Lower bounds for a polynomial in terms of its coefficients

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Abstract

- We determine new sufficient conditions in terms of the coefficients for a polynomial $f \in \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$ of degree $2d$ ($d \geq 1$) to be a sum of squares of polynomials, thereby strengthening results of Lasserre [6] and of Fidalgo and Kovacec [2].
- Exploiting these results, we determine, for any polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$ whose highest degree term is an interior point in the cone of sos forms of degree $2d$, a real number r such that $f - r$ is a sum of squares of polynomials.
- Actually, we determine three different real numbers r having this property.
- The existence of such a number r was proved earlier by Marshall [8], but no estimates for r were given.
- We also determine lower bounds (more precisely, three lower bounds) for any polynomial f whose highest degree term is positive definite.

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1 Introduction

• Fix a non-constant polynomial $f \in \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \cdots, X_n]$, where $n \geq 1$ is an integer number, and define

$$f_* := \inf\{f(\underline{a}) \mid \underline{a} \in \mathbb{R}^n\}.$$

• Denote the cone of all sos polynomials by $\sum \mathbb{R}[\underline{X}]^2$ and define

$$f_{sos} := \sup\{r \in \mathbb{R} \mid f - r \in \sum \mathbb{R}[\underline{X}]^2\}. \tag{1}$$

• One can prove that $f_{sos} \leq f_*$. Computing f_* is difficult in general, and one of the successful approaches is to compute f_{sos} instead. This is accomplished by using *semidefinite programming* (SDP) which is a polynomial time algorithm [5] [9].

• When is a given polynomial $f \in \mathbb{R}[\underline{X}]$ a sum of squares? One obvious necessary condition is that $f \geq 0$ on $\mathbb{R}^n$, but there is a well known result due to Hilbert [3] that this necessary condition is not sufficient in general.

• In this paper we are interested in some recent results, due to Lasserre [6] and to Fidalgo and Kovacec [2], which give sufficient conditions on the coefficients for a polynomial to be a sum of squares. We establish new and improved versions of these results; see Ths. 2.3 and 2.5 and Cors. 2.4 and 2.6.

• Let $\deg(f) = 2d$, $d \geq 1$, and decompose f as $f = f_0 + \cdots + f_{2d}$ (the homogeneous decomposition of f), where f_i is a form of degree i, $i = 0, \ldots, 2d$.

• We denote the cone of all positive semidefinite forms and sos forms of degree $2d$ by $P_{2d,n}$ and $\Sigma_{2d,n}$, respectively. We denote by $P^{\circ}_{2d,n}$ and $\Sigma^{\circ}_{2d,n}$ the interior of $P_{2d,n}$ and $\Sigma_{2d,n}$, more precisely, the interior in the subspace of $\mathbb{R}[\underline{X}]$ consisting of forms of degree $2d$.

• A necessary condition for $f_* \neq -\infty$ is that $f_{2d} \in P_{2d,n}$. A sufficient condition for $f_* \neq -\infty$ is that $f_{2d} \in P^{\circ}_{2d,n}$. A necessary condition for $f_{sos} \neq -\infty$ is that $f_{2d} \in \Sigma_{2d,n}$. A sufficient condition for $f_{sos} \neq -\infty$ is that $f_{2d} \in \Sigma^{\circ}_{2d,n}$ [8, Prop. 5.1].

• We apply Cors. 2.4 and 2.6 to determine, assuming that $f_{2d} \in \Sigma^{\circ}_{2d,n}$, two lower bounds for f_{sos}, which we denote by r_L and r_{FK} respectively; see Ths. 3.1 and 3.2. Yet another lower bound for f_{sos}, which we denote by r_{dmt}, is obtained by applying [2, Th. 2.3] directly; see Th. 3.3. The bounds r_L, r_{FK} and r_{dmt} are not comparable; see Ex. 4.2. If we assume only that $f_{2d} \in P^{\circ}_{2d,n}$ then it is still possible to determine lower bounds for f_*, in a similar way, but these may not be lower bounds for f_{sos}; see Th. 4.3.

• We introduce notation that we will need. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. For $\underline{X} = (X_1, \ldots, X_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $\underline{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Every polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$ can be written in the form

$$f = f_0 + \sum_{\alpha \in \Omega(f)} f_\alpha \underline{X}^\alpha + \sum_{i=1}^{n} f_{2d,i} X_i^{2d}, \tag{2}$$

where $f_0, f_{2d,i} \in \mathbb{R}$ and, for each $\alpha \in \Omega(f)$, $0 \neq f_\alpha \in \mathbb{R}$, $0 < |\alpha| \leq 2d$, and $\alpha \notin \{2d\epsilon_1, \ldots, 2d\epsilon_n\}$, where $\epsilon_i = (\delta_{i1}, \ldots, \delta_{in})$, and

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Let $\Delta(f) = \{\alpha \in \Omega(f) \mid f_\alpha \underline{X}^\alpha \text{ is not a square in } \mathbb{R}[\underline{X}]\} = \{\alpha \in \Omega(f) \mid \text{ either } f_\alpha < 0 \text{ or } \alpha_i \text{ is odd for some } i \in \{1, \ldots, n\}\}$. Since our polynomial f is usually fixed, we will often denote $\Omega(f)$ and $\Delta(f)$ just by Ω and Δ for short.

