Local-global principles for étale cohomology

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1 Overview

Local-global principles have long been an important part of number theory, beginning with the Hasse principle for quadratic forms over number fields F. This classical form of the principle, which asserts that a quadratic form is isotropic over F if and only if it is isotropic over each completion, holds as well for one-variable function fields over finite fields, and thus for all global fields. Other versions of the local-global principle apply to rational points on varieties, and to elements of cohomology groups (or sets). Such a principle relates an object over a field F to the objects it induces over the completions of F at its discrete valuations.

In [3], we obtained results about quadratic forms and about Brauer groups over higher dimensional fields, such as function fields over the p-adics. These results were proven with the use of a new type of local-global principle for this higher dimensional situation, concerning homogeneous spaces for rational connected linear algebraic groups. That principle was obtained using the technique of patching over fields, a method developed in [2] that was an outgrowth of patching methods that had relied on formal or rigid geometry. In that approach, the completions being considered are not at discrete valuations, but rather correspond to "patches" on the curve.

Using [3], Colliot-Thélène, Parimala and Suresh proved analogous local-global principles in terms of completions with respect to discrete valuations, in the case of quadric hypersurfaces (for application to quadratic forms) and for certain cases in which the homogeneous space is a torsor [1]. The validity of such a local-global principle in general remains open, but [1] motivated work of the three of us on local-global principles for torsors. In particular, we recently showed that local-global principles in terms of patches hold for torsors even for disconnected rational groups provided that the reduction graph of the given curve is a tree.

Since torsors are classified by the first étale cohomology group, this raises the question of whether local-global principles can be shown for higher étale cohomology groups, provided that the linear algebraic group is abelian (so that H^i is defined for $i > 1$). This is the subject of our project at BIRS.

2 Goals

Given a field F and a linear algebraic group G over F, by a local-global principle for étale cohomology we will mean an assertion that the natural map

$$H^i(F,G) \to \prod_{\xi\in\Xi} H^i(F_\xi,G)$$

has trivial kernel, where $\{F_\xi \,|\, \xi \in \Xi\}$ is a collection of fields containing F that are obtained using completions. Here we assume that G is abelian if $i > 1$; while if $i = 1$, this is a local-global principle for torsors. If F is a global field, the natural collection of overfields is the set of completions of F at discrete valuations. For function fields F over a higher dimensional base K (e.g. $K = \mathbb{Q}_p$), it appears that this set of completions will in general be insufficient, though in special cases it may suffice (as shown in [1] for $i = 1$, in the context of quadratic forms, as mentioned above). Instead, in the case that K is a complete discretely valued field, we consider another choice of Ξ, viz. a set obtained by patches, as in [2] and [3] (also see below).

The desired principle would apply not only to p-adic fields, but more generally to complete discretely valued fields, in particular including n-local fields, paralleling results on homogeneous spaces in [3]. Applications of the desired principle could include local-global principles for structures classified by invariants in higher étale cohomology groups, e.g. for octonion and Albert algebras.

3 Scientific progress at BIRS

We worked to prove a local-global principle for étale cohomology in the case of commutative linear algebraic groups G over function fields F of curves over complete discretely valued fields K. Since we were relying on results in [3], we assumed that the group G is rational, in the sense that each connected component is a birational to some $\mathbb{A}^n_K$. This is not a very restrictive hypothesis.

As in [3], our local-global principle is framed in terms of patches. That is, we consider a regular projective model $\widehat{X}$ of F over the ring of integers of K, say with closed fiber X, and we take a non-empty set $\mathcal{P}$ of closed points of X that includes every point at which two components of X meet. For each $P \in \mathcal{P}$, we take F_P to be the fraction field of the complete local ring of $\widehat{X}$ at P. For each component U of the complement of $\mathcal{P}$ in X, we take F_U to be the fraction field of the completion of the subring ring of F consisting of rational functions on $\widehat{X}$ that are regular on U; and we let $\mathcal{U}$ be the set of such components U. For each branch $\wp$ of X at a point $P \in \mathcal{P}$ along a component $U \in \mathcal{U}$, we also consider the fraction field $F_\wp$ of the complete local ring at $\wp$; and we let $\mathcal{B}$ be the set of branches. (See [3] for more details about this set-up.)

In this framework, we prove in particular a local-global principle for the n-th cohomology of the group $\mu_\ell^{\otimes^{n-1}}$. Namely, we show that for each $n > 1$, the map

$$H^n_{\text{ét}}(F,\mu_\ell^{\otimes^{n-1}}) \to \prod_{\xi\in\mathcal{P}\cup\mathcal{U}} H^n_{\text{ét}}(F_\wp,\mu_\ell^{\otimes^{n-1}})$$

is injective. There are also similar results for other tensor powers.

In order to show this, we first prove a long exact Mayer-Vietoris sequence for étale cohomology, relating $H^i(F,G)$, $\prod_{\xi\in\Xi} H^i(F_\xi,G)$ and $\prod_{\wp\in\mathcal{P}} H^i(F_\wp,G)$. In fact, we show this holds when G is an arbitrary commutative rational linear algebraic group. The proof uses a new "auxiliary" Grothendieck topology, which we call the *patching-étale topology* and which combines the étale topology with the topology defined by patches. The argument also draws on results from [2].

The above local-global principle is then deduced from the Mayer-Vietoris sequence, together with the Bloch-Kato theorem recently proved by Voevodsky, Rost and Weibel (e.g. cf. [4]). Note here that $n \geq 2$.

In particular, there is a counterexample to the local-global principle for torsors (i.e., for H^1), with $G = \mathbb{Z}/2\mathbb{Z}$ and F the function field of a Tate curve (see [1] or [3]). As noted above, this can occur because the group is disconnected and the reduction graph is not simply connected. But the above construction gives another explanation for this failure of the local-global principle for H^1 in such an example. Namely, the cokernel of the map on the H^0 level is non-trivial, as can be seen from a combinatorial description in terms of the incidence geometry of the components of the closed fiber of a semistable model.

References

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