Borel measurable functionals on measure algebras

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1 Overview of the Field

Let A be a Banach algebra. Then there are two natural products, here denoted by $\Box$ and $\Diamond$, on the second dual A'' of A; they are the *Arens products*. For definitions and discussions of these products, see [3, 4, 5, 6], for example. We briefly recall the definitions. As usual, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\Phi \in A''$, define $\lambda \cdot \Phi \in A$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a,\, \lambda \cdot \Phi\rangle = \langle \Phi,\, a \cdot \lambda\rangle\,, \quad \langle a,\, \Phi \cdot \lambda\rangle = \langle \Phi,\, \lambda \cdot a\rangle \quad (a \in A)\,.$$

For $\Phi, \Psi \in A''$, define

$$\langle \Phi \,\Box\, \Psi,\, \lambda\rangle = \langle \Phi,\, \Psi \cdot \lambda\rangle \;\; (\lambda \in A')\,,$$

and similarly for $\Diamond$. Thus $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras each containing A as a closed subalgebra. The Banach algebra A is *Arens regular* if $\Box$ and $\Diamond$ coincide on A'', and A is *strongly Arens irregular* if $\Box$ and $\Diamond$ coincide only on A. A subspace X of A' is *left-introverted* if $\Phi \cdot \lambda \in X$ whenever $\Phi \in A''$ and $\lambda \in X$.

There has been a great deal of study of the two algebras $(A'', \Box)$ and $(A'', \Diamond)$, especially in the case where A is the group algebra $(L^1(G),\, \star)$ or the measure algebra $(M(G),\, \star)$ of a locally compact group G. For example, it has been known for a long time that $L^1(G)$ is strongly Arens irregular for each locally compact group G. On the other hand, each C^*-algebra is Arens regular.

Recently, the three participants have studied [5] the second dual of a semigroup algebra; here S is a semigroup, and our Banach algebra is $A = (\ell^{\,1}(S), \star)$. We see that the second dual A'' can be identified with the space $M(\beta S)$ of complex-valued, regular Borel measures on βS, the Stone–Čech compactification of S. In fact, $(\beta S, \Box)$ is itself a subsemigroup of $(M(\beta S), \Box)$. See [13] for background on $(\beta S, \Box)$.

Let G be a locally compact group. The algebra $M(G)$ has been much studied. This algebra is the multiplier algebra of the group algebra $L^1(G)$. Even in the case where G is the circle group $\mathbb{T}$, the Banach algebra $M(G)$ is very complicated; its character space is 'much larger' than the dual group $\mathbb{Z}$ of $\mathbb{T}$ [10].

Starting at a BIRS 'Research in Teams' in September, 2006, the three participants have been studying the algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$. Our work continued at other meetings, some at BIRS, and in 2007 and 2008 we established a number of other results that are contained in [6].

The first part of our memoir [6] studied the second dual space of $C_0(\Omega)$, where Ω is a locally compact space. This second dual is identified with $C(\widetilde{\Omega})$ for a certain hyper-Stonean space $\widetilde{\Omega}$; in particular, $\widetilde{\Omega}$ is compact and extremely disconnected. The space $C(\widetilde{\Omega})$ contains as a proper closed C^*-subalgebra the space $\kappa(B^b(\Omega))$, which is an isometric copy of $B^b(\Omega)$, the space of *bounded Borel* functions on Ω. The *Dixmier* *space*, $D(\Omega)$, of Ω is the quotient of $B^b(\Omega)$ by the ideal of functions that vanish on sets of measure 0. These latter spaces, and their relation to $C(\widetilde{\Omega})$, are themes of our work.

We then turned to the algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$ when G is a locally compact group. For example, [6] contains many results on the semigroup structure of $\widetilde{G}$, which is the natural analogue of βS in the non-discrete case. Indeed it is shown in [6, Chapter 8] that $(\widetilde{G}, \Box)$ is semigroup if and only if G is discrete, and in [6, Chapter 7] that the algebra $(M(\widetilde{G}), \Box)$ determines the locally compact group G.

Finally we mention some strong results [7, 8, 9] of M. Daws, who proved using techniques from the theory of Hopf–von Neumann algebras that the space $W := WAP(M(G))$ of weakly almost periodic functionals on the measure algebra $M(G)$ of a locally compact group G is a commutative C^*-algebra, so resolving a long-standing open question, and also that the character space Φ_W of W is a compact, semi-topological semigroup under the natural product associated with the Arens product on W'. This latter result shows that Φ_W is entirely analogous in the non-discrete case to the well-known weakly almost periodic compactification of a group. The corresponding results about the Fourier algebra $A(G)$ and the Fourier–Stieljes algebra $B(G)$ are apparently open.

The plan for the present research week had three aspects:

(1) Let $B^b(\Omega)$ be the C^*-algebra of bounded, Borel functions on a locally compact space Ω, as above, viewing the elements of this space as continuous linear functionals on the measure space $M(\Omega)$. The character spaces of $B^b(\Omega)$ and $D(\Omega)$ are denoted by Φ_b and Φ_D, respectively. The space Φ_b is a topological quotient of $\widetilde{\Omega}$ and is totally disconnected, but it is not a Stonean space; the latter space is a Stonean space. We would like to calculate the cardinalities of key subsets of these spaces; this would be analogous to work in [6]. We also wish to know for which locally compact spaces Ω the space Φ_D is hyper-Stonean.

