Exposed faces and projections of spectrahedra

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Spectrahedra

A spectrahedron is the set of solutions to a linear matrix inequality:
Let $A_0, \ldots, A_n \in \mathrm{Sym_k}(\mathbb{R})$ be symmetric $k \times k$-matrices, and let

$$A(\underline{t}) = A_0 + t_1 A_1 + \cdots + t_n A_n = \begin{pmatrix} \ell_{11}(\underline{t}) & \cdots & \ell_{1k}(\underline{t}) \\ \vdots & \ddots & \vdots \\ \ell_{k1}(\underline{t}) & \cdots & \ell_{kk}(\underline{t}) \end{pmatrix}$$

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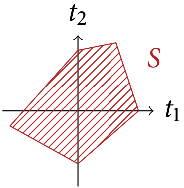
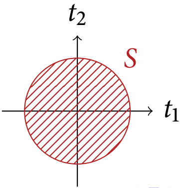


Polyhedra
$A(\underline{t})$ diagonal



$$A(t_1, t_2) = \begin{pmatrix} 1 - t_1 & t_2 \\ t_2 & 1 + t_1 \end{pmatrix}$$

Spectrahedra

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  Take the characteristic polynomial

  $$\det(A(\underline{t}) - sI_k) = (-1)^{k+1} s^k + c_{k-1}(\underline{t}) s^{k-1} + \cdots + c_o(\underline{t})$$

  with $c_i \in \mathbb{R}[\underline{t}]$, then

  $$S = \left\{ x \in \mathbb{R}^n \mid c_0(x) \geqslant 0, -c_1(x) \geqslant 0, \dots, (-1)^{k-1} c_{k-1}(x) \geqslant 0 \right\}$$

Projections of spectrahedra

$A_0, \ldots, A_m \in \mathrm{Sym}_k(\mathbb{R}), A(\underline{t}) = A_0 + t_1 A_1 + \cdots + t_m A_m$

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Let $\pi \colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear map. The image $\pi(S)$ is a projection of a spectrahedron.

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$$\pi(S) = \left\{ x \in \mathbb{R}^n \middle| \exists y \in \mathbb{R}^l \colon A_0 + x_1 A_1 + \cdots + x_n A_n + y_1 A_{n+1} + \cdots + y_l A_m \text{ is psd} \right\}.$$

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Projections of spectrahedra are also called semidefinitely representable sets or SDP (representable) sets.

The Lasserre relaxation

Let $C = \{x \in \mathbb{R}^n \mid p_1(x) \geqslant 0, \ldots, p_r(x) \geqslant 0\}$ be a basic closed semi-algebraic set. Always assume C convex with non-empty interior.

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Put $p_0 = 1$ and let

$$M = \left\{ \sum_{i=0}^{r} (s_{i1}^2 + \cdots + s_{iN}^2) p_i \;\middle|\; s_{ij} \in \mathbb{R}[\underline{t}] \right\}$$

be the quadratic module generated by $p_1, \ldots, p_r$. Write $\mathbb{R}[\underline{t}]_d$ for the space of polynomials of degree at most d.

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Note: $M_d \subsetneq M \cap \mathbb{R}[\underline{t}]_d$ in general.

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For $d \geqslant 1$, write $\mathbb{R}[\underline{t}]_d^\vee$ for the dual space of $\mathbb{R}[\underline{t}]_d$, i.e. the space of linear functionals $L\colon \mathbb{R}[\underline{t}]_d \to \mathbb{R}$, and put

$$\mathcal{L}_d = \left\{L \in \mathbb{R}[\underline{t}]_d^\vee \mid L(f) \geqslant 0 \text{ for all } f \in M_d \text{ and } L(1) = 1\right\}.$$

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Let $\pi \colon \mathbb{R}[\underline{t}]_d^\vee \to \mathbb{R}^n$ be given by $L \mapsto (L(t_1), \dots, L(t_n))$ and write $C_d = \pi(\mathcal{L}_d)$.

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$$C \subset \cdots \subset C_d \subset C_{d-1} \subset \cdots \subset C_1$$

Call C_d the Lasserre relaxation of degree d of C (w.r.t. $p_1, \ldots, p_r$).

