Spectrahedra and their Projections

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Convex Algebraic Geometry
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Introduction

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find the infimum/supremum that ℓ takes on S, and possibly a set of points where an optimum is attained.

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My talk will be about these kinds of sets.

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Let $A_0, \dots, A_n \in \mathrm{Sym}_k(\mathbb{R})$ be real symmetric matrices. For $x \in \mathbb{R}^n$ write

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Then

$$A(x) = A_0 + x_1 A_1 + x_2 A_2 = \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix} \succeq 0$$

if and only if $x_1^2 + x_2^2 \leq 1$.

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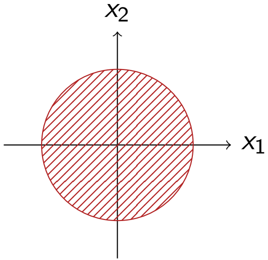
- A spectrahedron:



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- Spectrahedra have only exposed faces.

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  This follows from the fact that real symmetric matrices have only real Eigenvalues.

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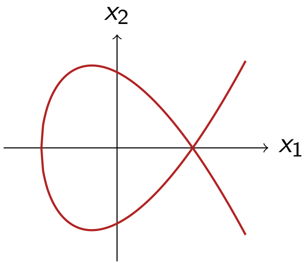

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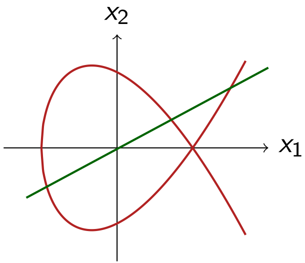

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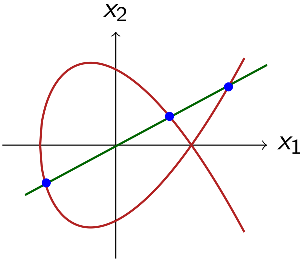

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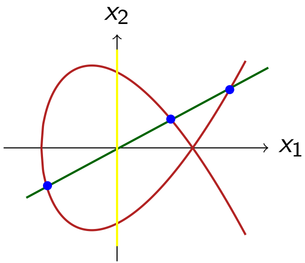

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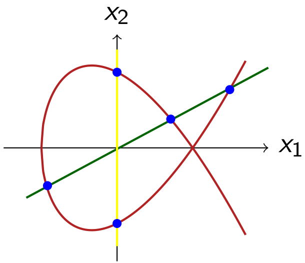

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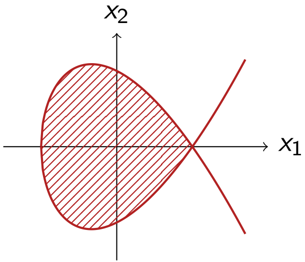

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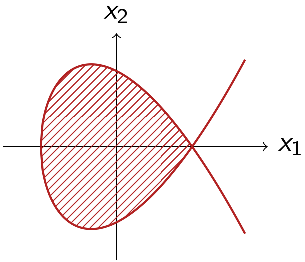

Theorem (Helton & Vinnikov, 2006)

Every spectrahedron is rigidly convex. Every rigidly convex set in $\mathbb{R}^2$ is a spectrahedron.

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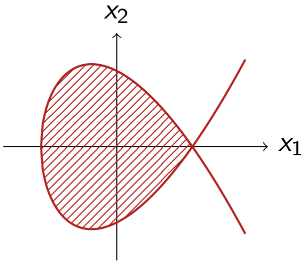

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This solves the Lax-Conjecture, as observed by Lewis, Parrilo & Ramana.

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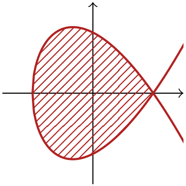

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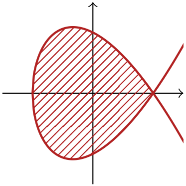

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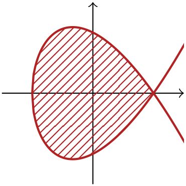

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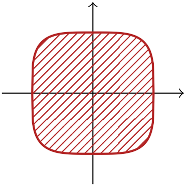

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$$S = L(\widetilde{S}) \text{ an sdr set}$$

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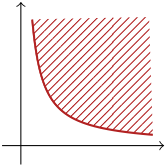

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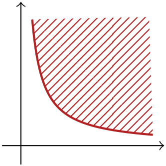

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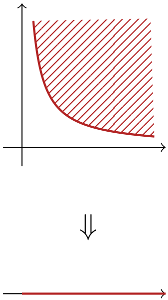

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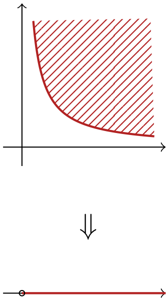

