3 Avoidance coupling

Another direct outcome [3]:

Two independent random walks on a graph are sure to collide at some time. However, if the random walks are not independent, it is sometimes possible to arrange them so that they never collide. As an application, one may envisage some anti-virus software moving from port to port in a computer system to check for incursions. It is natural to have such a program implement a random walk on the ports so as not to be predictable. If another program (possibly with a different purpose) also does a random walk on the ports, it may be desirable or even essential to prevent the programs from examining the same port at the same time.

We show that on the complete graph on n vertices, with or without loops, there is a Markovian coupling keeping apart $\Omega(n/\log n)$ random walks, taking turns to move.

4 Tokens on a graph

A problem that gave rise to the avoidance question above: Several tokens are located on a graph. At each step an adversary selects which one to move, while trying to avoid collisions asfor as long as possible. Is it possible to find the optimal strategy and worst locations for the tokens? In particular, does the worst configuration ever include more than 3 tokens?

This question is related to a number of problems on scheduling of random walks [4, 6, 5].

5 Random fault trees

Consider binary tree of depth n, and place randomly an and-gate or an or-gate at each node, independently at random. If the inputs at the leaves of the tree are random, $\{0, 1\}$ variables, so is the resulting value at the root. What is the distribution of the value at the root if we condition on the gates? Each of the 2^n input bits has probability 2^{-n} of influencing the output, but how sensitive is the output to changing some of the gates? What it other gates are also included?

6 The dark waiting room

A queueing system evolves according to the following rules. Several people are in a waiting room. Additional people arrive at some rate. At each round, each person can request service. If a unique request is received, it is filled and the person leaves. If more than one request is made, nothing happens.

There are many questions regarding this system. Is there a strategy such that every person is eventually served? If person requests service with probability $1/n$ where n is the number of people in the room then the number of people is recurrent as long as the rate of arrival is at most e^{-1}. However, if the number of people is not known, the problem is open.

A number of variations are also interesting: What if people are told how many requests were made at each round? What is the fastest strategy if there are n people in the room with no new arrivals?

7 Coupling TASEPs

Is there a coupling of two exclusion processes started from two different initial conditions, so that they are at the same state for all large times? In particular, does the so called standard coupling work? This question cames from recent advances related to these processes [1], and a positive answer would have some implications.

8 Allocations

The following problem has vexed researchers for a number of years. Given a Poisson point process in $\mathbb{R}^2$, an allocation is a rule to associate to each point of the process a set of area 1 in a translation invariant manner, so that the sets are a partition of the plane. A number of allocation rules had been known [7, 8], but it was not known how to allocate bounded, connected sets. We have found ways to construct such allocations. One of our constructions has the unusual properties that the sets of the partition even have disjoint closures.

References

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