Factorization and dilation problems for completely positive maps on von Neumann algebras

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Let (X, Σ, μ) be a probability space. An operator T on $L_\infty(X, \Sigma, \mu)$ is called <u>*Markov*</u> if T is a positive contraction, $T(1) = 1$ ($T^*(1) = 1$) and $\int Tf d\mu = \int f d\mu$, for all $f \in L_\infty(X, \Sigma, \mu)$. Then T extends to a positive contraction on $L_p(X, \Sigma, \mu)$, for all $p \geq 1$.

Theorem (Rota, 1961):

(a) $(T^n(T^*)^n)_{n\geq 1}$ admits a dilation in terms of a martingale.

(b) $(T^n(T^*)^n)(f)$ converges a.s., for all $f \in L_p(X, \Sigma, \mu)$, $p \geq 1$.

Idea of proof: On some probability space $(\Omega, \mathcal{F}, \nu)$, construct a Markov process associated with T. Imagine a particle located at $x_0 \in X$ at time $t = 0$, where $\mathrm{Prob}(x_0 \in A_0) = \mu(A_0)$. At $t = 1$ the particle jumps to a new location $x_1 \in X$, with $\mathrm{Prob}(x_1 \in A_1) = T(\chi_{A_1})(x_0)$. From x_1, the particle jumps at $t = 2$ to $x_2 \in X$, with probability that only depends on x_1, not on x_0. And so on.

<u>Model</u>: $\Omega := X^{\mathbb{N}}$ path (trajectory) space, $\mathcal{F}$ = product σ-algebra on Ω, ν = Markov measure on $\mathcal{F}$. For $n \geq 0$, X_n is a random variable given by $(x_n)_{n\geq 0} \in \Omega \mapsto x_n \in X$. Time evolution β is the shift operator on Ω. Define $\hat{\mathcal{F}}_0 := \{A_0 \times X \times X \times \ldots : A_0 \in \Sigma\}$ and for $n \geq 1$, $\mathcal{F}_n := \{X \times \ldots \times X \times S : S \in \mathcal{F}\}$. Clearly

$$\ldots \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \subseteq \ldots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 := \mathcal{F}.$$

Let $\iota : \Omega \to X$ be defined by $\iota(x_0, x_1, \ldots) := x_0$. Then ι extends to an isomorphism $\iota : L_p(\Omega, \hat{\mathcal{F}}_0, \nu) \to L_p(X, \Sigma, \mu)$, and we get

$$(\iota^* \circ (T^n(T^*)^n) \circ \iota)(f) = \hat{\mathbb{E}}(\mathbb{E}_n(f)), \quad n \geq 1$$

for all $f \in L_p(\Omega, \hat{\mathcal{F}}_0, \nu)$, where $\mathbb{E}_n := \mathbb{E}(\,\cdot\,|\mathcal{F}_n)$, $\hat{\mathbb{E}} := \mathbb{E}(\,\cdot\,|\hat{\mathcal{F}}_0)$.

Definition (Anantharaman-Delaroche, 2004):

Let (M, ϕ) and (N, ψ) be von Neumann algebras with normal, faithful states ϕ, ψ. A linear map $T\colon M \to N$ is called (ϕ, ψ)-*Markov map* if

- T is completely positive
- $T(1_M) = 1_N$
- $\psi \circ T = \phi$
- $T \circ \sigma_t^\phi = \sigma_t^\psi \circ T\,, \quad t \in \mathbb{R}\,.$

If $(M, \phi) = (N, \psi)$, then T is called a ϕ-*Markov* map on M.

Remark: A (ϕ, ψ)-*Markov map* $T\colon M \to N$ has an *adjoint* (ψ, ϕ)-Markov map $T^*\colon N \to M$, uniquely determined by

$$\psi(yT(x)) = \phi(T^*(y)x)\,, \quad x \in M\,, y \in N\,.$$

A noncommutative Kolmogorov-Daniell construction

Given a ϕ-Markov map T on (M, ϕ), find a von Neumann algebra P with a n. f. state χ, a time evolution endomorphism $\beta : P \to P$ and a normal, injective $*$-homomorphism $J_0 : M \hookrightarrow P$ such that

$$\beta \text{ is } \chi - \text{Markov} \;\;, \quad J_0 \text{ is } (\phi, \chi) - \text{Markov} \tag{1}$$

and, if $\mathbb{E}_{n]}$ and $\mathbb{E}_{[n}$ are the conditional expectations on $P_{n]}$ and $P_{[n}$, respectively, where $P_{n]} := \bigvee_{k \le n} J_k(M)$, $P_{[n} := \bigvee_{k \ge n} J_k(M)$ and $J_k := \beta^k \circ J_0$, then $(P, \beta, J_0, (\mathbb{E}_{n]})_{n \ge 0})$ is a quantum Markov process satisfying for all $n \ge 0$

$$\begin{aligned} \mathbb{E}_{n]} \circ J_q &= J_n \circ T^{q-n}\,, \quad q \ge n && (2) \\ \mathbb{E}_{[n} \circ J_0 &= J_n \circ (T^*)^n\,. && (3) \end{aligned}$$