• Let $\widetilde{f}(\underline{X}, Y) = Y^{2d} f(\frac{X_1}{Y}, \ldots, \frac{X_n}{Y})$. From (2), it is clear that

$$\widetilde{f}(\underline{X}, Y) = f_0 Y^{2d} + \sum_{\alpha \in \Omega} f_\alpha \underline{X}^\alpha Y^{2d - |\alpha|} + \sum_{i=1}^{n} f_{2d,i} X_i^{2d}$$

is a form of degree $2d$, called the homogenization of f. We have the following well-known result:

Proposition 1.1. *f is sos if and only if $\widetilde{f}$ is sos.*

Proof. See [7, Prop. 1.2.4]. □

• For a (univariate) polynomial of the form $p(t) = t^n - \sum_{i=0}^{n-1} a_i t^i$, where each a_i is nonnegative and at least one a_i is nonzero, we denote by $C(p)$ the unique positive root of p [10, Th. 1.1.3]. For any polynomial $q(t) = \sum_{i=0}^{n} b_i t^i$, $b_n \neq 0$, the roots of q are bounded in absolute value by $C(t^n - \sum_{i=0}^{n-1} \frac{|b_i|}{|b_n|} t^i)$. By convention, $C(t^n) := 0$.

• There are various upper bounds for $C(p)$ which are expressible in an elementary way in terms of the coefficients of p, for example,

Proposition 1.2. *Suppose $p(t) = t^n - \sum_{i=0}^{n-1} a_i t^i$, where each a_i is nonnegative and at least one a_i is nonzero. Then*

(1) $C(p) \leq \max\{1, a_0 + a_1 + \cdots + a_{n-1}\}$,

(2) $C(p) \leq \max\{a_0, 1 + a_1, 1 + a_2, \ldots, 1 + a_{n-1}\}$,

(3) $C(p) \leq 2\max\{a_{n-1}, (a_{n-2})^{1/2}, (a_{n-3})^{1/3}, \ldots, (a_0)^{1/n}\}$.

Proof. Bounds (1) and (2) are due basically to Cauchy. See [1] for these bounds and for other bounds of this sort. See [4, Ex. 4.6.2: 20] for bound (3). □

2 Sufficient conditions for a polynomial to be sos

- We make use of the following result:

Theorem 2.1 (Reznick)**.** *Suppose* $p(\underline{x}) = \sum_{i=1}^{n} a_i x_i^{2d} - 2d x_1^{a_1} \cdots x_n^{a_n}$*,* $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$*,* $|a| = 2d$*. Then* p *is sobs.*

- Notes:

— sobs := sum of binomial squares, i.e., a sum of squares of the form $(\beta \underline{x}^b - \gamma \underline{x}^c)^2$ with $\beta, \gamma \in \mathbb{R}$ and $c, d \in \mathbb{N}^n$.

— Th. 2.1 can be deduced from results of Reznick in [11] and [12], specifically, from [12, Th. 2.2 and Th. 4.4]. A direct elementary proof of Th. 2.1 is given below. If one only wants to prove that p is sos the proof is even simpler.

Proof. By induction on n. If $n = 1$ then $p = 0$ and the result is clear. Assume now that $n \geq 2$. By induction on n we can assume each a_i is strictly positive.

Case 1: Suppose $\exists\ i_1 \neq i_2$ with $a_{i_1} \leq d$ and $a_{i_2} \leq d$. Decompose $a = (a_1, \ldots, a_n)$ as $a = b + c$ with $b, c \in \mathbb{N}^n$, $b_{i_1} = 0$, $c_{i_2} = 0$ and $\sum_{i=1}^n b_i = \sum_{i=1}^n c_i = d$. Then $(\underline{x}^b - \underline{x}^c)^2 = \underline{x}^{2b} - 2\underline{x}^b \underline{x}^c + \underline{x}^{2c} = \underline{x}^{2b} - 2\underline{x}^a + \underline{x}^{2c}$, so

$$p = \sum_{i=1}^{n} a_i x_i^{2d} - 2d\underline{x}^a = \sum_{i=1}^{n} a_i x_i^{2d} - d[\underline{x}^{2b} + \underline{x}^{2c} - (\underline{x}^b - \underline{x}^c)^2]$$

$$= \frac{1}{2}[\sum_{i=1}^{n} 2b_i x_i^{2d} - 2d\underline{x}^{2b}] + \frac{1}{2}[\sum_{i=1}^{n} 2c_i x_i^{2d} - 2d\underline{x}^{2c}] + d(\underline{x}^b - \underline{x}^c)^2.$$

Each term is sobs, by induction on n.