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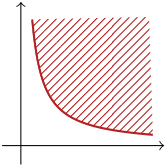

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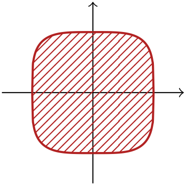

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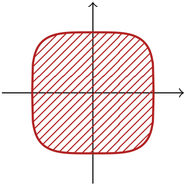

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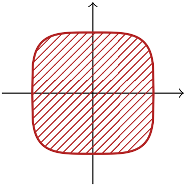

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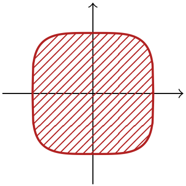

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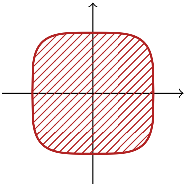

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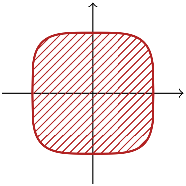

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Introduction: Semidefinitely Representable Sets

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Introduction: Semidefinitely Representable Sets

Question/Conjecture (Nemirovski, Helton & Nie):

Is every convex semi-algebraic set semidefinitely representable?

Constructions

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- The convex hull of a finite union of sdr sets is sdr.

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This observation gives us a method to construct sdr sets!

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- So $L(p)_d := \mathrm{QM}(p)_d \cap \mathbb{R}[x]_1$ is a semidefinitely representable subset of $\mathbb{R}[x]_1$.

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Thus

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If there is some $d \in \mathbb{N}$ *such that every* $\ell \in \mathbb{R}[x]_1$ *that is nonnegative on* S *belongs to* $\mathrm{QM}(p)_d$ *then* $\overline{\mathrm{conv}(S)}$ *is semidefinitely representable.*

Constructions II: The Lasserre-Parrilo Method

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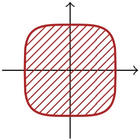

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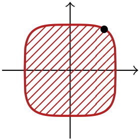

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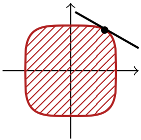

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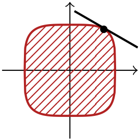

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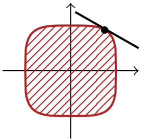

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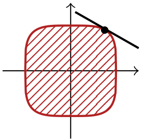

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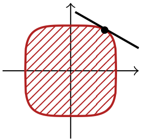

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If S is convex and $\mathrm{QM}(p)_d$ contains every linear polynomial that is nonnegative on S, then S is sdr.

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But the converse is true in a more general context!

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Now let $q^{(j)} = (q_1^{(j)}, \ldots, q_k^{(j)})$ be k-tuples of polynomials, i.e. $q_i^{(j)} \in \mathbb{R}[x]$.

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Now let $q^{(j)} = (q_1^{(j)}, \ldots, q_k^{(j)})$ be k-tuples of polynomials, i.e. $q_i^{(j)} \in \mathbb{R}[x]$. The polynomial

$$\sum_j q^{(j)} A(x) q^{(j)^t} \in \mathbb{R}[x]$$

is nonnegativ on S, as long as $\sum_j q^{(j)} B_i q^{(j)^t} = 0$ for all $i = 1, \ldots, m$.

Interlude: An Interesting Quadratic Module

Let S be semidefinitely representable, so

$$S = \{x \in \mathbb{R}^n \mid \exists y : \; A_0 + x_1 A_1 + \cdots + x_n A_n + y_1 B_1 + \cdots + y_m B_m \succeq 0\},$$

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So let $\mathrm{QM}(A)$ be the quadratic module generated by these polynomials.

Interlude: An Interesting Quadratic Module

Formally:

$$\mathrm{QM}(A) = \{\sum_j q^{(j)} A(x) q^{(j)} + \sigma \mid q^{(j)} \in \mathbb{R}[x]^k, \sum_j q^{(j)} B_i q^{(j)^t} = 0 \text{ for all } i,$$
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- Whenever S is bounded, then $\mathrm{QM}(A)$ is Archimedean, and thus contains every polynomial p with $p \geq \varepsilon$ on S for some $\varepsilon > 0$.