Note: Such a construction for a unital completely positive map on a unital C^*-algebra M satisfying (2) has been carried out by Sauvageot (1986). However, condition (1) does not appear to be satisfied. Also, here we insist on (3) being satisfied, as well, since then we obtain

$$J_0 \circ T^n \circ (T^*)^n = \mathbb{E}_{0]} \circ \mathbb{E}_{[n} \circ J_0 \,.$$

Further, since $J_0^* = J_0^{-1} \circ \mathbb{E}_{0]}$, this implies that

$$T^n \circ (T^*)^n = J_0^* \circ \mathbb{E}_{[n} \circ J_0 \,, \quad n \geq 1 \,.$$

A similar reasoning as in the proof of the classical theorem of Rota (using noncommmutative versions of martingale inequalities) yields convergence of $(T^n \circ (T^*)^n)(x)$ "a.s." $x \in L_p(M\,, \phi)$.

C. Anantharaman-Delaroche (2004) proved that a noncommutative Kolmogorov-Daniell construction satisfying all conditions $(1)-(3)$ is possible if and only if the ϕ-Markov map $T : M \to M$ is *factorizable.*

Definition (Anantharaman-Delaroche, 2004):
A (ϕ, ψ)-Markov map $T\colon M \to N$ is called *factorizable* if there exists a von Neumann algebra P with a normal, faithful state χ and injective $*$-homomorphisms $\alpha\colon M \to P$ and $\beta\colon N \to P$ such that

$$\alpha \text{ is } (\phi, \chi) - \text{Markov}, \;\; \beta \text{ is } (\psi, \chi) - \text{ Markov and } T = \beta^* \circ \alpha.$$

$$\begin{tikzcd} M \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & N \\ & P \arrow[ur, "\beta^*"'] & \end{tikzcd}$$

Note that $\beta^* = \beta^{-1} \circ \mathbb{E}_{\beta(N)}$.

Remark: By (2), $\mathbb{E}_{0]} \circ J_1 = J_0 \circ T$, which implies that $T = J_0^* \circ J_1$.

Remark: The set of factorizable ϕ-Markov maps on M is convex, and it is closed under composition and taking adjoints.

It can be shown that every Markov map between abelian von Neumann algebras is factorizable.

Problem (Anantharaman-Delaroche, 2004):
Is every Markov map factorizable?

Markov maps on $(M_n(\mathbb{C})\,,\tau_n))$

Here τ_n is the normalized trace on $M_n(\mathbb{C})$.

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,,\tau_n)$-Markov map, i.e., T is completely positive, $T(1) = 1$ and $\tau_n \circ T = \tau_n$. By a result of Choi (1973), T is completely positive if and only if

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})$$

where $a_1\,, \dots\,, a_d \in M_n(\mathbb{C})$ can be chosen to be linearly independent. Then, the condition $T(1) = 1$ is equivalent to $\sum_{i=1}^{d} a_i^* a_i = 1$, while the condition $\tau_n \circ T = \tau_n$ is equivalent to $\sum_{i=1}^{d} a_i a_i^* = 1$.

Result (Kümmerer, 1983): Every $(M_2(\mathbb{C})\,,\tau_2)$-Markov map lies in $\mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$, hence it is factorizable.

Theorem 1 (Haagerup-M.):

Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,, \tau_n)$-Markov map, written in the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,, \tag{4}$$

where $a_1\,, \ldots\,, a_d \in M_n(\mathbb{C})$ are linearly independent.

