Tropicalization of hyperbolic polynomials

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- A polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \ldots, z_n]$ is stable if

$$\mathbf{z} \in \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}^n \Longrightarrow P(\mathbf{z}) \neq 0$$

- For $P \in \mathbb{R}[\mathbf{z}]$ of degree d, stability is equivalent to that the homogenization $y^d P(z_1/y, \ldots, z_n/y)$ is hyperbolic with hyperbolicity cone containing $\mathbb{R}^n_{++} \times \{0\}$
- We are interested in combinatorial and geometric properties of the Taylor coefficients of stable polynomials
- Choe-Oxley-Sokal-Wagner (2004): Studied the support of homogeneous square-free stable polynomials
- Speyer (2005): Studied the tropicalization of homogeneous stable polynomials in three variables (Vinnikov polynomials). He obtained a new proof of Horn's conjecture on eigenvalues of sums of Hermitian matrices

Lemma

Let $A_1, \ldots, A_m$ be positive semidefinite Hermitian $n \times n$ matrices and H Hermitian. Then

$$P(\mathbf{z}) = \det(z_1 A_1 + \cdots + z_m A_m + H)$$

is stable

Proof. May assume A_j is PD for all j. Set $z_j = x_j + iy_j$, where $y_j > 0$. Then

$$\begin{aligned} P(\mathbf{z}) &= \det\left(i(\sum_j y_j A_j) + \sum_j x_j A_j + H \right) =: \det(iA + B) \\ &= \det(A) \det(iI + A^{-1/2} B A^{-1/2}) \neq 0 \end{aligned}$$

Conversely

Corollary to Lax Conjecture

If $P(x, y) \in \mathbb{R}[x, y]$ is stable of degree d then there are PSD matrices A, B of size $d \times d$ and Hermitian C such that

$$P(x, y) = \det(xA + yB + C).$$

- The converse fails for more than 2 variables

Theorem (Choe-Oxley-Sokal-Wagner, 2004)

The support, $\mathcal{B}$, of a stable, homogeneous and square-free polynomial

$$P(\mathbf{z}) = \sum_{S \in \binom{[n]}{r}} a(S) \prod_{j \in S} z_j$$

is the set of bases of a matroid

Bases exchange axiom:

$$S, T \in \mathcal{B}, i \in S \setminus T \implies \exists j \in T \setminus S \text{ such that } S \setminus \{i\} \cup \{j\} \in \mathcal{B}$$

Tropicalization

- Let $\mathbb{R}\{t\}$ be the real closed field

$$\mathbb{R}\{t\} = \{x(t) = \sum_{\alpha \in A} a_\alpha t^{-\alpha} \mid A \subset \mathbb{R} \text{ is well-ordered, and } a_\alpha \in \mathbb{R}\}$$

- The valuation $\nu : \mathbb{R}\{t\} \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\nu(x(t)) = \text{leading exponent of } x(t)$$

Tarski's principle

If an elementary statement is true in one real closed field, then it is true in every real closed field

Example

Let $A_1(t), \dots, A_m(t)$ be positive semidefinite hermitian $n \times n$ matrices and $H(t)$ hermitian (over $\mathbb{C}\{t\}$). Then

$$P(\mathbf{z}) = \det\left(z_1 A_1(t) + \cdots + z_m A_m(t) + H(t)\right)$$

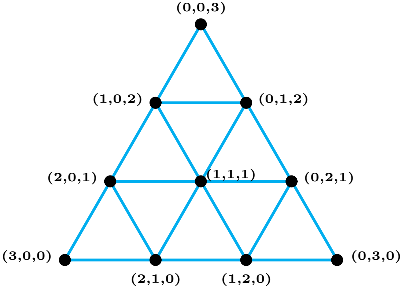
is stable

- Let $P(\mathbf{z}) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(t)\mathbf{z}^\alpha \in \mathbb{R}\{t\}[\mathbf{z}]$. The tropicalization, $\mathrm{trop}(P)$, of P is the map

$$\mathbb{N}^n \ni \alpha \mapsto \nu(a_\alpha(t)) \in \mathbb{R} \cup \{-\infty\}$$

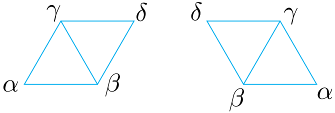
- We are interested in convexity properties of the tropicalization of spaces of stable polynomials

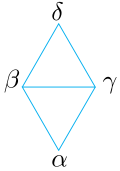
Let $\Delta_d = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 : \alpha_1 + \alpha_2 + \alpha_3 = d\}$