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Theorem
The Lasserre relaxtion of degree d *of* C *is exact if and only if* M_d *contains all* $\ell \in \mathbb{R}[\underline{t}]$ *of degree* 1 *such that* $\ell|_C \geqslant 0$.

Exposed faces

Let C be a convex subset of $\mathbb{R}^n$. A face of C is a convex subset F of C which is extremal, i.e. whenever $x, y \in C$ are such that $\frac{1}{2}(x+y) \in F$, then $x, y \in F$.

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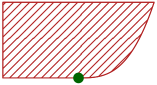
A face is called exposed if $F = \varnothing$ or if there exists a supporting hyperplane H of C such that $F = H \cap C$. (Equivalently: If there exists $\ell \in \mathbb{R}[\underline{t}]$ of degree 1 such that $\ell|_C \geqslant 0$ and $F = C \cap \{x \in \mathbb{R}^n \mid \ell(x) = 0\}$.



Spectrahedra vs. Convex semi-algebraic sets

All faces exposed?

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$$\left\{\text{Spectrahedra}\right\}$$

YES

Ramana & Goldman
2001

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$$\left\{\text{Spectrahedra}\right\} \subsetneq \left\{\begin{array}{l}\text{Rigidly}\\ \text{convex sets}\end{array}\right\}$$

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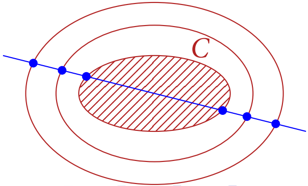
Rigid convexity: A set $C \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(C)$ is called rigidly convex if there exists a polynomial $p \in \mathbb{R}[\underline{t}]$ with $p(0)$ such that C is the closure of the connected component of $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ that contains 0 and such that the univariate polynomial $s \mapsto p(sv)$ has only real zeros, for every $v \in \mathbb{R}^n \setminus \{0\}$.

Every spectrahedron is rigidly convex. The converse is true for $n = 2$ (Helton & Vinnikov 2004)



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Renegar 2006

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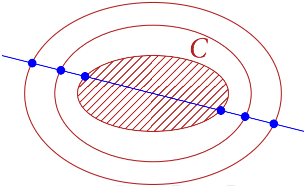

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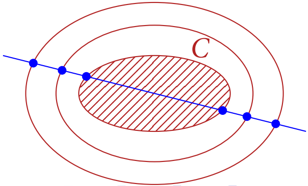

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All faces exposed?

YES

$\subsetneq$ $\left\{\begin{array}{l}\text{Lasserre-}\\ \text{exact sets}\end{array}\right\}$

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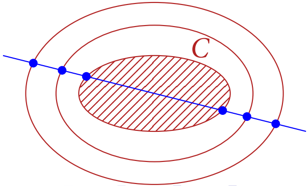

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All faces exposed?

$\{\text{Spectrahedra}\} \subsetneq \left\{\begin{array}{l}\text{Lasserre-}\\ \text{exact sets}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Projections of}\\ \text{spectrahedra}\end{array}\right\}$

YES (Spectrahedra), YES (Lasserre-exact sets), NO (Projections of spectrahedra)

$\{\text{Spectrahedra}\} \subsetneq \overset{?}{=} \left\{\begin{array}{l}\text{Rigidly}\\ \text{convex sets}\end{array}\right\}$

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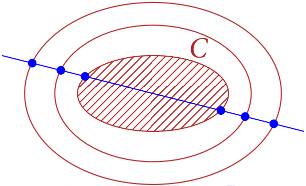
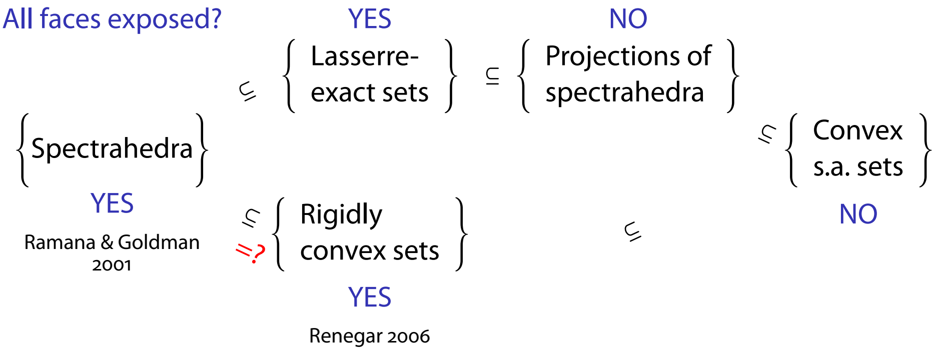