The following are equivalent:

1) T is factorizable

2) There exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and a unitary $u \in M_n(N)$ such that

$$Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1)u)\,, \quad x \in M_n(\mathbb{C})\,.$$

3) There exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and $v_1\,, \ldots\,, v_d \in N$ such that $u\colon = \sum_{i=1}^{d} a_i \otimes v_i$ is a unitary operator in $M_n(\mathbb{C}) \otimes N$ and

$$\tau_N(v_i^* v_j) = \delta_{ij}\,, \quad 1 \leq i, j \leq d\,.$$

Corollary 1:

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,, \tau_n)$-Markov map of the form (4), where $a_1\,, \ldots\,, a_d \in M_n(\mathbb{C})$. If $d \geq 2$ and the set

$$\{a_i^* a_j : 1 \leq i, j \leq d\}$$

is linearly independent, then T is not factorizable.

Proof of Corollary 1:

Assume that T is factorizable. By Theorem 1, there exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and $v_1, \ldots, v_d \in N$ such that $u := \sum_{i=1}^d a_i \otimes v_i$ is unitary. Since $\sum_{i=1}^d a_i^* a_i = 1$, it follows that

$$\sum_{i,j=1}^{d} a_i^* a_j \otimes (v_i^* v_j - \delta_{ij} 1_N) = u^* u - \left(\sum_{i=1}^{d} a_i^* a_i \right) \otimes 1_N = 0 .$$

By the linear independence of the set $\{a_i^* a_j : 1 \leq i, j \leq d\}$,

$$v_i^* v_j - \delta_{ij} 1_N = 0 , \quad 1 \leq i, j \leq d .$$

Since $d \geq 2$, it follows in particular that

$$v_1^* v_1 = v_2^* v_2 = 1 , \quad v_1^* v_2 = 0 .$$

Since N is finite, v_1 and v_2 are unitary operators, which gives rise to a contradiction. This proves that T is <u>not</u> factorizable.

Example 1 (Haagerup-M.): Set

$$a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $\sum_{i=1}^{3} a_i^* a_i = \sum_{i=1}^{3} a_i a_i^* = 1$. Hence the operator T defined by

$$Tx\colon = \sum_{i=1}^{3} a_i^* x a_i\,, \quad x \in M_3(\mathbb{C})$$

is a $(M_3(\mathbb{C})\,, \tau_3)$-Markov map. The set

$$\{a_i^* a_j : 1 \leq i, j \leq 3\}$$

is linearly independent. Hence, by Corollary 1, T is <u>not</u> factorizable.

Remark: Let $\mathcal{FM}(M_n(\mathbb{C}))$ be the set of factorizable $(M_n(\mathbb{C})\,, \tau_n)$-Markov maps. Since all automorphisms of $M_n(\mathbb{C})$ are inner,

$$\operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}))) \subseteq \mathcal{FM}(M_n(\mathbb{C})\,, \tau_n)\,. \tag{5}$$

Question: Is the inclusion (5) strict?

Proposition 1 (Haagerup-M.):

Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C}), \tau_n)$-Markov map written in the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i , \quad x \in M_n(\mathbb{C}) ,$$

where $a_1 , \ldots , a_d \in M_n(\mathbb{C})$ are linearly independent. Then the following conditions are equivalent:

(a) $T \in \text{conv}(\text{Aut}(M_n(\mathbb{C}))$.

(b) T satisfies condition 2) of Theorem 1 with N abelian.

(c) T satisfies condition 3) of Theorem 1 with N abelian.

Corollary 2:

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C}), \tau_n)$-Markov map of the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i , \quad x \in M_n(\mathbb{C}) ,$$

where $a_1 , \ldots , a_d \in M_n(\mathbb{C})$ are self-adjoint, $\sum_{i=1}^{d} a_i^2 = 1$ and satisfy $a_i a_j = a_j a_i$, $1 \leq i, j \leq d$. Then the following hold:

(a) T is factorizable.

(b) If $d \geq 3$ and the set $\{a_i a_j : 1 \leq i \leq j \leq d\}$ is linearly independent, then $T \notin \text{conv}(\text{Aut}(M_n(\mathbb{C})))$.

Schur multipliers

If $B = (b_{ij})_{i,j=1}^{n}$ is a positive semi-definite matrix, then the map

$$T_B : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$
$$T_B(x) \colon = (b_{ij} x_{ij})_{1 \le i,j \le n} \,, \quad x = (x_{ij})_{i,j=1}^{n} \in M_n(\mathbb{C})$$

is called the Schur multiplier associated to the matrix B. Note that T_B is completely positive. If, moreover,

$$b_{11} = b_{22} = \ldots = b_{nn} = 1 \,,$$

then $T_B(1) = 1$ and $\tau_n \circ T_B = \tau_n$. Hence T_B is $(M_n(\mathbb{C}) \,, \tau_n)$-Markov. There exist lin. independent $n \times n$ diagonal matrices $a_1 \,, \ldots a_d$ so that

$$T_B(x) = \sum_{i=1}^{d} a_i^* x a_i \,, \quad x \in M_n(\mathbb{C}).$$

If the entries of B are real, then $a_i^* = a_i$ and $\sum_{i=1}^{d} a_i^2 = 1$. By Corollary 2, T_B is factorizable. (This is a result of Ricard, 2007.)