Case 2: Suppose we are not in Case 1. Since there is at most one i satisfying $a_i > d$ it follows that $n = 2$, so $p = a_1 x_1^{2d} + a_2 x_2^{2d} - 2d x_1^{a_1} x_2^{a_2}$. We know that $p \geq 0$ on $\mathbb{R}^2$, by the arithmetic-geometric inequality. Since $n = 2$ and p is homogeneous, it follows that p is sos (dehomogenize p and apply [8], Prop. 1.2.1 and Prop. 1.2.4).

But we want to show p is sobs, which requires more work. Denote by $\mathrm{AGI}(2, d)$ the set of all homogeneous polynomials of the form $p = a_1 x_1^{2d} + a_2 x_2^{2d} - 2d x_1^{a_1} x_2^{a_2}$, $a_1, a_2 \in \mathbb{N}$, $a_1 + a_2 = 2d$. This set is finite. If $a_1 = 0$ or $a_1 = 2d$ then $p = 0$ which is trivially sobs. If $a_1 = a_2 = d$ then $p = d(x_1^d - x_2^d)^2$, which is also sobs. Suppose now that $0 < a_1 < 2d$, $a_1 \neq d$. Suppose $a_1 > a_2$ (The argument for $a_1 < a_2$ is similar.) Decompose $a = (a_1, a_2)$ as $a = b + c$, $b = (d, 0)$, $c = (a_1 - d, a_2)$. Expand p as in the proof of Case 1 to obtain

$$p = \frac{1}{2}[\sum_{i=1}^{2} 2b_i x_i^{2d} - 2d\underline{x}^{2b}] + \frac{1}{2}[\sum_{i=1}^{2} 2c_i x_i^{2d} - 2d\underline{x}^{2c}] + d(\underline{x}^b - \underline{x}^c)^2.$$

Observe that $\sum_{i=1}^{2} 2b_i x_i^{2d} - 2d\underline{x}^{2b} = 0$. Thus $p = \frac{1}{2}p_1 + d(\underline{x}^b - \underline{x}^c)^2$, where $p_1 := \sum_{i=1}^{2} 2c_i x_i^{2d} - 2d\underline{x}^{2c}$. If p_1 is sobs the p is also sobs. If p_1 is not sobs then we can repeat to get $p_1 = \frac{1}{2}p_2 + d(\underline{x}^{d'} - \underline{x}^{c'})^2$. Continuing in this way we get a sequence $p = p_0, p_1, p_2, \ldots$ with each p_i an element of the finite set $\mathrm{AGI}(2, d)$, so $p_i = p_j$ for some $i < j$. Since $p_i = 2^{i-j}p_j +$ a sum of binomial squares, this implies p_i is sobs and hence that p is sobs. □

Corollary 2.2 (Fidalgo-Kovacec [2, Th. 2.3]). *For a form* $p(\underline{X}) = \sum_{i=1}^{n} \beta_i X_i^{2d} - \mu \underline{X}^{\alpha}$ *such that* $\alpha_i > 0$ *and* $\beta_i \geq 0$ *for every* $i = 1, \ldots, n$ *and* $\mu \geq 0$ *if all* α_i *are even, the following are equivalent:*

i. p *is positive semidefinite.*

ii. $|\mu| \leq 2d \prod_{i=1}^{n} \left(\frac{\beta_i}{\alpha_i} \right)^{\frac{\alpha_i}{2d}}$.

iii. p *is sobs.*

iv. p *is sos.*

• Cor. 2.2 is an easy consequence on Th. 2.1. See [2] for the proof.

• In what follows we use Cor. 2.2 to improve on the sufficient conditions given in [6, Th. 3] and [2, Th. 4.3].

Theorem 2.3. *Suppose $f \in \mathbb{R}[\underline{X}]$ is a form of degree $2d$ and*

$$f_{2d,i} \geq \sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}, \quad i = 1, \ldots, n.$$

Then f is a sum of (binomial) squares.

Proof. We claim that

$$\sum_{i=1}^{n} |f_\alpha| \frac{\alpha_i}{2d} X_i^{2d} + f_\alpha \underline{X}^\alpha$$

is sobs, for each $\alpha \in \Delta$. It suffices to show that $\sum_{\alpha_i \neq 0} |f_\alpha| \frac{\alpha_i}{2d} X_i^{2d} + f_\alpha \underline{X}^\alpha$ is sobs, for each $\alpha \in \Delta$. Since

$$2d \prod_{\alpha_i \neq 0} \left(\frac{|f_\alpha| \frac{\alpha_i}{2d}}{\alpha_i} \right)^{\frac{\alpha_i}{2d}} = 2d \frac{|f_\alpha|}{2d} = |f_\alpha| \geq |f_\alpha|,$$

and since $f_\alpha < 0$ if all the α_i are even, by definition of Δ, this follows, as a consequence of Cor. 2.2. This proves the claim. Adding, as α runs through Δ, this implies

