Quantum error correction and operator algebras

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Quantum error correction

- Traditional view of QEC is in the Schrödinger picture for quantum time evolution (evolution of states):
- A quantum channel is a completely positive trace-preserving map
  \[ \mathcal{E} : \mathcal{B}_t(\mathcal{H}_1) \to \mathcal{B}_t(\mathcal{H}_2), \]
  with operators \( E_i \) (viewed as error operators in QEC) such that
  \[ \mathcal{E}(\rho) = \sum_i E_i \rho E_i^* \quad \forall \rho. \]
- Given such a channel, we look for a set \( S \) of states \( \rho \) (density operators) and a channel \( \mathcal{R} \) such that
  \[ (\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho \in S. \]
On the other hand, we can consider the Heisenberg picture which describes time evolution of *observables* via completely positive unital maps (dual maps)

\[ \mathcal{E}^\dagger : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1). \]

The “sharp” observables are given by self-adjoint operators \( X = \sum_k \lambda_k P_k \). The relationship between \( \mathcal{E} \), \( \mathcal{E}^\dagger \) is

\[ \text{Tr}(\mathcal{E}(\rho)P_k) = \text{Tr}(\rho \mathcal{E}^\dagger(P_k)), \]

which gives the probability that event \( k \) is measured after the evolution of the system with initial state \( \rho \).

Thus, the Schrödinger picture evolution \( \mathcal{E}_2 \circ \mathcal{E}_1 \) corresponds to \( (\mathcal{E}_2 \circ \mathcal{E}_1)^\dagger = \mathcal{E}_1^\dagger \circ \mathcal{E}_2^\dagger \) in the Heisenberg picture.
Thus we say $X = \sum_k \lambda_k P_k$ is a *correctable (sharp) observable* if there is a channel $\mathcal{R}$ such that

$$(\mathcal{R} \circ \mathcal{E})^\dagger(P_k) = P_k \quad \forall k.$$ 

This expression means that measuring $X$ before or after the action of $\mathcal{R} \circ \mathcal{E}$ would yield the same outcomes with the same probabilities no matter what the initial state was.
More generally, an observable is given by a *positive operator-valued measure* (POVM). In the case of a discrete measure, a POVM is specified by a family of positive operators $0 \leq A_k \leq I$, called *effects*, such that $\sum_k A_k = I$. If $A_k$ is a projection it is called a *sharp effect*.

Thus, we say an effect $A$ is *correctable for $\mathcal{E}$* if there is a channel $\mathcal{R}$ such that $(\mathcal{R} \circ \mathcal{E})^\dagger(A) = A$. And a POVM is correctable if all its effects are correctable.
**Question:** What are correctable effects for a given channel $E$?

**Investigate:** Suppose $P$ is a correctable sharp effect. Then there is an effect $0 \leq B \leq I$ such that $P = E^\dagger(B)$ ($B = R^\dagger(P)$ will do). Then we have

$$P^\perp E^\dagger(B)P^\perp = 0$$

$$\Rightarrow BE_i P^\perp = 0 \quad \forall i$$

$$\Rightarrow BE_i = BE_i P \quad \forall i$$

Similarly (since $E^\dagger$ is unital) we have $E_i P = BE_i P$, and hence

$$BE_i = E_i P \quad \forall i.$$ 

Thus $E_i^* E_j P = E_i^* BE_j = PE_i^* E_j$, and we see that if $P$ is correctable for $E$, then

$$[P, E_i^* E_j] = 0 \quad \forall i, j.$$
Correction of von Neumann algebras

**Theorem**

A sharp effect $P$ is correctable for the channel $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$ if and only if

$$[P, E_i^* E_j] = 0 \quad \text{for all } i, j.$$ 

For sufficiency, an explicit recovery operation $\mathcal{R}$ can be constructed and (an important point for practical purposes) the same recovery operation works for any channel $\mathcal{E}'$ with operators $E'_i$ that belong to the span of the $E_i$. (In many situations, the precise $E_i$ may not be known, but often the operator system they generate is.)
The commutant of the operators $E_i^* E_j$ is a von Neumann algebra, and hence the effects it contains are the closed convex hull of its projections. Since all projections in this algebra are corrected by $\mathcal{R}$, so are all the effects it contains, and thus we have the following:

**Corollary**

The set of effects spanning the von Neumann algebra

$$\mathcal{A} = \{ A \in \mathcal{B}(\mathcal{H}_1) : [A, E_i^* E_j] = 0 \text{ for all } i, j \}$$

are all corrected by the channel $\mathcal{R}$ constructed in the theorem above. Moreover, this algebra contains all the correctable sharp effects for $\mathcal{E}$. 
(Shor, Steane, Knill-Laflamme, Bennett-DiVincenzo-Smolin-Wootters, Gottesman, etc) A code is given by a subspace \( \mathcal{H}_0 \subseteq \mathcal{H}_1 \), \( \dim \mathcal{H}_0 < \infty \). Then \( \mathcal{H}_0 \) is correctable for \( \mathcal{E} \) if \( \exists \mathcal{R} \) such that \( \mathcal{R}(\mathcal{E}(\rho)) = \rho \ \forall \rho \) supported on \( \mathcal{H}_0 \).
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- In other words, $\mathcal{R}'(\mathcal{E}(V\rho V^*)) = \rho, \ \forall \rho \in \mathcal{B}_t(\mathcal{H}_0)$, where $\mathcal{R}'(\rho) = V^*\mathcal{R}(\rho)V$ and $V : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1$. 

- Thus $\mathcal{H}_0$ is correctable for $\mathcal{E}$ if $\mathcal{B}_t(\mathcal{H}_0)$ is correctable for $\mathcal{E}$ (in our algebraic sense) if $\{V^*\mathcal{E}(i)\mathcal{E}(j)V\} = \mathcal{B}_t(\mathcal{H}_0)$; i.e., $\exists \lambda_{ij}$ such that $V^*\mathcal{E}(i)\mathcal{E}(j)V = \lambda_{ij}I$, which is exactly the Knill-Laflamme condition for QEC.
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(K.-Laflamme-Poulin, Klappenecker, Sarvepalli, Aly, Nielsen, etc) A *subsystem code* is defined through a subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$ with a particular subsystem decomposition $\mathcal{H}_0 = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $V : \mathcal{H}_0 \hookrightarrow \mathcal{H}_1$. Then $\mathcal{H}_A$ is *correctable* for $\mathcal{E}$ if $\exists R$ such that $\forall \rho \in \mathcal{B}_t(\mathcal{H}_A), \forall \tau \in \mathcal{B}_t(\mathcal{H}_B), \exists \tau' \in \mathcal{B}_t(\mathcal{H}_B)$ for which

$$R(\mathcal{E}(V(\rho \otimes \tau)V^*)) = \rho \otimes \tau'.$$
Subsystem codes

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One can show that this is equivalent, in our framework, to the case where the correctable algebra $\mathcal{A}$ is any type I finite-dimensional factor

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes I_B.$$
Type I infinite dimensional example

- Let $\mathcal{H}_0 \subseteq \mathcal{H}_1$ with $\dim \mathcal{H}_0 = \infty = \dim \mathcal{H}_0^\perp$. Let $\{P_i\}_{i=0}^\infty$ be projections onto mutually orthogonal subspaces $\{\mathcal{H}_i\}_{i=0}^\infty$ and partial isometries $V_i$ such that $V_i^* V_i = P_0$ and $V_i V_i^* = P_i$. 

Suppose we have probabilities $p_i \geq 0; \sum p_i = 1$. Then we have a channel, $E(\rho) = \sum p_i V_i \rho V_i^* : \mathcal{B}(\mathcal{H}_0) \rightarrow \mathcal{B}(\mathcal{H}_1)$. Then $V_i^* V_j = \delta_{ij} I_{\mathcal{H}_0}$, and so $\{V_i^* V_j\}' = \mathcal{B}(\mathcal{H}_0)$. Thus, $\mathcal{B}(\mathcal{H}_0)$ is a type I infinite dimensional code, the natural generalization of finite-dimensional codes. In fact this is the prototypical example in continuous variable QEC (Braunstein, Lloyd-Slotine).
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Type II example: irrational rotation algebra

Consider two unitaries $U, V$ on infinite dimensional space such that $UV = e^{2\pi i \theta} VU$ with $\theta$ irrational. We can take $U = e^{ia\hat{x}}$, $V = e^{ib\hat{p}}$, where $\hat{x}, \hat{p}$ are position and momentum operators on $L^2(\mathbb{R})$ satisfying the canonical commutation relations $[\hat{x}, \hat{p}] = i\mathbf{1}$. 
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We can consider a noise model with errors $I$, $U$, $V$ as the possible errors. Thus, to find the correctable algebra we compute the commutant of $\{U, V\}$. 
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We can consider a noise model with errors $I, U, V$ as the possible errors. Thus, to find the correctable algebra we compute the commutant of $\{U, V\}$.

But, in the concrete case above, this commutant is generated by unitaries $U' = e^{i(a/\theta)\hat{x}}$ and $V' = e^{i(b/\theta)\hat{p}}$, and is a factor of type II (Faddeev), and hence we find a naturally arising type II correctable algebra.

