PENTAGRAM MAP, COMPLETE INTEGRABILITY AND CLUSTER MANIFOLDS

SOPHIE MORIER-GENOUD, VALENTIN OVSIENKO, AND SERGE TABACHNIKOV

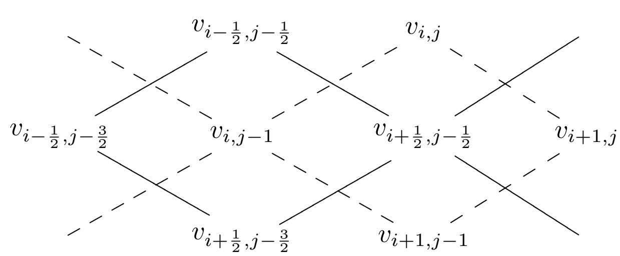
The pentagram map, T, is a remarkable dynamical system introduced by Richard Schwartz in 1992 and studied in a series of articles, see [5] and references therein. The pentagram map acts on the space $\mathcal{C}_n$ of n-gons in the projective plane modulo projective equivalence. Given an n-gon P, the corresponding n-gon $T(P)$ is generated by the intersection points of consecutive shortest diagonals of P. The most remarkable property of the pentagram map is its complete integrability. Conjectured by Schwartz about 20 years ago, it was recently proved in [3], for a larger space of *twisted n-gons*. Integrability of T on the initial space $\mathcal{C}_n$ remains a challenging problem.

The main purpose of this Workshop was to study the space $\mathcal{C}_n$ within the modern framework of *cluster algebra* recently developed by Fomin and Zelevinsky [2] as a powerful tool for the study of many classes of algebraic manifolds. This approach leads in particular to very special coordinate systems on $\mathcal{C}_n$ related to interesting algebraic and combinatorial structures. The space $\mathcal{C}_n$ is an algebraic manifold which is a close relative of the moduli space $\mathcal{M}_{0,n}$ of genus 0 curves with n marked points. This viewpoint closely relates the project to a fundamental domain of algebraic geometry.

The results obtained during and in the summer after the Workshop led to an article "2-frieze patterns and the cluster structure of the space polygons" currently submitted for publication. Below, we outline the main results and methods.

• The first theorem states that the space $\mathcal{C}_n$ is, indeed, a cluster manifold.

The main ingredient of our construction of the cluster structure on $\mathcal{C}_n$ is a (quite unexpected) relation to the classical notion of *friezes* developed by Coxeter and Conway [1]. A generalized version of the Coxeter-Conway friezes that we call a 2-*frieze pattern* an infinite grid (of numbers, or polynomials, rational functions, etc.) $(v_{i,j})_{(i,j)\in\mathbb{Z}^2}$ and $(v_{i+\frac{1}{2},j+\frac{1}{2}})_{(i,j)\in\mathbb{Z}^2}$ organized as follows



and such that every entry is equal to the determinant of the 2×2-matrix formed by its four neighbours:

$$v_{i,j-1} = v_{i-\frac{1}{2},j-\frac{3}{2}}\, v_{i+\frac{1}{2},j-\frac{1}{2}} - v_{i-\frac{1}{2},j-\frac{1}{2}}\, v_{i+\frac{1}{2},j-\frac{3}{2}}. \tag{1}$$

The relation between the space of n-gons $\mathcal{C}_n$ and the 2-friezes is as follows. As shown in [3], the space $\mathcal{C}_n$ can be identified with the space of difference equations of the form

$$V_i = a_i\, V_{i-1} - b_i\, V_{i-2} + V_{i-3}, \tag{2}$$

where $a_i, b_i \in \mathbb{R}$ (or $\mathbb{C}$) are n-periodic: $a_{i+n} = a_i$ and $b_{i+n} = b_i$, for all i, such that all the solutions are periodic. In other words, we consider the difference equations (2) with trivial monodromy.

In order to obtain a relation to 2-friezes, we assume: $v_{i,i} = a_i$, $v_{i-\frac{1}{2},i-\frac{1}{2}} = b_i$, and form a 2-frieze bounded from above a row of 1's:

$$
\begin{array}{ccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & b_1 & & a_1 & & b_2 & & a_2 & & b_3 & \cdots \\
& \cdots & b_1b_2 - a_1 & & a_1a_2 - b_2 & & b_2b_3 - a_2 & & \cdots & & \\
& & \vdots & & \vdots & & \vdots & & & &
\end{array}
\tag{3}
$$

The rest of the 2-frieze is determined with the help of the rule (1).