$$\sum_{i=1}^{n} (\sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}) X_i^{2d} + \sum_{\alpha \in \Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Since $f_{2d,i} \geq \sum_{\alpha\in\Delta} |f_\alpha|\frac{\alpha_i}{2d}$, for each i,

$$\sum_{i=1}^{n} f_{2d,i}X_i^{2d} - \sum_{i=1}^{n}(\sum_{\alpha\in\Delta} |f_\alpha|\frac{\alpha_i}{2d})X_i^{2d} = \sum_{i=1}^{n}(f_{2d,i} - \sum_{\alpha\in\Delta} |f_\alpha|\frac{\alpha_i}{2d})X_i^{2d}$$

is sobs. Adding again, this implies that

$$\sum_{i=1}^{n} f_{2d,i}X_i^{2d} + \sum_{\alpha\in\Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Finally, since the remaining terms $f_\alpha\underline{X}^\alpha$, $\alpha \in \Omega\backslash\Delta$, are squares of monomials, by definition of Δ, this implies that f is sobs. □

Corollary 2.4. *For any polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$, if*

$$\text{(L1)} \quad f_0 \geq \sum_{\alpha\in\Delta} |f_\alpha|^{\frac{2d-|\alpha|}{2d}} \quad \textit{and} \quad \text{(L2)} \quad f_{2d,i} \geq \sum_{\alpha\in\Delta} |f_\alpha|\frac{\alpha_i}{2d}, \quad i = 1,\ldots,n,$$

then f is a sum of squares.

Proof. Apply Th. 2.3 to the homogenization $\tilde{f}$ of f to conclude that $\tilde{f}$ is sos. Consequently, by Prop. 1.1, f is also sos. □

• In [6, Th. 3], it is proved that if

$$f_0 \geq \sum_{\alpha\in\Delta} |f_\alpha| \text{ and } f_{2d,i} \geq \sum_{\alpha\in\Delta} |f_\alpha| \frac{|\alpha|}{2d}, \quad i = 1, \dots, n,$$

then f is a sum of squares. Since $1 \geq \frac{2d-|\alpha|}{2d}$ and $\frac{|\alpha|}{2d} \geq \frac{\alpha_i}{2d}$, it is clear that Cor. 2.4 improves on [6, Th. 3].

Theorem 2.5. *Suppose $f \in \mathbb{R}[\underline{X}]$ is a form of degree $2d$ and*

$$\min_{i=1,\dots,n} f_{2d,i} \geq \frac{1}{2d} \sum_{\alpha\in\Delta} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}.$$

Then f is a sum of (binomial) squares.

Here, $\alpha^\alpha := \alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}$ (the convention being that $0^0 := 1$).

Proof. Let $e_\alpha := \frac{1}{2d} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}$. We claim that

$$e_\alpha \sum_{i=1}^{n} X_i^{2d} + f_\alpha \underline{X}^\alpha$$

is sobs, for each $\alpha \in \Delta$. Since $e_\alpha \geq 0$, $e_\alpha \sum_{\alpha_i = 0} X_i^{2d}$ is sobs, so it suffices to show that $e_\alpha \sum_{\alpha_i \neq 0} X_i^{2d} - f_\alpha \underline{X}^\alpha$ is sobs. Since

$$2d \prod_{\alpha_i \neq 0} (\frac{e_\alpha}{\alpha_i})^{\frac{\alpha_i}{2d}} = \frac{2de_\alpha}{(\alpha^\alpha)^{\frac{1}{2d}}} = |f_\alpha| \geq |f_\alpha|,$$

and since $f_\alpha < 0$ if all the α_i are even, by definition of Δ, this follows from Cor. 2.2. This proves the claim. Adding, this implies

$$\sum_{\alpha \in \Delta} e_\alpha \sum_{i=1}^n X_i^{2d} + \sum_{\alpha \in \Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Since $f_{2d,i} \geq \sum_{\alpha \in \Delta} e_\alpha$, for each i,

$$\sum_{i=1}^n f_{2d,i} X_i^{2d} - \sum_{\alpha \in \Delta} e_\alpha \sum_{i=1}^n X_i^{2d} = \sum_{i=1}^n (f_{2d,i} - \sum_{\alpha \in \Delta} e_\alpha) X_i^{2d}$$

is sobs. Adding again, this implies

$$\sum_{i=1}^n f_{2d,i} X_i^{2d} + \sum_{\alpha \in \Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Finally, since the remaining terms $f_\alpha \underline{X}^\alpha$, $\alpha \in \Omega \backslash \Delta$, are squares of monomials, this implies f is sobs. □