Interlude: An Interesting Quadratic Module

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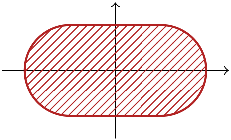

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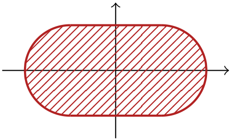

The corresponding spectrahedron in $\mathbb{R}^3$ is defined by the linear matrix polynomial

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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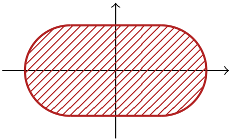

So every polynomial that is strictly positive on S is a sum of squares plus a polynomial of the following form:

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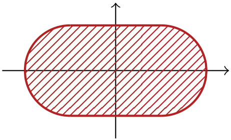

So every polynomial that is strictly positive on S is a sum of squares plus a polynomial of the following form:

$$\sum_j q_1^{(j)^2} + q_2^{(j)^2} + q_3^{(j)^2} + q_4^{(j)^2} + x_1(q_3^{(j)^2} - q_4^{(j)^2}) + x_2(q_1^{(j)^2} - q_2^{(j)^2}),$$

where $q_i^{(j)} \in \mathbb{R}[x_1, x_2]$ with $\sum_j 2q_1^{(j)} q_2^{(j)} - q_3^{(j)^2} - q_4^{(j)^2} = 0$.

Constructions II: The Results of Helton & Nie

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Theorem (Helton & Nie)

If for each p_i, the negative Hessian matrix is either a sum of squares of polynomial matrices, or positive definite on the tangent space of p_i at each point of S, then S is semidefinitely representable.

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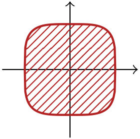

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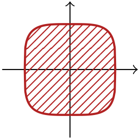

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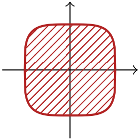

For the Hessian of $p = 1 - x_1^4 - x_2^4$ we find

$$-\nabla^2 p = \begin{pmatrix} 12x_1^2 & 0 \\ 0 & 12x_2^2 \end{pmatrix}$$

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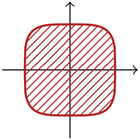

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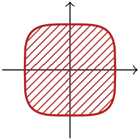

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$\Rightarrow S$ is sdr.

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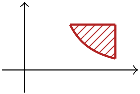

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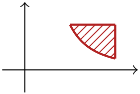

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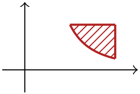

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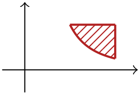

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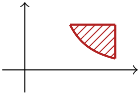

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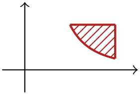

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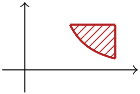

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So S is sdr.

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Both results use representations of nonnegative polynomials as sums of squares, together with degree bounds (Kuhlmann, Marshall & Schwartz; Scheiderer).

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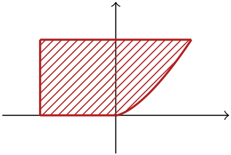

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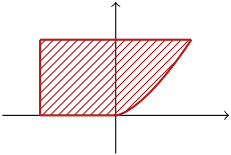

  S is bounded by segments of rational curves, and thus semidefinitely representable.

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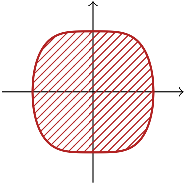

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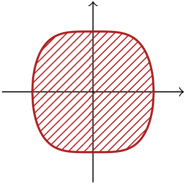

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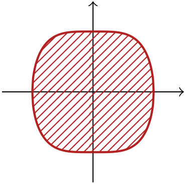

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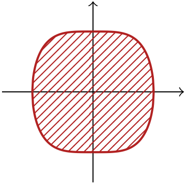

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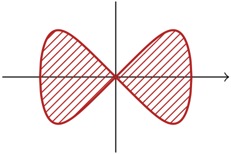

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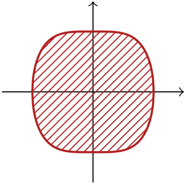

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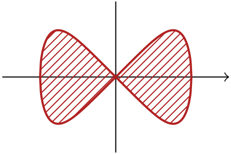

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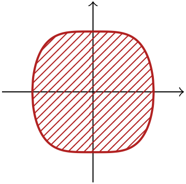

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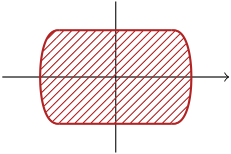

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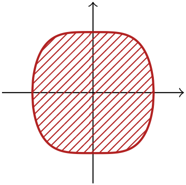

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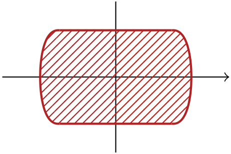

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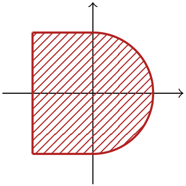

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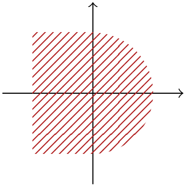

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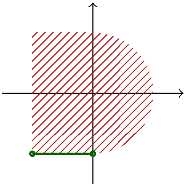

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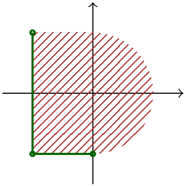