(2) Let G be a locally compact group. The maximal introverted translation-invariant subspace X of $B^b(G)$ is well-defined, and then $(X', \Box)$ is a Banach algebra. In this case, X is a closed subspace of $B^b(G)$. It is not clear whether X is always a C^*-subalgebra of $B^b(G)$; if so, the character space Φ_X of X is a compact right-topological semigroup analogous to a Stone–Čech compactification. We planned to investigate the linear space X and the compact space Φ_X in various cases. We question whether W and $\kappa(B^b(G))$ are introverted subspaces of $C(\widetilde{G})$.

(3) We planned to investigate which functions on $\widetilde{G}$ belong to the above-mentioned space W. For example, we do not know whether or not the characteristic function of the character space Φ of $L^\infty(G)$ belongs to W. This question is related to questions about the products $\varphi \Box \psi$ for $\varphi, \psi \in \widetilde{G}$ that were studied in [6]; these questions seem to have independent interest, and are related to problems about the products of singular measures on G. We also wished to determine the topological structure of the character space Φ_W.

2 Recent Developments and Open Problems

We cite two recent results.

1) A dramatic recent result of Losert, Neufang, Pachl, and Steprans [15] establishes that $M(G)$ is strongly Arens irregular for each locally compact group G.

2) An impressive calculation of Budak, Işik, and Pym [2] discusses how many points are required to 'determine the topological centre' of the Banach algebras $L^1(G)$ for all locally compact groups and $\ell^1(S)$ for various semigroups S. These results are related to those in [5].

3 Presentation Highlights

Since this was a workshop for three people assembled for 'Research in teams', there were no formal presentations.

4 Scientific Progress Made

We made progress in three related areas.

1) The following question was not specifically mentioned in the proposal, but seems to be a key preliminary to our work. *Let X be a compact space such that $C(X)$ is isometrically isomorphic to the second dual space of a Banach space. Is it necessarily true that there is a locally compact space Ω such that $X = \widetilde{\Omega}$?*

In Banff, we established the following result. Here D represents the collection of extreme points of the unit ball of $N(X)$, the space of normal measures on X. As in [6], D can be identified with the set of isolated points of X.

Let X be a compact space such that $C(X)$ is isometrically isomorphic to the second dual space of a separable Banach space. Then either D is countable and $C(X)$ is isometrically isomorphic to $C(\beta\mathbb{N}) = c_0''$, or D has cardinality $\mathfrak{c}$ and $C(X)$ is isometrically isomorphic to $(C[0,1])''$.

However, after our stay in Banff, we discovered that this result is already contained in an old paper of H. Elton Lacey [14], with a different proof in [12]; we are grateful to F. Dashiell and T. Schlumprecht for a discussion of the literature. It seems that our proof is quite elementary, and avoids the appeal to some deep results in Banach space theory that Lacey makes. It is not clear whether our proof is sufficiently different from existing proofs to justify publication.

The question in the case when the space $C(X) = E''$ for a space E that is not separable remains open.

The analogous question in the isomorphic theory of Banach spaces was resolved in a similar way by Stegall [17]; for related work, see [11].

2) Let G be a locally compact group. We asked whether or not the space $B^b(G)$ is left-introverted for each locally compact group G. We proved that this is not the case, at least when the group G is infinite and metrizable. The easy proof uses a strong, old result of Rudin from [16]. We do not yet know the answer in the case where G is not metrizable.

3) Again, let G be a locally compact group. We considered the space W of weakly almost periodic functionals on $M(G)$. We proved that W can be identified with the space of functions $F \in C(\widetilde{G})$ such that $F(\varphi \,\Box\, \psi) = F(\varphi \diamond \psi)$ for all $\varphi, \psi \in \widetilde{G}$, a result also contained in the work of Daws; our proof of this result uses a theorem of Bourgain and Talagrand [1]. Using this result, we can easily see that $\kappa(B^b(G)) \cap W$ is a C^*-subalgebra of $C(\widetilde{G})$, a result that follows from Daws' work, give examples of various elements that belong to the space W, but not to $\kappa(B^b(G))$, and can show that W is 'large' in some sense; we hope to make this statement more precise.

5 Outcome of the Meeting

The three participants are continuing to work on the mentioned problems, and expect our work to lead to a publication in due course. In particular, we shall meet again in Leeds in December 2010 to attempt to make further progress

Lau will attend a conference on *Harmonic analysis* at the Chern Institute of Mathematics, Nankai University, Tianjin, China in June, 2011,

Dales, Lau, and Strauss will attend the *20th International Conference on Banach algebras in Waterloo, Canada*, 3-10 August, 2011, and will have discussions on their work there.

A proposal has been made to the *Fields Institute* in Toronto for a Thematic Program on *Banach algebras and harmonic analysis* in the second half of 2013; each of the three participants is an organiser or co-organiser of this programme. In particular, this proposal suggests a workshop on *Topological centres*; this topic is closely related to our present work. It is expected that many questions related to our work, and related matters, will be discussed during this semester.

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