- In [2, Th. 4.3] it is proved that if $f \in \mathbb{R}[\underline{X}]$ is any form of degree $2d$ and

$$\min_{i=1,\ldots,n} f_{2d,i} \geq \frac{1}{n}(\frac{n}{2d})^{2d} \sum_{\alpha \in \Delta} |f_\alpha| \alpha^\alpha$$

then f is a sum of squares. Using $\alpha^\alpha \geq (\frac{2d}{n})^{2d}$, one sees immediately that

$$\frac{1}{n}(\frac{n}{2d})^{2d} \alpha^\alpha \geq \frac{1}{2d}(\alpha^\alpha)^{\frac{1}{2d}}.$$

Consequently, Th. 2.5 improves on [2, Th. 4.3]. The fact that $\alpha^\alpha \geq (\frac{2d}{n})^{2d}$ is an immediate consequence of the fact that the minimum value of the function

$$G(t_1, \ldots, t_n) := t_1^{t_1} \cdots t_n^{t_n}$$

on the compact subset of $\mathbb{R}^n$ defined by $t_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n t_i = 2d$ is equal to $(\frac{2d}{n})^{2d}$, the minimum occurring at the point $t_1 = \cdots = t_n = \frac{2d}{n}$.

Corollary 2.6. *If $f \in \mathbb{R}[\underline{X}]$ is a polynomial of degree $2d$ and*

$$\text{(FK)} \qquad \min_{i=1,\ldots,n} \{f_{2d,i}, f_0\} \geq \frac{1}{2d} \sum_{\alpha \in \Delta} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}} (2d - |\alpha|)^{\frac{2d-|\alpha|}{2d}}$$

then f is a sum of squares.

Proof. Homogenize f and apply Th. 2.5 and Prop. 1.1. □

• Recall that $\Sigma^\circ_{2d,n}$ (resp., $P^\circ_{2d,n}$) denotes the interior of the cone $\Sigma_{2d,n}$ (resp., $P_{2d,n}$) in the real vector space consisting of forms of degree $2d$. The following result is well-known. It is proved, for example, in [8, Prop. 5.3(2)].

Corollary 2.7. $X_1^{2d} + \cdots + X_n^{2d} \in \Sigma^\circ_{2d,n}$.

Proof. Let $f(\underline{X}) = X_1^{2d} + \cdots + X_n^{2d} + h(\underline{X})$ where $h(\underline{X})$ is any form of degree $2d$ whose coefficients have absolute value $\le \epsilon$ where ϵ is some small positive real. Applying Th. 2.3 or Th. 2.5, one sees that f is sos, for ϵ sufficiently small. □

Remark 2.8. Let C be a cone in a finite dimensional real vector space V. Let C° denote the interior of C. If $f \in C^\circ$ and $g \in V$ then $g \in C^\circ$ iff $g - \epsilon f \in C$ for some real $\epsilon > 0$.

Proof. Suppose $g - \epsilon f \in C$. Let $h \in V$. Since f belongs to the interior of C, there exists some real $\delta > 0$ such that $f + \frac{\delta}{\epsilon} h \in C$. Then $g + \delta h = (g - \epsilon f) + \epsilon(f + \frac{\delta}{\epsilon} h) \in C$. This proves that g belongs to the interior of C. The other implication is clear. □

• It follows from Cor. 2.7 and Rem. 2.8 that a form f of degree $2d$ is an interior point of $\Sigma_{2d,n}$ iff $f - \epsilon \sum_{i=1}^n X_i^{2d} \in \Sigma_{2d,n}$ for some real $\epsilon > 0$.

• Ths. 2.3 and 2.5 provide sufficient conditions for $f \in \Sigma^{\circ}_{2d,n}$ to hold and have the nice additional property of allowing computation of ϵ:

Corollary 2.9. *If f is a form of degree $2d$ and $\epsilon := \max\{\epsilon_1, \epsilon_2\} > 0$ where*

$$\epsilon_1 := \min_{i=1,\dots,n}(f_{2d,i} - \sum_{\alpha\in\Delta} |f_\alpha|\frac{\alpha_i}{2d}),\ \ \epsilon_2 := \min_{i=1,\dots,n} f_{2d,i} - \frac{1}{2d}\sum_{\alpha\in\Delta}|f_\alpha|(\alpha^\alpha)^{\frac{1}{2d}},$$

then $f \in \Sigma^{\circ}_{2d,n}$ and $f - \epsilon\sum_{i=1}^{n} X_i^{2d} \in \Sigma_{2d,n}$.

Proof. Applying Th. 2.3 or Th. 2.5 (depending on whether $\epsilon = \epsilon_1$ or $\epsilon = \epsilon_2$) to the form $f - \epsilon\sum_{i=1}^{n} X_i^{2d}$, we see that $f - \epsilon\sum_{i=1}^{n} X_i^{2d}$ is sos. □

3 Determining lower bounds

• In this section we assume $f_{2d} \in \Sigma^{\circ}_{2d,n}$ and we use Cor. 2.4 and Cor. 2.6 to produce concrete lower bounds for f_{sos}, which we denote by r_L and r_{FK}, respectively. We also apply Cor. 2.2 more or less directly to produce another concrete lower bound for f_{sos}, which we denote by r_{dmt}.

