Representing Polyhedra by Few Polynomials

Martin Henk

Banff, February, 2010

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- Martin Grötschel Impact for hard combinatorial optimization problems?, Constructions?, Approximations by polynomial inequalities?

- Bröcker, Scheiderer, '84,...,'89. Every basic closed semi-algebraic set $S \subset \mathbb{R}^n$ can be represented by at most $n(n+1)/2$ polynomial inequalities, i.e., there exist $p_1, \ldots, p_{n(n+1)/2} \in \mathbb{R}[x]$ such that

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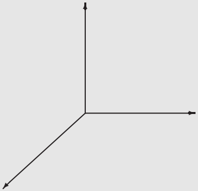
In the case of basic open semi-algebraic sets, n polynomials suffice, and both bounds are best possible.

- Open: For instance, the positive orthant $\{x \in \mathbb{R}^n : x_i > 0,\ 1 \leq i \leq n\}$ cannot be described by less than n strict polynomial inequalities.



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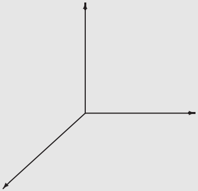

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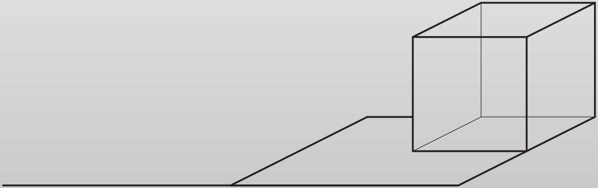

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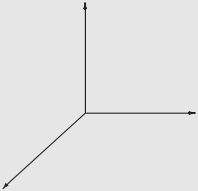

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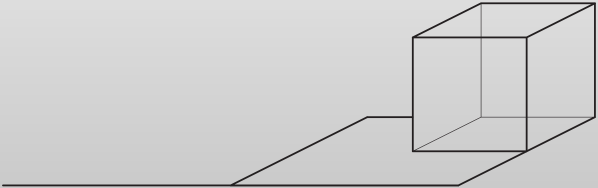

- Can the bound be improved, e.g., for convex sets?

Consequences for polyhedra

- Every polyhedron

$$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i,\ 1 \leq i \leq m\},$$

  given as the intersection of finitely many linear inequalities, can be described by at most $n(n+1)/2$ polynomial inequalities. The interior of a polyhedron can even be described by n polynomials.

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It depends...!

(Trivial) Examples

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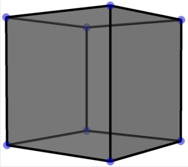
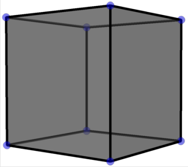
- The (regular) n-cube



$$C_n = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1, \, 1 \leq i \leq n\}$$

(Trivial) Examples

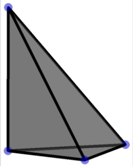
- The (regular) n-cube (or any other parallelepiped)



$$C_n = \{x \in \mathbb{R}^n : -1 \leq x_i \leq 1,\ 1 \leq i \leq n\}$$
$$= \{x \in \mathbb{R}^n : (x_i)^2 \leq 1,\ 1 \leq i \leq n\}.$$

(Trivial) Examples

- The n-simplex



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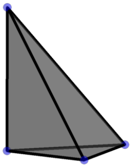
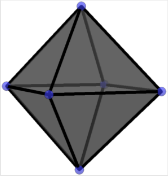


$$T_n = \{x \in \mathbb{R}^n : x_i \geq 0,\ x_1 + \cdots + x_n \leq 1\}$$
$$= \left\{x \in \mathbb{R}^n : x_i \left(1 - \sum_{k=i}^n x_k\right) \geq 0,\ 1 \leq i \leq n\right\}.$$

- The regular n-crosspolytope



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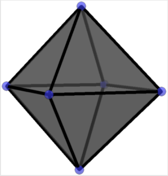

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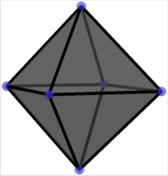

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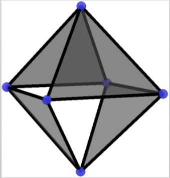
  - $p_{1/2}$ = product of 4 facet defining inequalities which do not have an edge in common.



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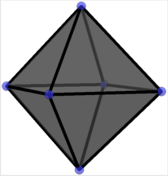

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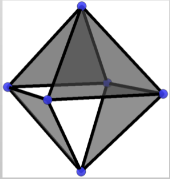
  - p_0 = circumsphere of $C_3^\star$.



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- For bi-pyramids?