We are particularly interested in 2-friezes bounded from above and from below by two rows of 1's:

$$
\begin{array}{ccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & b_1 & a_1 & b_2 & a_2 & b_3 & \cdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
$$

that we call *closed*. We call the *width* of a closed 2-frieze the number of the rows between the two rows of 1's.

• Our next result states that a $2n$-periodic 2-frieze (3) is closed if and only if the difference equation (2) has trivial monodromy.

This theorem allows us to identify three spaces: the space $\mathcal{C}_n$, the space of difference equations (2) with monordomy and the the space of closed 2-friezes.

It should be stressed that the notion of 2-friezes has already appeared in the literature [4] but have not been studied in detail. The above result is new, this is a generalization of a classical Coxeter-Conway theorem.

The structure of cluster manifold on the space of 2-friezes is defined in terms of "zig-zag coordinates". Draw an arbitrary *double zig-zag* and denote by $(x_1, \ldots, x_{n-4}, y_1, \ldots, y_{n-4})$ the entries lying on this double zig-zag:

$$
\begin{array}{ccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\
& & x_1 & y_1 & & & \\
& & & x_2 & y_2 & & \\
& & x_3 & y_3 & & & \\
& & \vdots & \vdots & & & \\
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
$$

in such a way that x_i stay at the entries with integer indices and y_i stay at the entries with half-integer indices. Applying the recurrence relations, complete the 2-frieze pattern by rational functions in x_i, y_j. For example, in the case of width 1 we get:

$$
\begin{array}{ccccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & x & y & \frac{y+1}{x} & \frac{x+y+1}{xy} & \frac{x+1}{y} & x & y & \cdots \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots
\end{array}
$$

We hope that the defined cluster structure will help us to prove integrability of the pentagram map restricted to $\mathcal{C}_n$.

In the second part of our work we study *integral* closed 2-friezes, i.e., consisting of positive integers. For instance, the 2-frieze pattern

$$\begin{array}{cccccccccccc} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 2 & 3 & 2 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{array}$$

is the unique integral 2-frieze pattern of width 1. The following 2-frieze is of width 2:

$$\begin{array}{cccccccccccccc} \ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ \ldots & 1 & 3 & 5 & 2 & 1 & 3 & 5 & 2 & 1 & 3 & 5 & 2 & \ldots \\ \ldots & 5 & 2 & 1 & 3 & 5 & 2 & 1 & 3 & 5 & 2 & 1 & 3 & \ldots \\ \ldots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \end{array}$$

The classification of integral 2-friezes is a fascinating problem formulated in [4]. This problem remains open.

We present an inductive method of constructing a large number of closed positive integral 2-frieze patterns. Consider two closed positive integral 2-frieze patterns of widths $n-4$ and $k-4$, respectively, with coefficients

$$b_1,\ a_1,\ b_2,\ a_2, \ldots, b_n,\ a_n \qquad b'_1,\ a'_1,\ b'_2,\ a'_2, \ldots, b'_k,\ a'_k.$$

We call the *connected summation* the following way to glue them together and obtain a 2-frieze pattern of width $n+k-7$.

(1) Cut the first one at an arbitrary place, say between b_2 and a_2.
(2) Insert $2(k-3)$ integers: $a'_2,\ b'_3, \ldots, a'_{k-2},\ b'_{k-1}$.
(3) Replace the three left and the three right neighbouring entries by:

$$\begin{array}{lcl} (b_1,\ a_1,\ b_2) & \to & (b_1 + b'_1, \quad a_1 + a'_1 + b_2\, b'_1, \quad b_2 + b'_2) \\ (a_2,\ b_3,\ a_3) & \to & (a_2 + a'_{k-1}, \quad b_3 + b'_k + a_2\, a'_k, \quad a_3 + a'_k), \end{array} \tag{4}$$

leaving the other $2(n-3)$ entries $b_4, a_4, \ldots, b_n, a_n$ unchanged.

We prove the following statement.

• The connected summation yields a closed positive integral 2-frieze of width $n+k-7$.

The classical Coxeter-Conway integral frieze patterns were classified with the help of a similar stabilization procedure. In particular, a beautiful relation with triangulations of an n-gon (and thus with the Catalan numbers) was found making the result most attractive. The above procedure of connected summation is a step towards classification of integral 2-frieze patterns.

References

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