• Our lower bounds r_L, r_{FK} and r_{dmt} depend on the coefficients f_α, $\alpha \in \Delta$, $|\alpha| < 2d$, and ϵ, where ϵ is such that $\epsilon > 0$ and $f_{2d} - \epsilon\sum_{i=1}^{n} X_i^{2d} \in \Sigma_{2d,n}$. Existence of ϵ is a consequence of Cor. 2.7 and Rem. 2.8. I'll say more about ϵ and r_L, r_{FK} and r_{dmt} in Section 4.

• We use Cor. 2.4 to produce a concrete lower bound r_L for f_{sos} as follows:

Theorem 3.1. *If $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then $f_{sos} \geq r_L$, where*

$$r_L := f_0 - \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|},$$

$$k := \max_{i=1,\dots,n} C(t^{2d} - \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{\alpha_i}{2d} \epsilon^{-\frac{|\alpha|}{2d}} t^{|\alpha|})$$

and $\epsilon > 0$ is such that $f_{2d} - \epsilon \sum_{i=1}^{n} X_i^{2d} \in \Sigma_{2d,n}$.

• Notes:

— Th. 3.1 proves in particular that if $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then $f_{sos} \neq -\infty$, i.e., it provides another proof of [8, Prop. 5.1].

— If $\ell \geq k$ then

$$f_0 - \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} \ell^{|\alpha|} \leq f_0 - \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|} = r_L.$$

In this way, by taking ℓ to be an upper bound for k computed using Prop. 1.2, we obtain a lower bound $f_0 - \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} \ell^{|\alpha|}$ for f_{sos} which is expressible in an elementary way in terms of ϵ and the coefficients f_α, $\alpha \in \Delta$, $|\alpha| < 2d$.

Proof. Since $f_{2d} \in \Sigma^{\circ}_{2d,n}$, by Cor. 2.7 and Rem. 2.8, there exists $\epsilon > 0$ such that $f_{2d} = \epsilon(X_1^{2d} + \cdots + X_n^{2d}) + g$ for some $g \in \Sigma_{2d,n}$. Scaling suitably ($X_i \mapsto \frac{X_i}{\sqrt[2d]{\epsilon}}$), we can assume that $\epsilon = 1$. Let $\hat{f} := f - g$. Decomposing $\hat{f}$ as in equation (2) yields

$$\hat{f} = f_0 + \sum_{\alpha \in \Omega, |\alpha| < 2d} f_\alpha \underline{X}^\alpha + \sum_{i=1}^{n} X_i^{2d}. \tag{3}$$

If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $\hat{f} - r_L = \hat{f} - f_0$ is sos, using equation (3) and the definition of Δ, so $f - r_L$ is also sos and the result is clear. Thus we can assume $\{\alpha \in \Delta \mid |\alpha| < 2d\} \neq \emptyset$, so $k > 0$. Scaling by $X_i \mapsto kX_i$, and rewriting condition (L2) of Cor. 2.4 for the polynomial $\hat{f}(k\underline{X}) - r$, using equation (3), yields

$$k^{2d} \geq \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{\alpha_i}{2d} k^{|\alpha|}, \quad i = 1, \ldots, n.$$

By definition of k, $k^{2d} \geq \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{\alpha_i}{2d} k^{|\alpha|}$ for all i, so condition (L2) holds for $\hat{f}(k\underline{X}) - r$. Rewriting condition (L1) of Cor. 2.4 for the polynomial $\hat{f}(k\underline{X}) - r$, we see that if $r \leq r_L$ then (L1) holds for $\hat{f}(k\underline{X}) - r$ so $\hat{f} - r$ is sos and hence also $f - r$ is sos. □

- In a similar way, we use Cor. 2.6 to produce a concrete lower bound r_{FK} for f_{sos}:

Theorem 3.2. *If* $f_{2d} \in \Sigma^{\circ}_{2d,n}$ *then* $f_{sos} \geq r_{FK}$*, where* $r_{FK} := f_0 - k^{2d}$*,*

$$k := C(t^{2d} - \sum_{i=1}^{2d-1} b_i t^i),$$
$$b_i := \frac{1}{2d}(2d-i)^{\frac{2d-i}{2d}} \epsilon^{-\frac{i}{2d}} \sum_{\alpha \in \Delta, |\alpha| = i} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}, \; i = 1, \dots, 2d-1$$

and $\epsilon > 0$ *is given as in Th. 3.1.*

- Note: If $\ell \geq k$ then

$$f_0 - \sum_{i=1}^{2d-1} b_i \ell^i \leq f_0 - \sum_{i=1}^{2d-1} b_i k^i = f_0 - k^{2d} = r_{FK}$$

so, using Prop. 1.2 again, we get another lower bound for f_{sos} expressible in an elementary way in terms of ϵ and the coefficients f_α, $\alpha \in \Delta$, $|\alpha| < 2d$.