Dimension 2

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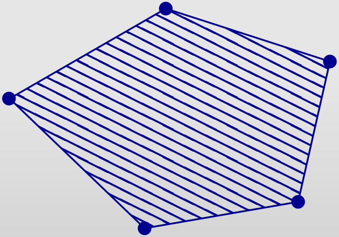
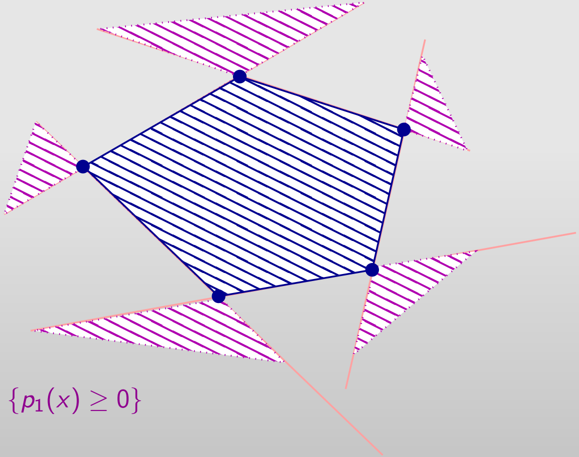
- Bernig, 1998. For each (bounded) polygon we can construct 2 polynomial inequalities representing the polygon.

- Let $P = \{x \in \mathbb{R}^2 : \langle a_i, x \rangle \leq b_i, 1 \leq i \leq m\}$ be a polygon.



- Let $P = \{x \in \mathbb{R}^2 : \langle a_i, x \rangle \leq b_i, 1 \leq i \leq m\}$ be a polygon.

$$p_1(x) = (b_1 - \langle a_1, x \rangle) \cdot (b_2 - \langle a_2, x \rangle) \cdot \ldots \cdot (b_m - \langle a_m, x \rangle)$$



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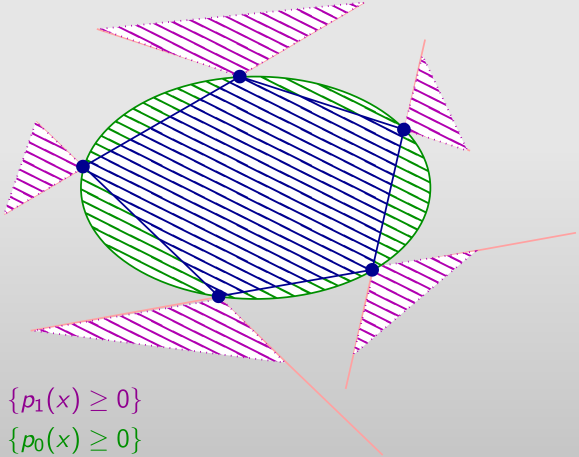

$\{p_1(x) \ge 0\}$
$\{p_0(x) \ge 0\}$

- $p_0(x)$ is of the form

$$p_0(x) = 1 - \sum_{i=1}^{m} \lambda_i \left[\frac{\langle w_i, x\rangle - l_i}{u_i - l_i}\right]^{2k},$$

where w_i are normal vectors of support hyperplanes of the vertices,

$$l_i = \min_{x \in P} \langle w_i, x\rangle, \quad u_i = \max_{x \in P} \langle w_i, x\rangle$$

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- In particular, the degree depends on metric properties of the polygon.

- The obvious generalization of that 2-dimensional approach to consider polynomials

$$p_k(x) = \prod \text{support hyperplanes of } k\text{-faces}, \quad k = 1, \dots, n-1,$$
$$p_0(x) = \text{concave polynomial through the vertices}$$

  does not work for $n \geq 3$ (see, e.g., crosspolytope).

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- Consequence: Let

$$S = \left\{ x \in \mathbb{R}^n : f_1(x) \geq 0, \dots, f_m(x) \geq 0 \right\}$$

with $\deg(f_i) \leq d$. Then we can find $2\binom{n+d}{n} - 2$ polynomials representing the set S.

Simple polytopes seem to be simpler

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- Rough idea:
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  $$P = \{x \in \mathbb{R}^n : l_i(x) \geq 0,\, 1 \leq i \leq m\}.$$
  - Let
  $$\sigma_j(x) = \sum_{\substack{J \subseteq \{1,\ldots,m\} \\ \#J = j}} \prod_{k \in J} l_k(x)$$
  be the j-th elementary symmetric polynomial of $l_1(x), \ldots, l_m(x)$.

- $P = \{x \in \mathbb{R}^n : \sigma_i(x) \geq 0,\ 1 \leq i \leq m\}$

- Let $x \in \mathbb{R}^n$ such that $\sigma_i(x) \geq 0$, $1 \leq i \leq m$. Let

$$f(t) = \prod_{i=1}^{m} (l_i(x) + t) = \sum_{i=0}^{m} \sigma_i(x)\, t^{m-i}.$$

All coefficients are non-negative and hence, the roots $-l_i(x)$, $1 \leq i \leq m$, are non-positive, i.e., $x \in P$.

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    If P is simple then there exists an $\epsilon > 0$ such that for $x \in P + \epsilon B_n$
$$\sigma_i(x) \geq 0,\ 1 \leq i \leq m - n.$$

- Let $x \in P$. Since P is simple, there exist at most n linear forms $l_i(x)$ vanishing at x.

