CONES OF CONVEX FORMS

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Abstract. These are rough notes of my talk on 2/17/10 at BIRS 10w5007, Convex Algebraic Geometry. Please write me if you find mistakes or want to see more detail. (I didn't know I was going to be writing this when I went to the conference.) Proofs that seem ok when given verbally seem fragile in print, so I had to put in more details. I've added a few new examples, or assertions of examples, where appropriate. The organization and transitions all need work.

1. Let $F_{n,m}$ denote the vector space of real homogeneous polynomials in n variables of degree m. A function p is *convex* if for all $x, u \in \mathbb{R}^n$, when we define $\phi_{x,u}(t) = p(x + tu)$, we have $\phi''_{x,u}(0) \ge 0$. Convexity is determined by the behavior of p on all two dimensional subspaces of $\mathbb{R}^n$. Thus if $q(x) = p(Mx)$, where M is an invertible linear change of variables, then p is convex if and only if q is convex. If we write $p_i = \frac{\partial p}{\partial x_i}$, etc., then

$$\phi''(0; x, u) = Hes(p; x, u) := \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}(x) u_i u_j. \tag{1}$$

If $p \in F_{n,m}$, then $Hes(p; x, u)$ is a bihomogeneous form in (x, u) so p is convex if and only if $Hes(p)$ is psd. Now (new definition), we say that p is *definitely convex* if $Hes(p)$ is positive definite; in this case, for $x, u \in S^{n-1}$, $Hes(p; x, u)$ will achieve a positive minimum. For all integers r, $(\sum x_j^2)^r$ is definitely convex.

2. Let $K_{n,m}$ denote the set of convex forms in $F_{n,m}$. This talk was motivated by a longstanding question of Pablo Parrilo about whether there exist convex forms which are not sos and by a recent preprint of Greg Blekherman, *Convex forms that are not sums of squares*, arXiv: 0910.065v1 [math.AG] 5 Oct 2009, showing that "most" convex forms are not sos (if you fix degree and go to a large enough number of variables). No specific examples were given there, or here, sorry to say. The reasons it is hard to find an example include the following: (i) Our intuition overemphasizes the monomials in which the exponents are large and deemphasizes the ones which are smaller. For example, 99.66% of the monomials of a quartic in 100 variables are either $x_i x_j x_k x_\ell$ or $x_i^2 x_j x_k$. (ii) The examples of psd forms which are not sos either have many zeros (see #5 below for why those can't be convex) or have few terms, and these usually can't be convex even if they are obviously sos. (iii) Greg's talk (the day after mine) offered another intuition: look for a form which is close to constant on the unit sphere; that is, one which is essentially a perturbation of $(\sum_j x_j^2)^{m/2}$.

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This gives another way to get a handle on what convex forms might be even if we don't know yet how to make them sos. It should also be mentioned that $K_{n,m}$ is different from $P_{n,m}$ and $\Sigma_{n,m}$ in a fundamental way with respect to homogenization and dehomogenization. These don't matter for psd and sos polynomials and forms, and one can go back and forth. These properties are decidedly false for convex forms. Trivially, $t^2 - 1$ is a convex polynomial and can't be homogenized to a convex form. More seriously, $t^4 + 12t^2 + 1$ is a very convex polynomial, and $x^4 + 12x^2y^2 + y^4$ is very not convex. The cones $P_{n,m}$ and $\Sigma_{n,m}$ are also closed under multiplication, and $K_{n,m}$ isn't. One can look at x^2 and y^2 or see #8 below.

3. If $p, q \in K_{n,m}$ and $\lambda > 0$, then $p + q, \lambda p \in K_{n,m}$, so $K_{n,m}$ is a convex cone. If $p^{(\ell)} \to p$ coefficient-wise, and $p^{(\ell)} \in K_{n,m}$, then $Hes(p^\ell) \to Hes(p)$ pointwise, hence $p \in K_{n,m}$, so $K_{n,m}$ is a closed convex cone. If p is definitely convex, then the usual perturbation argument (with $x, u \in S^{n-1}$) shows that p is interior to $K_{n,m}$, whereas if p is convex but not definitely convex, and $Hes(p; \bar{x}, \bar{u}) = 0$ for some $\bar{x}, \bar{u} \in S^{n-1}$, then $Hes(p - t(\sum x_j^2)^{m/2}, \bar{x}, \bar{u}) < 0$ for $t > 0$, Thus the interior of $K_{n,m}$ consists precisely of the definitely convex forms in $F_{n,m}$.