A function $h : \Delta_d \to \mathbb{R}$ is a hive if all rhombus inequalities are satisfied:

$$h(\beta) + h(\gamma) \geq h(\alpha) + h(\delta)$$





				0				
			15		23			
		24		36		41		
	29		43		53		56	
31		45		56		65		67

Horn's problem

Characterize all triples of vectors $\alpha, \beta, \gamma \in \mathbb{R}^n$ such that there are two Hermitian $n \times n$ matrices A, B such that

- α are the eigenvalues of A
- β are the eigenvalues of B
- γ are the eigenvalues of $A + B$
- Call (α, β, γ) a Horn triple

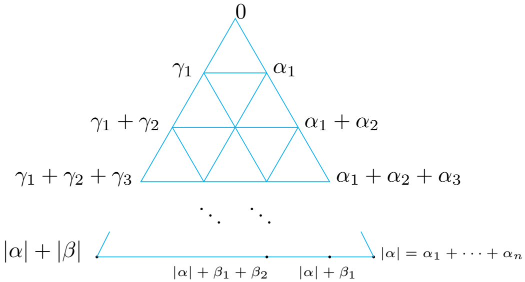
- Solved by Klyachko and Knutson–Tao in the late 90's
- Knutson–Tao's characterization involves hives

Let $\alpha, \beta, \gamma \in \mathbb{R}^n$ be such that

$$\alpha_1 \geq \alpha_2 \geq \cdots, \quad \beta_1 \geq \beta_2 \geq \cdots, \quad \gamma_1 \geq \gamma_2 \geq \cdots, \quad |\alpha| + |\beta| = |\gamma|$$

We want to determine if (α, β, γ) is a Horn triple. Mark the boundary of Δ_n as

$$\partial(\alpha, \beta, \gamma) :=$$

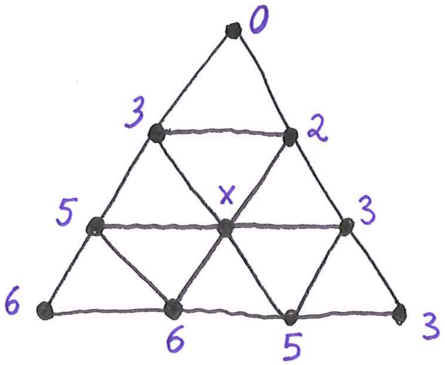


Theorem (Knutson-Tao)

(α, β, γ) is a Horn triple if and only if $\partial(\alpha, \beta, \gamma)$ can be completed to a hive

Example

Is $(2,1,0),(2,1,0),(3,2,1)$ a Horn triple?



Yes, let $4 \leq x \leq 5$

- Let $\mathcal{H}_3^d$ be the space of all stable polynomials $P \in \mathbb{R}\{t\}[x, y, z]$ with support Δ_d
- By the now resolved Lax Conjecture

$$P(x, y, z) = \det(xA(t) + yB(t) + zC(t))$$

where $A(t), B(t), C(t)$ are positive definite
- Let Hive_d be the space of all hives on Δ_d

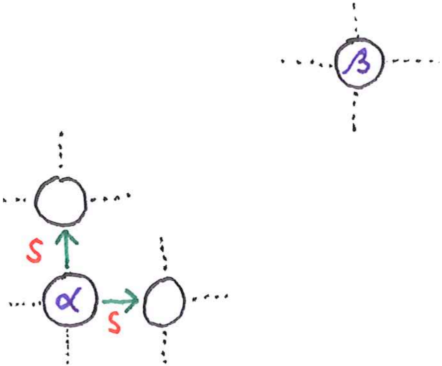
Theorem (Speyer)

$$\operatorname{trop}(\mathcal{H}_3^d) = \text{Hive}_d$$

What about other spaces of stable polynomials?

M-concave functions (Murota)

- Let $\alpha, \beta \in \mathbb{Z}^n$ and $|\alpha| := \sum_{i=1}^{n} |\alpha_i|$
- A step from α to β is an $s \in \mathbb{Z}^n$ such that $|s| = 1$ and $|\alpha + s - \beta| = |\alpha - \beta| - 1$. Indicate this by $\alpha \xrightarrow{s} \beta$

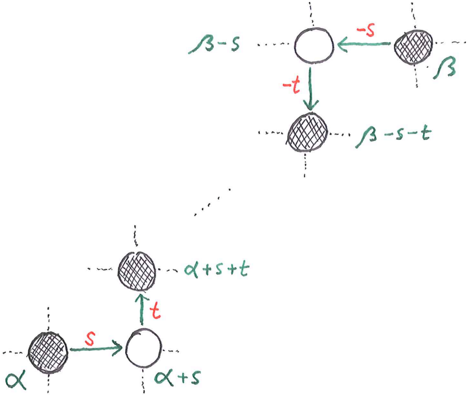


A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is *M-concave* if it satisfies