Example 2 (Haagerup-M.): Let $\beta = 1/\sqrt{5}$ and set

$$B \colon = \begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & \beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix} .$$

Then T_B satisfies the hypotheses of Corollary 2, hence T_B is a factorizable Markov map on $M_6(\mathbb{C})$, but $T_B \notin \mathrm{conv}(\mathrm{Aut}(M_6(\mathbb{C})))$.

Example 3 (Haagerup-M.): Let $0 < s < 1$ and set

$$B(s)\colon = \begin{pmatrix} 1 & \sqrt{s} & \sqrt{s} & \sqrt{s} \\ \sqrt{s} & s & s & s \\ \sqrt{s} & s & s & s \\ \sqrt{s} & s & s & s \end{pmatrix} + (1-s)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \omega & \overline{\omega} \\ 0 & \overline{\omega} & 1 & \omega \\ 0 & \omega & \overline{\omega} & 1 \end{pmatrix},$$

where $\omega = e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$ and $\overline{\omega}$ is its complex conjugate. Then $B(s)$ is positive semi-definite matrix of rank 2 (cf. Christensen and Vesterstrøm). Moreover,

$$T_{B(s)}(x) = \sum_{i=1}^{2} a_i(s)^* x a_i(s)\,, \quad x \in M_4(\mathbb{C})\,,$$

where $a_1(s) = \mathrm{diag}(1\,, \sqrt{s}\,, \sqrt{s}\,, \sqrt{s})\,, a_2(s) = \sqrt{1-s}\,\mathrm{diag}(0\,,1\,,\omega\,,\overline{\omega})$. The set $\{a_i^* a_j : i,j = 1,2\}$ is linearly independent, hence $T_{B(s)}$ is <u>not</u> factorizable, by Corollary 1.

Furthermore, set

$$L = \frac{dB(s)}{ds}_{|s=1} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 3-i\sqrt{3} & 3+i\sqrt{3} \\ 1 & 3+i\sqrt{3} & 0 & 3-i\sqrt{3} \\ 1 & 3-i\sqrt{3} & 3+i\sqrt{3} & 0 \end{pmatrix}.$$

Then

$$N(t) := \left(e^{-L_{ij}t}\right)_{1\le i,j\le 4}\,, \quad t \ge 0$$

is a semigroup of positive definite matrices having 1 on the diagonal. Hence

$$T(t)\colon = T_{N(t)}\,, \quad t \ge 0$$

is a semigroup of Schur multipliers which are $(M_4(\mathbb{C}), \tau_4)$-Markov maps. For $t > 0$, $N(t)$ has rank 4, and therefore Corollary 1 cannot be applied. Using a different method we can obtain from Theorem 1 that there exists $t_0 > 0$ such that $T(t)$ is <u>not</u> factorizable, for any $0 < t < t_0$.

Remarks:

By a result of Kümmerer and Maassen (1987), it follows that if

$$T(t) := e^{-Lt}, \quad t \geq 0$$

is a one-parameter semigroup of $(M_n(\mathbb{C}), \tau_n)$-Markov maps satisfying

$$T(t)^* = T(t), \quad t \geq 0,$$

then $T(t) \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$, for all $t \geq 0$. In particular,

$$T(t) \text{ is factorizable}, \quad t \geq 0.$$

In very recent work, Junge, Ricard and Shlyakhtenko have generalized the result of Kümmerer and Maassen, by showing that if $(T_t)_{t \geq 0}$ is a strongly continuous one-parameter semigroup of (M, τ_M)-Markov maps (with $T_0 = \mathrm{id}_M$) on an <u>arbitrary</u> finite von Neumann algebra M with a faithful, normal tracial state τ_M, satisfying

$$T(t)^* = T(t), \quad t \geq 0,$$

<u>then</u> $T(t)$ is factorizable, for all $t \geq 0$. This result has been obtained independently (by different methods) by Yoann Dabrowski.