4. Since $\phi_{x,x}(t) = (1+t)^m p(x)$, $\phi''_{x,x}(0) = m(m-1)p(x) \ge 0$, hence p is psd. That is, $K_{n,m} \subset P_{n,m}$. If $p(x) = \left(\sum_j \alpha_j x_j\right)^m$, then $\phi_{x,u}(t) = ((x+tu)\cdot\alpha)^m = (x\cdot\alpha + t(u\cdot\alpha))^m$, so $Hes(p; x, u) = m(m-1)(x\cdot\alpha)^{m-2}(u\cdot\alpha)^2 \ge 0$. Thus all sums of m-th powers are convex; that is, $Q_{n,m} \subset K_{n,m}$. (See *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc., Volume 96, Number 463, March, 1992 (MR 93h.11043) for much more on $P_{n,m}$ and $Q_{n,m}$. I have scanned this Memoir and put it on my website http://www.math.uiuc.edu/∼reznick/memoir.html).

5. Convex forms are almost positive definite: if $p(a) = 0$, then p vanishes to m-th order in every direction at m. Proof: Wlog, suppose $p(1, 0, \dots, 0) = 0$ and suppose that x_1 occurs in some term of p. Let $r \ge 1$ be the largest power of x_1 that appears and suppose the associated terms in p are $x_1^r q(x_2, ..., x_n)$. We assume $q \ne 0$, so after a linear change, we may assume that $q(1, 0, \dots, 0) = 1$. This would mean that $p(x_1, x_2, 0, \dots, 0) = x_1^r x_2^{m-r}$ + lower order terms in x_1, x_2. Taking the 2×2 Hessian, we see that the leading term in x_1 and then x_2 is:

$$
\begin{aligned}
&r(r-1)x_1^{r-2}x_2^{m-r}(m-r)(m-r-1)x_1^r x_2^{m-r-2} - (r(m-r)x_1^{r-1}x_2^{m-r-1})^2 \\
&\qquad = -rm(m-1)x_1^{2r-2}x_2^{2m-2r-2}
\end{aligned}
\tag{2}
$$

which can't be positive, contradiction. Observe that forms in $Q_{n,m}$ have the same property.

6. The following result is in the literature somewhere. I don't know much convex analysis, but it seems to be the same as Cor. 15.3.1 in Rockafellar's *Convex Analysis* (1970); the notes attribute it to a 1951 paper of E. Lorch which I don't understand. V. I. Dmitriev (see below) attributes the observation to his advisor Selim Krein (1969). I think it is also true if p is any positive definite function which is homogeneous of

degree m, whether polynomial or not. Let

$$f(x_2,\dots,x_n) = p(1,x_2,\dots,x_n). \tag{3}$$

THM: Suppose p is a positive definite form in $F_{n,m}$. Then $p(x_1,\dots,x_n)$ is convex if and only if $f^{1/m}(x_2,\dots,x_n)$ is convex.

The theorem is true, but the proof I gave in the talk is incomplete. Here's a better one. We will prove the theorem pointwise. If we can show that $Hes(p;a,u) \ge 0$ for all $(a_1,\dots,a_n)$ with $a_1 \ne 0$ iff $f^{1/m}$ is convex, then by continuity, $Hes(p;a,u) \ge 0$ for all a and we're done. So assume a is given with $a_1 \ne 0$, and by homogeneity, we may assume that $a_1 = 1$. Let

$$\begin{aligned} \tilde{p}(x_1,x_2\dots,x_n) &= p(x_1,x_2+a_2x_1,\dots x_n+a_nx_1), \\ \tilde{f}(x_2,\dots,x_n) &= \tilde{p}(1,x_2,\dots x_n) = f(1,x_2+a_2,\dots,x_n+a_n) \end{aligned} \tag{4}$$

Then p and $f^{1/m}$ are convex if and only if $\tilde{p}$ and $\tilde{f}$ are convex (since they involve linear changes of variable), and we can drop the tildes and assume that we are looking at $a = (1,0,\dots,0)$. Furthermore, we are looking at a two dimensional vector space and by making a change of variables in $(x_2,\dots,x_n)$, we may assume without loss of generality that this two-dimensional space is $(x,y,0,\dots,0)$. Now, resuming the talk's argument, suppose

$$p(x,y,0,\dots,0) = a_0x^m + \binom{m}{1}a_1x^{m-1}y + \binom{m}{2}a_2x^{m-2}y^2 + \dots. \tag{5}$$