- Hence at least $m - n$ linear forms are positive at x and so

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- Thus by continuity we can find an $\epsilon > 0$ such that for all $x \in P + \epsilon B_n$

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- Thus

$$P = \{x \in \mathbb{R}^n : \sigma_{m-n+i+1}(x) \geq 0,\, 0 \leq i \leq n-1,\, p_\epsilon(x) \geq 0\},$$

where $\{x \in \mathbb{R}^n : p_\epsilon(x) \geq 0\}$ is a "good" approximation of P.

- A simple polytope $P = \{x \in \mathbb{R}^n : l_i(x) \geq 0,\, 1 \leq i \leq m\}$ is described by the n polynomial inequalities

$$p_i(x) := \sigma_{m-n+i+1}(x) \geq 0,\, 1 \leq i \leq n-1, \quad p_0(x) \geq 0,$$

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- In particular, $p_i(x)$ vanishes on the i-faces of P, $i = 0, \ldots, n-1$.

Example

- For a regular simplex $P \subseteq \mathbb{R}^3$ we can choose

$$l_1(x) = 1 + x_1 - x_2 + x_3, \quad l_2(x) = 1 - x_1 + x_2 + x_3$$
$$l_3(x) = 1 + x_1 + x_2 - x_3, \quad l_4(x) = 1 - x_1 - x_2 - x_3.$$

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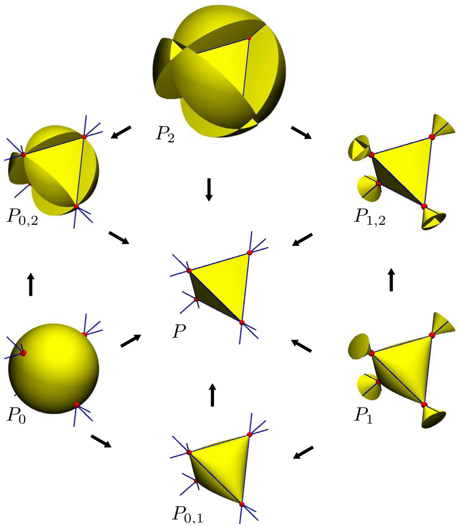
-

$$\begin{aligned} p_2 &= l_1\, l_2\, l_3\, l_4 \\ p_1 &= l_1\, l_2\, l_3 + l_1\, l_2\, l_4 + l_1\, l_3\, l_4 + l_2\, l_3\, l_4 \\ &= 4\,(1 - x_1^2 - x_2^2 - x_3^2 - 2\, x_1\, x_2\, x_3) \\ p_0 &= 3 - x_1^2 - x_2^2 - x_3^2. \end{aligned}$$

- For $J \subset \{0,1,2\}$ let $P_J = \{x \in \mathbb{R}^3 : p_j(x) \geq 0, j \in J\}$



Simplex

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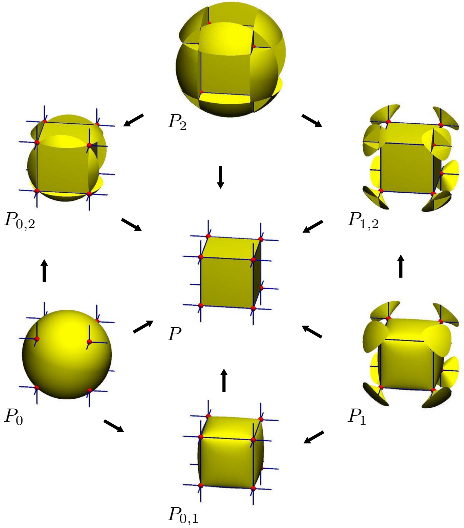

Cube

The general case

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  For every 3-dimensional polyhedra we can construct 3 polynomials representing the polyhedra.

Averkov&Bröcker, 2010. Let

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be a basic closed semi-algebraic set.

- If all $f_i(x)$ are linear, i.e., S is a polyhedron, then S can be represented by n polynomials.

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- Let d be the maximal number of polynomials vanishing at a point. Then there exist $d+1$ polynomials $p_0, \ldots, p_d$ representing S.

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  - If there are only finitely many points where d polynomials $f_i(x)$ vanish then d polynomials suffice.
- The proofs are "semi-effective".
  - Separation theorems based on Stone-Weierstrass approximation.

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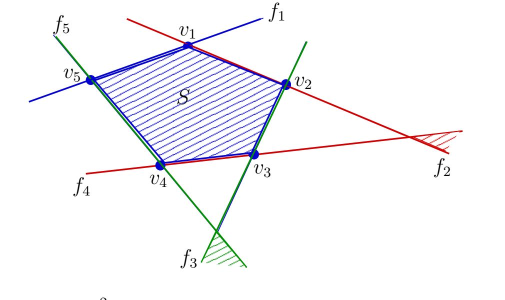

$S = \{x \in \mathbb{R}^2 : f_2(x) \cdot f_4(x) \geq 0,\ f_3(x) \cdot f_5(x) \geq 0,\ f_1(x) \geq 0\}$

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- Averkov&Bey, 2010. $d \leq \max\left\{\frac{m}{k}, \log_2(m)\right\}$, and it is best possible for any k among a certain family of polynomials.

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Thank you for your attention!