Further related results

Dykema and Jushenko (2009) considered the following sets for $n \geq 1$:

$$\mathcal{F}_n := \overline{\bigcup_{k\geq 1} \{B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_k(u_i u_j^*)\,, u_1, \ldots, u_n \in \mathcal{U}(M_k(\mathbb{C}))\}}$$

$$\mathcal{G}_n := \Big\{B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_M(u_i u_j^*)\,, u_1, \ldots, u_n \in \mathcal{U}(M)\,, \text{ for some } (M\,, \tau_M) \text{ von Neumann algebra with n.f. tracial state } \tau_M\Big\}$$

By results of Kirchberg (1993), Connes' embedding conjecture holds if and only if

$$\mathcal{F}_n = \mathcal{G}_n\,, \quad \text{for all } n \geq 1\,.$$

Consider further the set

$$\Theta := \{B = (b_{ij}) \in M_n(\mathbb{C}) : B \text{ positive semidefinite}\,,\ b_{ii} = 1\,, 1 \leq i \leq n\}\,.$$

It is clear that

$$\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n\,, \quad n \geq 1\,.$$

Question: Is it true that $\mathcal{F}_n = \Theta_n$, for all $n \geq 1$?

Dykema and Jushenko proved that the answer is NO if $n \geq 4$. More precisely, in the case $n = 4$, they proved that $\mathcal{G}_4$ has no extreme points of rank 2, while there are extreme points of rank 2 in Θ_4. Hence $\mathcal{G}_4 \neq \Theta_4$.

Connection with factorizability

As a consequence of Theorem 1,

$$\mathcal{G}_n = \{B \in \Theta_n : T_B \text{ is factorizable}\}\,, \quad n \geq 1\,.$$

On the connection between Anantharaman-Delaroche's work and Kümmerer's work (Communicated by Claus Koestler, May 2008)

Definition (Kümmerer, JFA 1985):

Let (M, ϕ) be a von Neumann algebra with a normal, faithful state ϕ. A ϕ-Markov map $T\colon M \to M$ has a *dilation* if there exists

- (N, ψ) von Neumann algebra with a normal faithful state ψ
- $i\colon M \to N$ (ϕ, ψ)-Markov injective $*$-homomorphism
- $\alpha \in \mathrm{Aut}(N, \psi)$

such that $T^n = i^* \circ \alpha^n \circ i$, for all $n \geq 1$.

$$\begin{tikzcd} N \arrow[r, "\alpha^n"] & N \arrow[d, "\iota^*"] \\ M \arrow[u, "\iota"] \arrow[r, "T^n"] & M \end{tikzcd}$$

Combining results from Anantharaman-Delaroche (2004) with results from Kümmerer's unpublished Habilitationsschrift (1986), one gets the following

Theorem (Anantharaman-Delaroche, 2004 + Kümmerer, 1986):

Let $T\colon M \to M$ be a ϕ-Markov map. The following are equivalent:

(1) T is factorizable.

(2) T has a dilation.

Proof:

The implication (2) $\Rightarrow$ (1) is elementary, because if (2) holds, then

$$T = i^* \circ (\alpha \circ i)\,,$$

where both $\alpha \circ i$ and i are (ϕ, ψ)-Markov injective $*$-homomorphisms of M into N.

We now show that (1) $\Rightarrow$ (2).

Anantharaman-Delaroche (2004) proved that if T is factorizable, then there exists (N, ψ) a von Neumann algebra N with a normal, faithful state ψ, an injective $*$-homomorphism $i\colon M \to N$ which is (ϕ, ψ)-Markov, and a (ψ, ψ)-Markov injective $*$-homomorphism $\beta : N \to N$ such that

$$T^n = i^* \circ \beta^n \circ i\,, \quad n \geq 1\,.$$

However, using a result of Kümmerer from his Habilitationsschrift (1986), one can extend β to a $\tilde{\psi}$-preserving automorphism α on a larger von Neumann algebra $\tilde{N}$, namely

$$(\tilde{N}\,, \tilde{\psi}) = \text{inductive limit of } \; (N\,, \psi) \xrightarrow{\beta} (N\,, \psi) \xrightarrow{\beta} \ldots$$

such that $\tilde{i} : M \to N \subseteq \tilde{N}$ becomes a $(\phi, \tilde{\psi})$-Markov injective $*$-homomorphism and

$$T^n = (\tilde{i})^* \circ \alpha^n \circ \tilde{i}\,, \quad n \geq 1.$$

Hence T has a dilation.

In his Habilitationsschrift (1986), Kümmerer constructs examples of τ_n-Markov maps on $M_n(\mathbb{C})$ having <u>no</u> dilation. His examples are similar to our examples 1 and 3, but he does not consider the one-parameter semigroup case.