$\alpha, \beta \in \mathbb{Z}^n$ and $\alpha \xrightarrow{s} \beta$

$$\Longrightarrow$$

$\exists$ step t, $\alpha + s \xrightarrow{t} \beta$,
such that
$f(\alpha) + f(\beta) \leq$
$f(\alpha{+}s{+}t){+}f(\beta{-}s{-}t)$



Properties of *M*-concave *f*

- Global maximum $\Longleftrightarrow$ local maximum ($|\cdot| \leq 2$)
- The naive algorithm for finding maximum converges after $O(n^2 D)$ evaluations of f, where

$$D = \max\{|\alpha - \beta| : \alpha, \beta \in \mathrm{supp}(f)\}$$

A polynomial $P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \mathbf{z}^\alpha$ has constant parity if

$$a_\alpha a_\beta \neq 0 \implies |\alpha| \equiv |\beta| \mod 2$$

Theorem (B.)

Suppose $P(\mathbf{z}) \in \mathbb{R}\{t\}[\mathbf{z}]$ is stable and has constant parity. Then $\mathrm{trop}(P)$ is M-concave

- The support of $f : \mathbb{N}^n \to \mathbb{R} \cup \{-\infty\}$ is

$$\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : f(\alpha) \neq -\infty\}$$

- Suppose $\mathrm{supp}(f) = \Delta_d$. Then f is M-concave if and only if f is a hive.

- The tropical Grassmannian, $\mathrm{Gr}(r,n)$, can be defined via Plücker coordinates as the space of all mappings f_A

$$\binom{[n]}{r} \ni S \mapsto \nu(A(S)) \in \mathbb{R} \cup \{-\infty\}$$

  where A runs over all $r \times n$ matrices over $\mathbb{C}\{t\}$, and $A(S)$ is the $r \times r$ minor of A with rows indexed by S
- The Dressian, $\mathrm{Dr}(r,n)$, can be defined as the space of M-concave functions on $\mathbb{N}^n$ with support contained in $\binom{[n]}{r}$. Also called valuated matroids
- Let $\mathcal{H}_{r,n}^{SF}$ be the space of all stable polynomials (over $\mathbb{R}\{t\}$) with support contained in $\binom{[n]}{r}$. Then

$$\mathrm{Gr}(r,n) \subsetneq \mathrm{trop}(\mathcal{H}_{r,n}^{SF}) \subsetneq \mathrm{Dr}(r,n)$$

$\mathrm{Gr}(r,n) \subset \mathrm{trop}(\mathcal{H}_{r,n}^{SF})$:

- Let $Z = \mathrm{diag}(z_1, \ldots, z_n)$. Then, if $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{C}\{t\}^{r\times n}$

$$P_A(\mathbf{z}) := \det(AZA^*) = \text{by Cauchy-Binet} = \sum_{S\in\binom{[n]}{r}} A(S)\overline{A(S)}\prod_{j\in S} z_j$$

$$P_A(\mathbf{z}) = \det\left(\sum_{j=1}^{n} z_j \mathbf{a}_j \overline{\mathbf{a}_j}^T\right)$$

- Thus P_A is stable and $\mathrm{trop}(P_A) = 2f_A$

- Let $\text{Hive}_{r,n}$ be the space of all M-concave functions with support $\Delta_{r,n} = \{\alpha \in \mathbb{N}^n : |\alpha| = r\}$
- Let $\mathcal{H}_{r,n}$ be the space of stable polynomial with support $\Delta_{r,n}$
- Hence $\mathrm{trop}(\mathcal{H}_{r,n}) \subseteq \text{Hive}_{r,n}$
- We have $\mathrm{trop}(\mathcal{H}_{r,n}) = \text{Hive}_{r,n} \quad$ for $n = 1, 2, 3$
- However $\mathrm{trop}(\mathcal{H}_{3,7}) \subsetneq \text{Hive}_{3,7}$

Regress to $n = 2$

- A homogeneous polynomial $P(x, y)$ is stable if and only if all zeros of

$$P(x, 1) := \sum_{j=0}^{r} a_j x^j$$

are real and non-positive

- Newton inequalities: $a_j^2 \geq a_{j-1} a_{j+1}$ for all j

Hence $\nu(a_k) \geq (\nu(a_{k-1}) + \nu(a_{k+1}))/2$. Converse:

Lemma (Hardy, Hutchinson)

If $a_0, \ldots, a_r$ are positive and $a_k^2 \geq 4 a_{k-1} a_{k+1}$ for all k then all zeros of

$$a_0 + a_1 x + \cdots + a_r x^r$$

are real and non-positive

Thus if $h(k)$ is concave, then $\sum_{k=0}^{r} 4^{-\binom{k}{2}} t^{h(k)} x^k y^{r-k}$ is stable. Hence $\mathrm{trop}(\mathcal{H}_{2,n})$ is the space of concave sequences

Proof of Speyer's theorem

If $\nu(a_0(t)), \ldots, \nu(a_r(t))$ satisfy $2\nu(a_k(t)) > \nu(a_{k-1}(t)) + \nu(a_{k+1}(t))$, then

$$P(x, y) = \sum_{k=0}^{r} a_k(t) x^k y^{r-k}$$

is stable

Lemma (B.)

Let

$$P(x, y, z) = \sum_{\alpha \in \Delta_n} a_\alpha x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}, \quad \text{where } a_\alpha > 0$$

Then P is stable if and only if the following polynomials have only real zeros

- $x \mapsto P(x, 1, \lambda)$, for all $\lambda > 0$
- $y \mapsto P(1, y, \lambda)$, for all $\lambda > 0$
- $z \mapsto P(1, \lambda, z)$, for all $\lambda > 0$

- We need to prove that all hives are tropicalizations of stable polynomials
- It is enough to prove it for strict hives, that is, hives for which the rhombus inequalities are strict
- By Tarski's principle and the previous Lemma, it is enough to prove that if $\mathrm{trop}(P)$ is a strict hive then

$$P(1, \lambda, z) =: \sum_{k=0}^{n} a_k(t) z^k$$

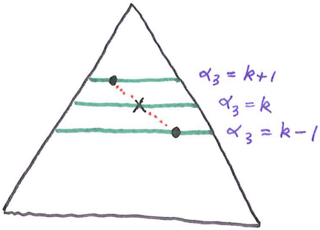
has only real zeros
- In view of the Hardy–Hutchinson lemma it suffices to prove that

$$2\nu(a_k(t)) > \nu(a_{k-1}(t)) + \nu(a_{k+1}(t)), \quad \text{for all } k \quad (*)$$

- Let $h = \mathrm{trop}(P)$ and $C = \nu(\lambda)$, then

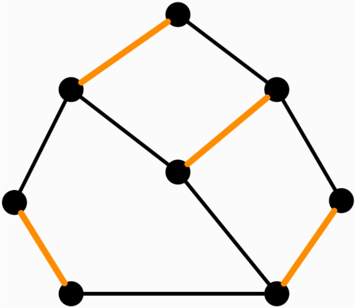
$$\nu(a_k(t)) = \max\{h(\alpha) + \alpha_2 C : \alpha \in \Delta_n \text{ and } \alpha_3 = k\}$$

- $\alpha \mapsto h(\alpha) + \alpha_2 C$ is also a strict hive. Thus $(*)$ follows from concavity



Maximal matching problem

- Let $G = (V, E)$ be a graph
- A subset $F \subseteq E$ is a matching if each vertex is contained in at most one edge in F



- Let $\mu : E \to \mathbb{R}$ be a weight function
- We want to maximize the quantity

$$\mu(F) = \sum_{e \in F} \mu(e), \quad F \text{ is a matching}$$

Let $\lambda : E \to \mathbb{R}^+$ and define

$$F_\lambda(z_1, \ldots, z_n) = \sum_{F \text{matching}} (-1)^{|F|} \prod_{e=ij \in F} \lambda(e) z_i z_j$$

Heilmann-Lieb Theorem

$F_\lambda(z_1, \ldots, z_n)$ is stable

Let $\lambda(e) = t^{\mu(e)}$ and apply Tarski's principle. Let $\nu : \{0,1\}^V \to \mathbb{R} \cup \{-\infty\}$ be defined by

$$\nu(S) = \max\{\mu(F) = \sum_{e \in F} \mu(e) : \bigcup F = S\}$$

Then

$$\operatorname{trop}\left(\sum_{F \text{matching}} (-1)^{|F|} t^{\mu(F)} \prod_{e=ij \in F} z_i z_j \right) = \nu$$

Hence

$$\nu(S) = \max\{\mu(F) = \sum_{e \in F} \mu(e) : \bigcup F = S\}$$

is M-concave.