Spectrahedra vs. Convex semi-algebraic sets



All faces exposed?

YES

$\left\{\begin{array}{l}\text{Lasserre-}\\ \text{exact sets}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Projections of}\\ \text{spectrahedra}\end{array}\right\}$

NO

$\{\text{Spectrahedra}\} \subsetneq \left\{\begin{array}{l}\text{Lasserre-}\\ \text{exact sets}\end{array}\right\}$

$\left\{\begin{array}{l}\text{Projections of}\\ \text{spectrahedra}\end{array}\right\} \subsetneq \left\{\begin{array}{l}\text{Convex}\\ \text{s.a. sets}\end{array}\right\}$

YES

Ramana & Goldman 2001

$\{\text{Spectrahedra}\} \begin{array}{c}\subsetneq\\ \overset{?}{=}\end{array} \left\{\begin{array}{l}\text{Rigidly}\\ \text{convex sets}\end{array}\right\} \subsetneq \left\{\begin{array}{l}\text{Convex}\\ \text{s.a. sets}\end{array}\right\}$

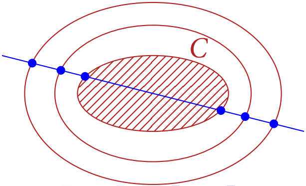
NO

YES

Renegar 2006

Rigid convexity: A set $C \subset \mathbb{R}^n$ with $0 \in \operatorname{int}(C)$ is called rigidly convex if there exists a polynomial $p \in \mathbb{R}[\underline{t}]$ with $p(0)$ such that C is the closure of the connected component of $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ that contains 0 and such that the univariate polynomial $s \mapsto p(sv)$ has only real zeros, for every $v \in \mathbb{R}^n \setminus \{0\}$.

Every spectrahedron is rigidly convex. The converse is true for $n = 2$ (Helton & Vinnikov 2004)



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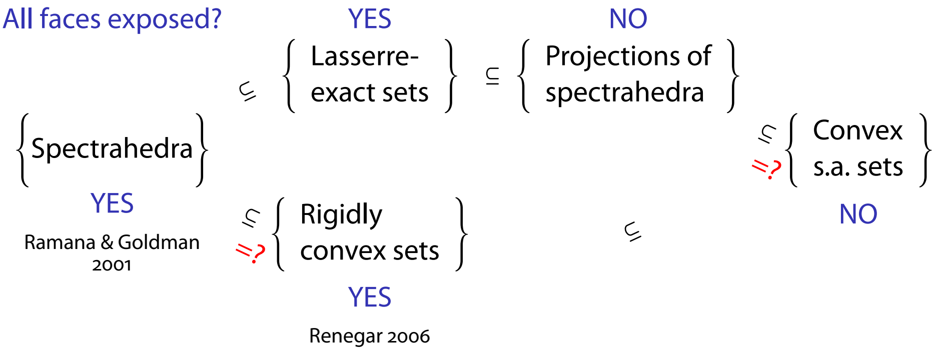

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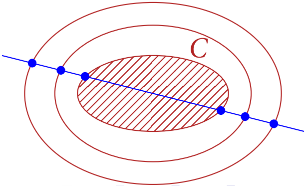

Main result

$C = \{x \in \mathbb{R}^n \mid p_1(x) \geqslant 0, \dots, p_r(x) \geqslant 0\}$ convex with non-empty interior.
$M = \left\{\sum_{i=0}^r (s_{i1}^2 + \dots + s_{iN}^2) p_i \,\middle|\, s_{ij} \in \mathbb{R}[\underline{t}]\right\}$
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$\mathcal{L}_d = \left\{L \in \mathbb{R}[\underline{t}]_d^\vee \mid L(f) \geqslant 0 \text{ for all } f \in M_d \text{ and } L(1) = 1\right\}$
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(Example of such C by Gouveia (2009)).