Proposition (Kümmerer, 1986):

(1) Let $T\colon M_3(\mathbb{C}) \to M_3(\mathbb{C})$ be the τ_3-Markov map

$$Tx\colon = \sum_{i=1}^{3} a_i^* x a_i\,, \quad x \in M_3(\mathbb{C})$$

where

$$a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then T has <u>no</u> dilation.

(2) Let $n \geq 4$ and $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the τ_n-Markov map

$$Tx\colon = \sum_{i=1}^{2} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})$$

where

$$a_1 = \operatorname{diag}\left(1\,, \frac{1}{\sqrt{2}}\,, \frac{1}{\sqrt{2}}\,, 0\,, \ldots\,, 0\right), a_2 = \operatorname{diag}\left(0\,, \frac{1}{\sqrt{2}}\,, \frac{i}{\sqrt{2}}\,, 1\,, \ldots\,, 1\right).$$

Then T is a Schur multiplier which has <u>no</u> dilation.

The noncommutative Rota dilation property

Definition (Junge, Le Merdy, Xu, 2006):

Let (M, τ) be a (finite) von Neumann algebra with a normal, faithful tracial state τ. A τ-Markov map $T\colon M \to M$ has the *Rota dilation property* if there exists

- N von Neumann algebra with a normal faithful tracial state τ_N
- $(N_n)_{n\geq 1}$ decreasing sequence of von Neumann subalgebras of N
- $i\colon M \hookrightarrow N$ trace-preserving embedding

such that for all $n \geq 1$, $T^n = i^* \circ E_{N_n} \circ i$, where E_{N_n} is the trace-preserving conditional expectation of N onto N_n.

$$\begin{tikzcd} M \arrow[rr, "T^n"] \arrow[dr, "\iota"'] & & M \\ & N \xrightarrow{\mathbb{E}_{N_n}} N_n \arrow[ur, "\iota^*"'] & \end{tikzcd}$$

Remark: If $T\colon M \to M$ has the Rota dilation property, then T is positive (as an operator on $L_2(M, \tau)$) and it is factorizable, since

$$T = i^* \circ E_{N_1} \circ i = (E_{N_1} \circ i)^* \circ (E_{N_1} \circ i)\,.$$

The following is an example of a factorizable trace-preserving Markov map on $M_2(\mathbb{C})$ which does not have the Rota dilation property. Set

$$T\left(x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix}, \quad x \in M_2(\mathbb{C})\,.$$

Then $T \in \mathrm{Aut}(M_2(\mathbb{C}))$, and hence it is factorizable, but T is not positive (as an operator on $L_2(M_2(\mathbb{C}), \tau_2)$).

Theorem (Anantharaman-Delaroche, 2004):

If $T\colon M \to M$ is a factorizable Markov map and $T^* = T$, then T^2 has the Rota dilation property.

Remark: If M is abelian, then any Markov map T on M is factorizable. If, moreover, $T = T^*$, then the Rota dilation for T^2 in above theorem can be chosen such that N is abelian. This is the classical Rota dilation theorem.

Theorem 2 (Haagerup-M.):

For some large $n \in \mathbb{N}$, there exists a Markov map T on $(M_n(\mathbb{C}), \tau_n)$ such that $T^* = T$, but T^2 is not factorizable. In particular, T^2 does not have the Rota dilation property.

Remark: By the result of Junge, Ricard, Shlyakhtenko/ Dabrowski, if $(T_t)_{t\geq 0}$ is a strongly cont. semigr. of self-adj. (M, τ_M)-Markov maps on (M, τ_M), then $T_t = (T_{t/2})^2$ has Rota dilation property for all $t \geq 0$.

Theorem 3 (Haagerup-M.):

Let M be a finite von Neumann algebra with normal faithful tracial state τ, and let $S\colon M \to M$ be a τ-Markov map on M. TFAE:

(1) S has the Rota dilation property

(2) S has a Rota dilation of order 1

(3) $S = T^*T$, where $T\colon M \to N$ is a factorizable (τ, τ_N)-Markov map, for some vN alg. N with a normal faithful tracial state τ_N.