Then the Hessian matrix, evaluated at $(1,0)$ is

$$m(m-1)\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \tag{6}$$

and since $a_0 = p(1,0,\dots,0) > 0$, p is convex at a if and only if $a_0a_2 \ge a_1^2$. On the other hand, we have

$$f(t) = p(1,t) = a_0 + \binom{m}{1}a_1t + \binom{m}{2}a_2t^2 + \dots \tag{7}$$

and a routine computation shows that

$$(f^{(1/m)})''(0) = (m-1)a_0^{1/m-2}(a_0a_2 - a_1^2). \tag{8}$$

A more complicated proof computes the Hessian of p, uses the Euler pde's $mp = \sum x_ip_i$ and $(m-1)p_i = \sum x_jp_{ij}$ to replace p_{11} and p_{1j} with p, the p_i's with $i \ge 2$ and the p_{ij}'s with $i,j \ge 2$. The discriminant with respect to u_1 is a positive multiple (after a change of variables) of the Hessian of $f^{1/m}$. In the interest of time and aesthetics, I won't write it down.

7. Greg Blekherman pointed out that the theorem below is a special case of a result in my *Uniform denominators in Hilbert's Seventeenth Problem*, Math. Z., 220 (1995), 75-98 (MR 96e:11056), see .../paper30.pdf, which is true, but the proof here is much shorter. As with #6., it's probably true if p is not a form, but merely is, say

C^3 on the unit sphere, to ensure that the given bounds exist. This has the amusing implication that if p is a positive definite and somewhat smooth function on the unit sphere, then it is the restriction to the unit sphere of a convex function. This can't possibly be a new theorem.

THM. If p is a positive definite form of degree m, then there exists N so that $p_N := (\sum_j x_j^2)^N p$ is convex.

PF. Since p is pd, it is bounded away from 0 on S^{n-1} and so there are uniform upper bounds T for $|\nabla_u(f)(x)/f(x)|$, for $x, u \in S^{n-1}$ and U for $|\nabla_u^2(f)(x)/f(x)|$, for $x, u \in S^{n-1}$. Since $\sum x_i^2$ is rotation invariant, it suffices to show that p_N is convex at $(1, 0, \dots, 0)$ in the 2-dimensional space $(x, y, 0, \dots, 0)$. Suppose p has degree m. We claim that if $N > (T^2 + U)/2$, then p_N is convex. In view of the last theorem, it suffices to show that $p_N^{1/(2N+m)}(1, t, 0, \dots, 0)$ is convex at $(1, 0, \dots, 0)$. Writing down the relevant Taylor series, we have

$$(1 + t^2)^{N/(2N+m)}(1 + \alpha t + \beta t^2/2 + \dots)^{1/(N+2m)} \tag{9}$$

where $|\alpha| \le T$ and $|\beta| \le U$. By expanding the product, a standard computation shows that the second derivative is

$$\frac{2N}{2N+m} + \frac{1}{2N+m}b - \frac{2N+m-1}{(2N+m)^2}a^2 \ge \frac{1}{2N+m}\left(2N - U - T^2)\right) \ge 0. \tag{10}$$

There are may be errors of constants here, but the overall idea seems to work. In *Uniform denominators* it was proved that there exists such an N so that p_N is a sum of squares, but actually the proof showed $p_N \in Q_{n,m+2N} \subset K_{n,m+2N}$ and was much less elementary.

8. If $a > 0$, then $x^2 + ay^2$ is convex, but if $n \ge 1$ and $(x^2 + y^2)^r(x^2 + ax^2) \in K_{2,2r+2}$ for all a, then since the cone is closed, we would have $x^2(x^2+y^2)^r$ being convex, which violates #5. It is thus of vague computational interest to determine the interval I_r so that $(x^2 + y^2)^r(x^2 + ax^2) \in K_{2,2r+2} \iff a \in I_r$. A straightforward and hopefully correct computation shows that $a \in I_r \iff a + 1/a \le 8r + 18 + 8/r$. In particular, for $r = 1$, $(x^2 + y^2)(x^2 + ay^2) \in K_{4,4} \iff 17 - 12\sqrt{2} \le a \le 17 + 12\sqrt{2}$.