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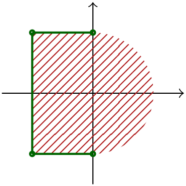

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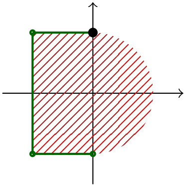

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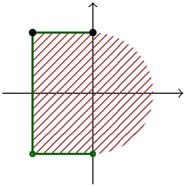

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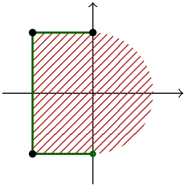

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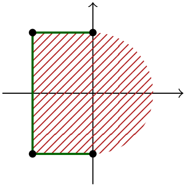

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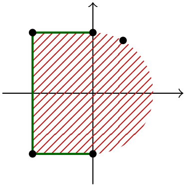

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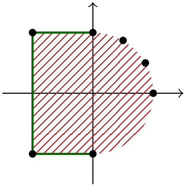

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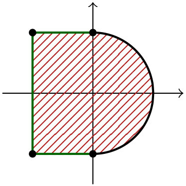

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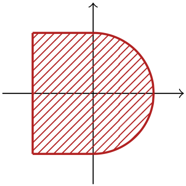

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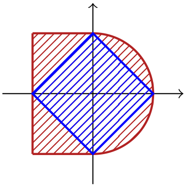

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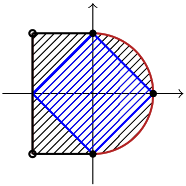

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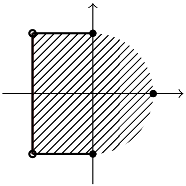

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Take $x \in \text{relint}(S)$. Then $\text{relint}(S) = \{x\} \looparrowleft S$. □

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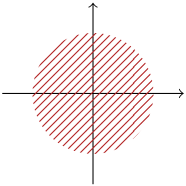

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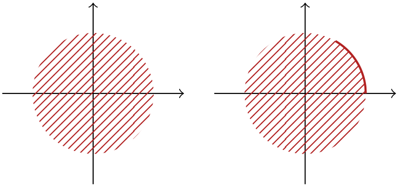

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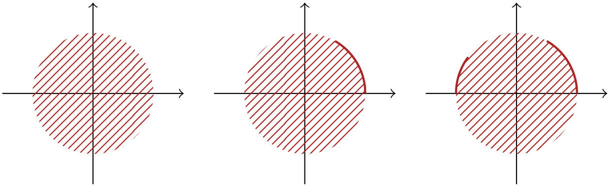

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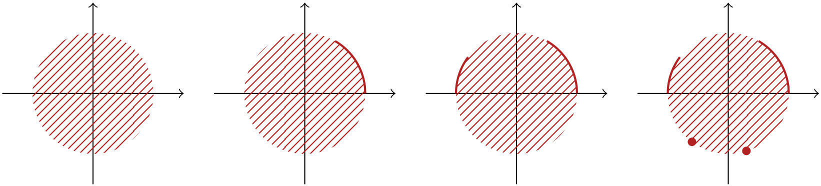

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Let $A \in \mathrm{Sym}_k(\mathbb{R})$ and $B \in \mathbb{R}^{m \times k}$. Let I_m denote the identity matrix of dimension m. Then the following are equivalent:

(i) *there is some $\lambda \in \mathbb{R}$ such that* $\left(\begin{array}{c|c} A & B^t \\ \hline B & \lambda \cdot I_m \end{array}\right) \succeq 0$

(ii) $A \succeq 0$ *and* $\ker A \subseteq \ker B$

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The case when S is sdr but not a spectrahedron can then be reduced to the above case. □

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The case when S is sdr but not a spectrahedron can then be reduced to the above case. □

Note: The proof gives an explicit construction of a spectrahedron projecting to $T \looparrowleft S$.

Non-Closed Sets: Sketch of Proof

Theorem

Suppose $T \subseteq S$ are both semidefinitely representable sets. Then

$$T \looparrowleft S$$

is again semidefinitely representable.

Proof of the Theorem.

The case when S is sdr but not a spectrahedron can then be reduced to the above case. □

Note: The proof gives an explicit construction of a spectrahedron projecting to $T \looparrowleft S$. One can for example see that rational coefficients in the representations of T and S are preserved.

Future Work/Open Problems

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- What about the complexity of semidefinite representations? How many additional variables are needed?

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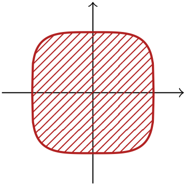
Can anyone prove that the set $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 + x_2^4 \leq 1\}$ is not the projection of a spectrahedron from $\mathbb{R}^3$?



Thank you for your attention!