Key Lemma in the proof of **Theorem 2**:

Let $n, d \in \mathbb{N}$ with $d \geq 5$ and set

$$Tx\colon = \sum_{i=1}^{n} a_i^* x a_i \,, \quad x \in M_n(\mathbb{C}) \,,$$

where $a_1 \,, \dots \,, a_d \in M_n(\mathbb{C})$ satisfy:

(1) $a_i = a_i^* \,, \, 1 \leq i \leq d$

(2) $\sum_{i=1}^{d} a_i^2 = 1$

(3) $a_i^2 a_j = a_j a_i^2 \,, \, 1 \leq i, j \leq d$

(4) $A\colon = \{a_i a_j : 1 \leq i, j \leq d\}$ is linearly independent

(5) $B\colon = \cup_{i=1}^{6} B_i$ is linearly independent, where

$B_1\colon = \{a_i a_j a_k a_l : i \neq j \neq k \neq l\} \,, B_2\colon = \{a_i a_j a_k^2 : i \neq j \neq k \neq k\} \,,$
$B_3\colon = \{a_i^3 a_j : i \neq j\} \,, B_4\colon = \{a_i a_j^3 : i \neq j\} \,, B_5\colon = \{a_i^2 a_j^2 : i < j\} \,,$
$B_6\colon = \{a_i^4 : 1 \leq i \leq d\} \,.$

Then T is a $(M_n(\mathbb{C}) \,, \tau_n)$-Markov map, but T^2 is not factorizable. In particular, T^2 does <u>not</u> have the Rota dilation property.

Remark: Operators $a_1 \,, \dots \,, a_d$ satisfying conditions $(1) - (5)$ can be realized in $L_\infty(S^{d-1}) \bar{\otimes} L(\mathbb{Z}_2 * \ldots * \mathbb{Z}_2)$ as

$$a_i = b_i \otimes u_i \,, \quad 1 \leq i \leq d$$

where $b_1 \,, \dots \,, b_d$ are the coordinate functions on S^{d-1} (the unit sphere in $\mathbb{R}^d$) and $u_1 \,, \dots \,, u_d \in L(\mathbb{Z}_2 * \ldots * \mathbb{Z}_2)$ are the self-adjoint unitaries corresponding to the generators $g_1 \,, \dots \,, g_d$ of $\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$. Using the fact that this group is residually finite, it is possible to get examples of $n \times n$ matrices $a_1 \,, \dots \,, a_d$ satisfying $(1) - (5)$ for large values of n.

Further results

Recall the noncommutative little Grothendieck inequality (cb-version):

Theorem (Pisier–Shlyakhtenko, 2002, Haagerup-M, 2008):
Let A be a C*-algebra. If $T : A \to OH(I)$ is a completely bounded linear map, then there exist states f_1, f_2 on A such that

$$\|T(x)\| \leq \sqrt{2}\|T\|_{\mathrm{cb}} f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}, \quad x \in A\,.$$

Problem: What is the best constant C_0 in the inequality

$$\|T(x)\| \leq C\|T\|_{\mathrm{cb}} f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}, \quad x \in A\,. \tag{6}$$

for all choices of A and T.

Note: $1 \leq C_0 \leq \sqrt{2}$.

Theorem 4 (Haagerup-M): $C_0 > 1$.

More precisely,

(1) There exists $T\colon M_3(\mathbb{C}) \to OH(\{1,2,3\})$ such that (6) does not hold with $C = 1$, for any choice of states f_1, f_2.

(2) There exists $T\colon l_\infty\{1,2,3,4\} \to OH(\{1,2\})$ such that (6) does not hold with $C = 1$, for any choice of states f_1, f_2.

Key Lemma in the proof of **Theorem 2**:

Let $(A\,,\tau)$ be a finite dimensional C^*-algebra with a faithful tracial state τ. Let $d \in \mathbb{N}\,,\ d \geq 2\,,$ and consider $a_1\,,\ldots\,,a_d \in A$ satisfying $\sum_{i=1}^d a_i^* a_i = \sum_{i=1}^d a_i a_i^* = dI\,,\ \tau(a_i^* a_j) = \delta_{ij}\,,$ for all $1 \leq i,j \leq d$ and, moreover, the sets $\{a_i^* a_j\,, 1 \leq i,j \leq d\}$ and $\{a_i a_j^*\,, 1 \leq i,j \leq d\}$ are linearly independent. Define $T : A \to OH(d)$ by

$$Tx\colon = (\tau(a_1^* x)\,,\ldots\,,\tau(a_d^* x))\,,\quad x \in A\,.$$

Then $\|T\|_{\mathrm{cb}} < 1$, while the best constant in the inequality

$$\|Tx\| \leq K f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}\,,\quad x \in A$$

(for all choices of states $f_1\, f_2 \in A$) is $K = 1$.