9. The work that has been done on $K_{n,m}$ has actually been done on $f^{1/m}$. Suppose $X =< u_1, \dots, u_n >$ is an n-dimensional vector space and a (non-homogeneous) polynomial $f(x_2, \dots, x_n)$ of degree m is used to define a proposed norm on X by

$$||u_1 + x_2u_2 + \cdots x_nu_n|| = f(x_2, \dots x_n)^{1/m} \tag{11}$$

These came up in my 1976 PhD thesis; when $m = 2$, this sort of norm is familiar from Hilbert space. It turns out that the triangle inequality is satisfied iff $f^{1/m}$ is convex. The relevant paper is *Banach spaces with polynomial norms*, Pacific J. Math., 82 (1979), 223-235 (MR 83c.46007), and is now scanned on my webpage as .../paper4.pdf .

Independently and somewhat earlier, V. I. Dmitriev started working on this subject and wrote two papers: in 1973 and 1991. (There are at least two different V. I. Dmitriev's MathSciNet; this one is at Kursk State Technical University.) The earlier paper is: MR0467523 (57 #7379) Dmitriev, V. I. *The structure of a cone in a five-dimensional space.* (Russian) Vorone. Gos. Univ. Trudy Naun.-Issled. Inst. Mat. VGU Vyp. 7 (1973), 13–22. I have ordered this from the inter-library but not seen it yet. The later paper is MR1179211 (93i:12003) Dmitriev, V. I. *Extreme rays of a cone of convex forms of the sixth degree in two variables* . (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1991, , no. 10, 28–35; translation in Soviet Math. (Iz. VUZ) 35 (1991), no. 10, 25–31, which I have seen, both in English and in Russian. All my information about the 1973 paper comes from the 1991 one. S. Krein had asked n 1969 for the extreme elements of $K_{n,m}$. (For $m = 2$, $P_{n,2} = K_{n,2} = Q_{n,2}$ so there is nothing to say.) In 1973, Dmitriev proved that $Q_{2,4} = K_{2,4}$ and that $Q_{2,4k} \subsetneq K_{2,4k}$ for $k > 1$. In 1991, Dmitriev completed the result for $K_{2,4k+2}$ and gave the extremal elements of $K_{2,6}$; see #14. Independently, in my thesis and the above paper, I found these results, with $x^{2k} + x^2y^{2k-2} + y^{2k} \in K_{2,2k} \setminus Q_{2,2k}$, along with an example in $K_{3,4} \setminus Q_{3,4}$. Somewhat sadly, Dmitriev writes in 1991 "I am not aware of any articles on this topic, except [his first one]."

It is worth emphasizing that since $P_{n,m} = \Sigma_{n,m}$ if $n = 2$ or $(n, m) = (3, 4)$, these examples seem to be useless in finding a convex form which is not sos.

10. For references to the inner product, see the Memoir. We write

$$x^i = x_1^{i_1} \cdots x_n^{i_n}, \qquad c(i) = \frac{(i_1 + \cdots + i_n)!}{i_1! \cdots i_n!} \tag{12}$$

and write $p \in F_{n,d}$ as

$$p(x) = \sum_i c(i)a(p;i)x^i \tag{13}$$

so that the coefficients have multinomial coefficients attached. For $\alpha \in \mathbb{R}^n$, define the form $(\alpha\cdot)^d$ by

$$(\alpha\cdot)^d(x) = \left(\sum_j \alpha_j x_j\right)^d = \sum_i c(i)\alpha^i x^i. \tag{14}$$

We define the classical (apolarity, Fischer, *obvious*) inner product by

$$[p, q] = \sum_i c(i)a(p;i)a(q;i). \tag{15}$$

We see that $[p, (\alpha\cdot)^d] = p(\alpha)$, and this proves immediately that as closed convex cones, $P_{n,m}$ and $Q_{n,m}$ are duals. By definition, $p \in \Sigma^*_{n,m}$ iff $[p, h^2] \ge 0$ for all forms $h \in F_{n,m/2}$. If we write such a form as $\sum t(j)x^j$, with monomials of degree $m/2$, then a calculation shows that

$$[p, h^2] = \sum_j \sum_k a(p; j + k)t(j)t(k), \tag{16}$$

and so $p \in F_{n,m/2}$ if and only if the above generalized Hankel quadratic form is psd. This, as we all found out in Banff, is a spectrahedron.