Proof. After scaling we can assume that $\epsilon = 1$ and $f_{2d} = X_1^{2d} + \cdots + X_n^{2d} + g$, where $g \in \Sigma_{2d,n}$. If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $b_i = 0$ for $i = 1, \dots 2d-1$, $k = 0$ (by definition of $C(t^{2d})$), so $r_{FK} = f_0$. In this case the result is clear. So we can assume $\{\alpha \in \Delta \mid |\alpha| < 2d\} \neq \emptyset$, so $k > 0$. Set $r = r_{FK}$. Rewriting condition (FK) for the polynomial $\hat{f}(k\underline{X}) - r$, where $\hat{f} := f - g$, yields the condition:

$$\min\{(f_0 - r), k^{2d}\} \geq \sum_{i=1}^{2d-1} b_i k^i. \tag{4}$$

By definition of k and r, (4) holds, in fact, $f_0 - r = k^{2d} = \sum_{i=1}^{2d-1} b_i k^i$. This proves that $\hat{f} - r$ is sos and hence also that $f - r$ is sos. □

- One can also apply Cor. 2.2 directly to obtain a lower bound r_{dmt} for f_{sos}.

Theorem 3.3. *If $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then*

$$f_{sos} \geq r_{dmt} := f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} (2d - |\alpha|) \left[\left(\frac{f_\alpha}{2d}\right)^{2d} ((\frac{t}{\epsilon})^{|\alpha|} \alpha^\alpha) \right]^{\frac{1}{2d-|\alpha|}},$$

where $t := |\{\alpha \in \Delta \mid |\alpha| < 2d\}|$ and $\epsilon > 0$ is given as in Th. 3.1.

Proof. Let $\Delta' = \{\alpha \in \Delta \mid |\alpha| < 2d\}$. After scaling, we can assume that $\epsilon = 1$. Let $\overline{f} = f_0 + \sum_{\alpha\in\Delta'} f_\alpha \underline{X}^\alpha + X_1^{2d} + \cdots + X_n^{2d}$ and let $F(\underline{X}, Y)$ denote the homogenization of

$\overline{f}(\sqrt[2d]{t}\underline{X}) - r$, where $r := f_0 - \sum_{\alpha\in\Delta'} r_\alpha$, each $r_\alpha \geq 0$. Then

$$\begin{aligned} F(\underline{X}, Y) &= (f_0 - r)Y^{2d} + \sum_{\alpha\in\Delta'}(X_1^{2d} + \cdots + X_n^{2d} + f_\alpha t^{|\alpha|/2d}\underline{X}^\alpha Y^{2d-|\alpha|}) \\ &= \sum_{\alpha\in\Delta'}(r_\alpha Y^{2d} + X_1^{2d} + \cdots + X_n^{2d} + f_\alpha t^{|\alpha|/2d}\underline{X}^\alpha Y^{2d-|\alpha|}). \end{aligned}$$

By Cor. 2.2, each term appearing in this sum will be sos if

$$|f_\alpha| t^{\frac{|\alpha|}{2d}} \leq 2d\left(\frac{r_\alpha}{2d - |\alpha|}\right)^{\frac{2d-|\alpha|}{2d}} \prod_{\alpha_i\neq 0}\left(\frac{1}{\alpha_i}\right)^{\frac{\alpha_i}{2d}},$$

or, equivalently, if

$$r_\alpha \geq (2d - |\alpha|)\left[\left(\frac{f_\alpha}{2d}\right)^{2d} t^{|\alpha|}\alpha^\alpha\right]^{\frac{1}{2d-|\alpha|}}.$$

Hence if $r \leq r_{dmt}$ then $\overline{f} - r$ is sos, so also $f - r$ is sos. $\square$

4 Further remarks

(1) The sufficient conditions given in Ths. 2.3 and 2.5 are not comparable. These conditions are also not necessary.

Example 4.1.

(a) $f(X,Y,Z) = X^4 + Y^4 + 4Z^4 + 4XZ^3$ is sos, by Th. 2.3, but Th. 2.5 does not apply.

(b) $f(X,Y,Z) = X^4 + Y^4 + Z^4 + \sqrt{8}XYZ^2$ is sos, by Th. 2.5, but Th. 2.3 does not apply.

(c) $f(X,Y,Z) = 16X^4 + Y^4 + 4Z^4 + 8XZ^3$ is sos, but neither Th. 2.3 nor Th. 2.5 applies.

(2) The bounds r_L, r_{FK} and r_{dmt} described in Ths. 3.1, 3.2 and 3.3 are not comparable.

Example 4.2.

(a) For $f(X,Y) = X^6 + Y^6 + 7XY - 2X^2 + 7$, we have $r_L \approx -1.124$, $r_{FK} \approx -0.99$ and $r_{dmt} \approx -1.67$, so $r_{FK} > r_L > r_{dmt}$.