Proof of Theorem 4: Use above Key Lemma with

(1) $d = 3\,,\ \tau = \tau_3\,,\ A = M_3(\mathbb{C})\,,$

$$a_1 = \sqrt{\frac{3}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,,\quad a_2 = \sqrt{\frac{3}{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$a_3 = \sqrt{\frac{3}{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(2) $d = 2\,,\ A = l_\infty(\{1,2,3,4\})\,,\ \tau(c) = \frac{1}{4}(c_1 + \ldots + c_4)\,,$ for all $c = (c_1\,,\ldots\,,c_4) \in \mathbb{C}^4\,,$

$$a_1\colon = \left(\sqrt{2}\,, \sqrt{\frac{2}{3}}\,, \sqrt{\frac{2}{3}}\,, \sqrt{\frac{2}{3}}\right)\,,\quad a_2\colon = \left(0, \frac{2}{\sqrt{3}}\,, \frac{2}{\sqrt{3}}\omega\,, \frac{2}{\sqrt{3}}\bar{\omega}\right)\,.$$

where $\omega^3 = 1\,.$

On the asymptotic quantum Birkhoff conjecture

Classical Birkhoff theorem (Birkhoff, 1946):

Every doubly stochastic matrix is a convex combination of permutation matrices.

Consider the abelian von Neumann algebra $D := l_\infty(\{1, 2, \ldots, n\})$ with trace given by $\tau(\{i\}) = 1/n$, $1 \leq i \leq n$. The positive unital trace-preserving maps on D are the linear operators on D which are given by doubly stochastic $n \times n$ matrices. Note that every automorphism of D is given by a permutation of $\{1, 2, \ldots, n\}$.

The quantum Birkhoff conjecture:

Does every *completely positive* unital trace-preserving map

$$T\colon (M_n(\mathbb{C}), \tau_n) \to (M_n(\mathbb{C}), \tau_n), \quad n \geq 1$$

lie in $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))$?

This turns out to be false for $n \geq 3$ (see, e.g, Example 1). For the case $n \geq 4$, this was first shown by Kümmerer and Maasen (1987), while the case $n = 3$ was settled by Landau-Streater (1993).

The asymptotic quantum Birkhoff conjecture (A. Winter, 2005):

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map, $n \geq 1$. Then

$$\lim_{k\to\infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0. \tag{7}$$

Theorem 5 (Haagerup-M):
Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map, $n \geq 1$. Then

$$d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T\,, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) \geq d_{\mathrm{cb}}(T,\ \mathcal{FM}(M_n(\mathbb{C})))\,.$$

In particular, if T is not factorizable, then

$$d_{\mathrm{cb}}(T\,, \mathcal{FM}(M_n(\mathbb{C}))) > 0\,,$$

since $\mathcal{FM}(M_n(\mathbb{C}))$ is closed. Therefore, the asymptotic quantum Birkhoff conjecture does not hold for $n \geq 3$.

Proof: We show that given $m, n \geq 1$, then for any τ_n-Markov map T on $M_n(\mathbb{C})$ and any τ_m-Markov map S on $M_m(\mathbb{C})$,

$$d_{\mathrm{cb}}(T \otimes S,\ \mathrm{conv}(\mathrm{Aut}(M_n \otimes M_m)) \geq d_{\mathrm{cb}}(T,\ \mathcal{FM}(M_n(\mathbb{C})))\,.$$

Let $i\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ be given by

$$i(x)\colon = x \otimes 1\,, \quad x \in M_n(\mathbb{C})\,.$$

It is easily checked that $i^*(T \otimes S)i = T$, where i^* is the adjoint of i. Since $\|i\|_{\mathrm{cb}} = \|i^*\|_{\mathrm{cb}} = 1$, we get

$$d_{\mathrm{cb}}(T \otimes S\,, \mathrm{conv}(\mathrm{Aut}(M_n \otimes M_m)) \geq \tag{8}$$

$$d_{\mathrm{cb}}(T,\ i^*\mathrm{conv}(\mathrm{Aut}(M_n \otimes M_m))i)\,.$$

Since for every $u \in \mathcal{U}(M_n \otimes M_m)$, the map $i^* \circ \mathrm{ad}(u) \circ i$ is factorizable, and $\mathcal{FM}(M_n(\mathbb{C}))$ is a convex set, we deduce that

$$i^*\mathrm{conv}(\mathrm{Aut}(M_n \otimes M_m)i \subset \mathcal{FM}(M_m(\mathbb{C}))\,,$$

which together with (8) completes the proof.