11. The inner product also has a differential interpretation. Let

$$D^i = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}. \tag{17}$$

Then a routine calculation shows that $q(D)p = d![p,q]$, and, as noted in the references, if $q = fg$, where $\deg g = r$, then

$$d![p,q] = q(D)p = f(D)(g(D)p) = (d-r)![g(D)p, f]. \tag{18}$$

After changing letters for clarity, the Hessian for p at β in the α direction is

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}(\beta)\alpha_i\alpha_j &= \left(\alpha_1\frac{\partial}{\partial x_1} + \cdots + \alpha_n\frac{\partial}{\partial x_n}\right)^2 (p)(\beta) \\ &= [(\alpha\cdot D)^2 p, (\beta\cdot)^{m-2}] = m(m-1)[(\alpha\cdot)^2(\beta\cdot)^{m-2}, p].\end{aligned} \tag{19}$$

In particular, it follows immediately that $K^*_{n,m}$ is the vector space spanned by the set of forms $\{(\alpha\cdot)^2(\beta\cdot)^{m-2} : \alpha, \beta \in \mathbb{R}^n\}$. Note that elements of $K^*_{n,m}$ can have at most two linear factors. For $k \ge 3$, $\prod_{j=1}^{k}(x - jy)^2 \in P_{2,2k} \setminus K^*_{2,2k}$, and this gives a nonconstructive proof that $Q_{2,2k} \subsetneq K_{2,2k}$. Since the extremal elements of $P_{2,4}$ are products of two linear squares, we get $Q_{2,4} = K_{2,4}$ and since $(x^2+y^2-z^2)^2$ is not a product of squares of linear factors, $Q_{3,4} \subsetneq K_{3,4}$. Similar arguments show that $Q_{n,m} \subsetneq K_{n,m}$ if $m \ge 4$ unless $(n,m) = (2,4)$.

12. The Memoir introduced the *blender*, an extremely natural mathematical object. A blender is a closed convex cone of forms in $F_{n,d}$ which is also closed under all invertible linear changes of variable. (Including, by continuity, non-invertible ones; cf. orbitopes.) It is not hard to show that if W is a blender and there exist $p, q \in W$ and $u, v \in \mathbb{R}^n$ so that $p(u) < 0 > q(v)$, then $W = F_{n,d}$. For an $n \times n$ matrix M, we let $p \circ M$ be the form defined by $(p \circ M)(x) = p(Mx)$. It is not hard to show that $[p \circ M, q] = [p, q \circ M^t]$, and so the dual cone of a blender is also a blender. Examples of blenders are all cones listed above. More generally, if m_i are positive even integers and $d_i > 0$ and $\sum m_i d_i = m$, then the cone generated by products $f_1^{m_1} \cdots f_r^{m_r}$, $\deg f_i = d_i$ also defines a blender. (The only issue is that it is closed, and this can be done similarly to the proof for $Q_{n,m}$.) In the Memoir, $K_{n,m}$ is denoted $N_{n,m}$ based on its origins in norms. One non-trivial blender is $W_\alpha \in F_{2,4}$ which is generated by $x^4 - \alpha x^2y^2 + y^4$ for $0 \le \alpha \le 2$. These are all the possible blenders of binary quartics. By taking $(x,y) \mapsto (x+y, x-y)$, $x^4 + \frac{12+2\alpha}{2-\alpha}x^2y^2 + y^4 \in W_\alpha$ (at least for $\alpha < 2$.) It can be shown, and will be written up elsewhere, that $W^*_\alpha = W_\beta$, where $\alpha^2 + \beta^2 = 4$,

13. Suppose $p \in F_{2,m}$. Let $A(p)(x,y)$ denote the 2×2 Hessian; it will be a form of degree $2m-4$. Equivalently, let $f(t) = p(1,t)$. Then up to a constant multiple of $t^{1/m-2}$, $(f^{1/m})''$ is $B(f)(t) := mf(t)f''(t) - (m-1)f'(t)^2$. This is nominally a polynomial of degree $2m-2$, but it is easy to check that the coefficients of t^{2m-2} and t^{2m-3} formally vanish and that this is the dehomogenization $A(p)(1,t)$, again up to

harmless multiple. We assume that p is not an m-th power of a linear form so it is positive definite. Observe that

$$
\begin{aligned}
p(t) = \sum_{k=0}^{m} \binom{m}{k} a_k t^k \implies p'(t) = m \sum_{k=0}^{m-1} \binom{m-1}{k} a_{k+1} t^k \\
\implies p''(t) = m(m-1) \sum_{k=0}^{m-2} \binom{m-2}{k} a_{k+2} t^k \\
\implies B(f)(t) = m^2(m-1)((a_0a_2 - a_1^2) + (m-2)(a_0a_3 - a_1a_2)t + \cdots .
\end{aligned}
\tag{20}
$$