(b) For $f(X,Y) = X^6 + Y^6 + 4XY + 10Y + 13$, $r_L \approx -0.81$, $r_{FK} \approx -0.93$ and $r_{dmt} \approx -0.69$, so $r_{dmt} > r_L > r_{FK}$.

(c) For $f(X,Y) = X^4 + Y^4 + XY - X^2 - Y^2 + 1$, $r_L \approx -0.125$, $r_{FK} \approx -0.832$ and $r_{dmt} \approx -0.875$, so $r_L > r_{FK} > r_{dmt}$.

(3) To be able to compute r_L, r_{FK} and r_{dmt} one needs to know ϵ and the coefficients f_α, $|\alpha| < 2d$. What can one do if ϵ is not given, i.e., if only the coefficients f_α, $|\alpha| \le 2d$ are given? Applying Cor. 2.9 to the form f_{2d} allows us to compute ϵ in certain cases: If $\epsilon := \max\{\epsilon_1, \epsilon_2\} > 0$ where

$$\epsilon_1 := \min_{i=1,\dots,n} (f_{2d,i} - \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha| \frac{\alpha_i}{2d}), \ \ \epsilon_2 := \min_{i=1,\dots,n} f_{2d,i} - \frac{1}{2d} \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}},$$

then $f_{2d} \in \Sigma^\circ_{2d,n}$ and $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in \Sigma_{2d,n}$.

(4) So far we have been assuming that $f_{2d} \in \Sigma^\circ_{2d,n}$ and we have used this assumption to determine lower bounds for f_{sos}. What can one say if one assumes only that $f_{2d} \in P^\circ_{2d,n}$? Suppose $\epsilon > 0$ is given such that $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in P_{2d,n}$. One can then define r_L exactly as in Th. 3.1, but using this new ϵ, i.e.,

$$r_L := f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|},$$

$$k := \max_{i=1,\dots,n} C(t^{2d} - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{\alpha_i}{2d} \epsilon^{-\frac{|\alpha|}{2d}} t^{|\alpha|}).$$

The r_L defined in this way might not be a lower bound for f_{sos} (it is even possible that

$f_{sos} = -\infty$), but it will be a lower bound for f_*. Similar remarks apply to the other bounds r_{FK} and r_{dmt}.

Theorem 4.3. *If $f_{2d} \in P^\circ_{2d,n}$ and $\epsilon > 0$ is such that $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in P_{2d,n}$ then r_L, r_{FK} and r_{dmt}, defined as in Ths. 3.1, 3.2 and 3.3, respectively, but using this new choice of ϵ, are lower bounds for f on $\mathbb{R}^n$.*

Proof. Argue as in the proof of Ths. 3.1, 3.2 and 3.3. The form g is no longer sos but it is positive semidefinite, which is all one needs for the conclusion. □

Note: In Th. 4.3, the largest possible choice for ϵ is the minimum value of the rational function $f_{2d}/\sum_{i=1}^n X_i^{2d}$ on the $n-1$-sphere

$$\mathbb{S}^{n-1} := \{\underline{a} \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 = 1\}.$$

(5) Denote by $\mathbb{R}[\underline{X}]_k$ the vector space of polynomials of degree $\leq k$. We know that for any $p \in P^\circ_{2d,n}$ and any $g \in \mathbb{R}[\underline{X}]_{2d-1}$, $(p+g)_* \neq -\infty$ and, for any $p \in \Sigma^\circ_{2d,n}$ and any $g \in \mathbb{R}[\underline{X}]_{2d-1}$, $(p+g)_{sos} \neq -\infty$. Note that if $p \in P_{2d,n}$ is not positive definite then there exists $\underline{0} \neq \underline{a} \in \mathbb{R}^n$ such that $p(\underline{a}) = 0$. Let $g(\underline{X}) = \sum_{i=1}^n a_i X_i$. Then $(p+g)(t\underline{a}) = t\|\underline{a}\|^2 \to -\infty$ as $t \to -\infty$, so $(p+g)_* = -\infty$. Therefore for any $p \in \partial P_{2d,n}$ ($\partial P_{2d,n}$ denotes the boundary of $P_{2d,n}$, i.e. $\partial P_{2d,n} = P_{2d,n} \setminus P^\circ_{2d,n}$), there exists $g \in \mathbb{R}[\underline{X}]_{2d-1}$, such that $(p+g)_* = -\infty$. The validity of the corresponding result for boundary points of $\Sigma_{2d,n}$ is unknown to the authors.

Question 4.4. Is it true that for any $p \in \partial\Sigma_{2d,n}$ there exists $g \in \mathbb{R}[\underline{X}]_{2d-1}$ such that $(p+g)_{sos} = -\infty$?

The answer to this question is 'yes' if $n \leq 2$ or $d = 1$ or ($n = 3$ and $d = 2$) by Hilbert's result [3]. In fact these are precisely the cases where $P_{2d,n}$ and $\Sigma_{2d,n}$ coincide.

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