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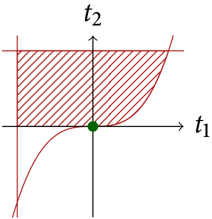

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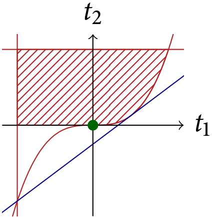

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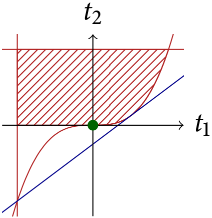

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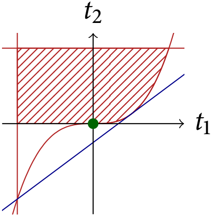

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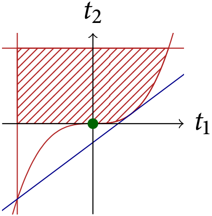

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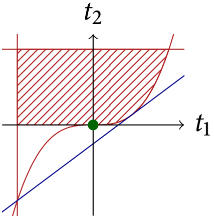

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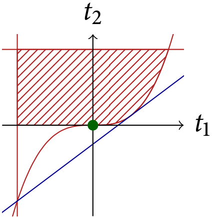

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Alternative proof by Gouveia.

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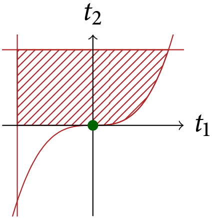

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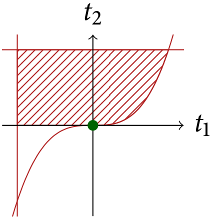

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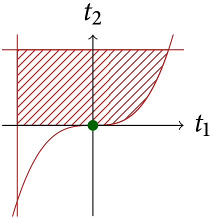

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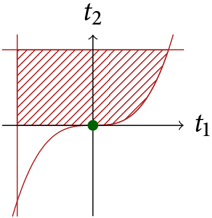
Examples

$C = \{x \in \mathbb{R}^n \mid p_1(x) \geqslant 0, \dots, p_r(x) \geqslant 0\}$ convex with non-empty interior.
$M = \left\{\sum_{i=0}^r (s_{i1}^2 + \cdots + s_{iN}^2)p_i \,\middle|\, s_{ij} \in \mathbb{R}[\underline{t}]\right\}$
$M_d = \left\{\sum_{i=0}^r (s_{i1}^2 + \cdots + s_{iN}^2)p_i \,\middle|\, (s_{i1}^2 + \cdots + s_{iN}^2)p_i \in \mathbb{R}[\underline{t}]_d \text{ for all } i\right\}$

Example: $p_1 = t_2, \quad p_2 = 1 - t_2, \quad p_3 = t_1 + 1, \quad p_4 = t_2 - t_1^3.$



But it follows from a result of Helton and Nie that C is the projection of a spectrahedron: Let $C_1 = [-1, 0] \times [0, 1]$ and $C_2 = \{p_1 \geqslant 0, p_2 \geqslant 0, p_4 \geqslant 0, t_1 \geqslant 0\}$. S is the convex hull of C_1 and C_2, C_1 is a spectrahedron, and the third Lasserre relaxation of C_2 is exact:

For any $\varepsilon \in [0, 1]$, we can write $\ell_\varepsilon = t_1^3 - 3\varepsilon^2 t_1 + 2\varepsilon^3 + (t_2 - t_1^3)$. The polynomial $t_1^3 - 3\varepsilon^2 t_1 + 2\varepsilon^3 \in \mathbb{R}[t_1]$ is non-negative on $[0, \infty)$ and is therefore contained in $\mathrm{QM}(t_1)_3 \subseteq \mathbb{R}[t_1]$ by a result of Kuhlmann, Marshall, and Schwartz.