It's easy to argue that if $A(p)$ is positive definite, or, equivalently, if $B(f)(t)$ has full degree and is positive, then p is not extremal in $K_{2,m}$. To look at extremal elements, we assume that one of these has a zero. If, say $B(f)(0) = 0$, then $B(f)'(0) = 0$ as well, so $a_0a_2 = a_1^2$ and $a_0a_3 = a-1a-2$. Since $a_0 > 0$, we can write $a_1 = ra_0$, from which it follows that $a_2 = r^2a_0$ and $a_3 = r^3a_0$. Under these substitutions, the coefficient of t^2 in $B(f)(t)$ is $m^2(m-1)\binom{m-3}{2}(a_4 - a_0r^4)$. In other words, zeros of $B(f)$ make f agree with an m-th power of a linear form for the first four terms of the Taylor series, and it's larger on the fifth term. When $m = 4$, this implies that $p(x,y) = a_0(x+ry)^4 + cy^4$ with $c \ge 0$, and this was (my original, at least) proof that $K_{2,4} = Q_{2,4}$.

14. In degree six, one can argue that if there is only one zero and if it has order two, then the form is not extremal. Thus an extremal sextic either has a zero of order four or more or two zeros. In the first case, a similar argument to above shows that $p(x,y) = a_0(x+ry)^6 + cy^6$ with $c > 0$. In the second case, we may make a change of variables to put two zeros at $(1,0)$ and $(0,1)$ and, having done that, we may scale them so that the coefficients of x^6 and y^6 are both 1. That is,

$$
p(x,y) = x^6 + 6a_1x^5y + 15a_2x^4y^2 + 20a_3x^3y^3 + 15a_4x^2y^4 + 6a_5xy^5 + y^6. \tag{21}
$$

The two assumed zeros show that $(a_1,a_2,a_3) = (r,r^2,r^3)$, and not assuming anything a priori, that $(a_5,a_4,a_3) = (s,s^2,s^3)$. The common value for a_3 shows that $r = s$. Let

$$
q_r(x,y) = x^6 + 6rx^5y + 15r^2x^4y^2 + 20r^3x^3y^3 + 15r^2x^2y^4 + 6rxy^5 + y^6. \tag{22}
$$

A calculation shows that

$$
\begin{aligned}
A(q_r)(x,y) = 900(1-r^2)x^2y^2 \times \\
6r^2(x^4+y^4) + (4r+20r^3)(x^3y+xy^3) + (1+15r^2+20r^4)x^2y^2
\end{aligned}
\tag{23}
$$

Consideration of $A(q_r(1,\pm1))$ and a little algebra show that this is psd if and only if $r = \pm1$ (and $q_r(x,y) = (x\pm y)^6$) or $|r| \le 1/2$. A little more work leads to a theorem found independently by me (in 1979) and Dmitriev (in 1991):

THM The extremal elements in $K_{2,6}$ are given by $(ax+by)^6$ and $\{q_r, |r| \le 1/2\}$.

Interestingly enough, $q_r(x+y,x-y)$ is an even form:

$$\begin{aligned} &2(1+r)(1+5r+10r^2)x^6+30(1-r^2)(1+2r)x^4y^2 \\ &+30(1-r^2)(1-2r)x^2y^4+2(1-r)(1-5r+10r^2)y^6 \end{aligned} \tag{24}$$

The boundary example, is $q_{-1/2}(x+y,x-y)=x^6+45x^2y^4+18y^6$, which scales to $x^6+5(3/2)^{2/3}x^2y^4+18y^6$. In this case, $A(q_{\pm 1/2})$ has an extra zero: $A(q_{-1/2})(x,y)$ is a positive multiple of $x^2(x-y)^2y^2(x^2-xy+y^2)$, so the extremal elements, even for $Q_{2,6}$, have varying algebraic patterns. One expects that some of the extremal elements of $K_{2,2k}$ will be hard to classify algebraically as k increases.

15. It is worth exploring the sections of $P_{2,6}=\Sigma_{2,6}$, $Q_{2,6}$ and $K_{2,6}$ of forms of the type $g_{A,B}(x,y)=x^6+15Ax^4y^2+15Bx^2y^4+y^6$. Pictures are in the Mathematica appendix at the end.

(i) If $g_{A,B}$ is on the border of the $P_{2,6}$ section then it clearly has a zero, and since it's psd, we can assume $(x+ry)^2$ is a factor. By the evenness, $(x-ry)^2$ must also be a factor, and since the third factor is even, the coefficients of x^6,y^6 force it to be $x^2+\frac{1}{r^4}y^2$. Thus, the border forms are $(x^2-r^2y^2)^2(x^2+\frac{1}{r^4}y^2)$, and the border is parameterized by

$$15A=\frac{1}{r^4}-2r^2, \qquad 15B=r^4-\frac{2}{r^2} \tag{25}$$

The curve is sketched in the appendix. Everything above it and to the right is in $P_{2,6}$.

(ii) If $g_{A,B}$ is in $Q_{2,6}=\Sigma^*_{2,6}$, then its generalized Hankel matrix or catalecticant is psd. This matrix is

$$\begin{vmatrix} 1 & 0 & A & 0 \\ 0 & A & 0 & B \\ A & 0 & B & 0 \\ 0 & B & 0 & 1 \end{vmatrix} \tag{26}$$

This is psd iff $A\ge B^2$ and $B\ge A^2$, so the section is the familiar calculus region between these two parabolas. This is sketched in the appendix.

(iii) It would be challenging in general to find this section for $K_{2,6}$ except that we know that all the extremal elements of $K_{2,6}$ which are not purely sixth powers have a representative in this section. We scale $q_r(x,y)$ so that the coefficients of x^6 and y^6 are both 1 and discover that the parameterization of the boundary is $\psi(r),\psi(-r)$, where

$$\psi(r)=\frac{(1-r)^{2/3}(1+r)^{1/3}(1+2r)}{(1+5r+10r^2)^{2/3}(1-5r+10r^2)^{1/3}} \tag{27}$$

The intercepts occur when $r=\pm\frac{1}{2}$ and, as noted earlier, represent $x^6+5(\frac{3}{2})^{2/3}x^2y^4+y^6$ (or with (x,y) reversed) and so are at $(18^{-1/3},0)$ and $(0,18^{-1/3})$. The point $(1,1)$ actually is smooth but of infinite curvature. The Taylor series of $\psi(r)$ at $r=0$ begins $1+\frac{16}{3}r^3-48r^4$, so locally, $x-y=\frac{32}{3}r^3$ and $x+y-2=-96r^4$, hence

$x+y-2 \approx c(x-y)^{4/3}$ for some $c<0$. The maximum value of $\psi(r)$ can be computed: $5^{-5/3}(1565+496\sqrt{10})^{1/3} \approx 1.000905$. Astonishingly, this can be found in my 1979 paper on p.232, done without Mathematica!

16. Some other interesting extremal elements of $K_{2,2k}$. In the 1979 paper, I asserted that $x^{2d}+\alpha x^{2k}y^{2d-2k}+y^{2d} \in K_{2,2d}$ if and only if $0 \le \alpha \le \alpha(d,k)$. It is more illuminating to rescale and assume that $p(x,y)=ax^{2d}+bx^{2k}y^{2d-2k}+cy^{2d}$ is in $K_{2,2d}$ but $A(p)(x,y)$ has a double zero at $(1,1)$. Under these conditions, it is not hard to find that

(28)

$$p(x,y)=k(2k-1)^2x^{2d}+d(2d-1)(2k-1)(2d-2k-1)x^{2k}y^{2d-2k}+(d-k)(2d-2k-1)^2y^{2d}.$$

Of particular interest is when $d=2k$, and the example simplifies to

$$x^{4k}+(8k-2)x^{2k}y^{2k}+y^{2k}. \tag{29}$$

In fact, $\alpha(d,k)$ grows at most linearly in d. Another interesting border example is

$$x^{6k}+(6k-1)(6k-3)x^{4k}y^{2k}+(6k-1)(6k-3)x^{2k}y^{2k}+y^{6k}. \tag{30}$$

Finally, trying to put four zeros on $A(p)(x,y)$ for octic p leads to

$$(x^2+y^2)^4+\frac{8}{\sqrt{7}}xy(x^2-y^2)(x^2+y^2)^2. \tag{31}$$

I will do my best to make this coherent by the time these notes become a paper. It is also true that $x^4+y^4+z^4+2x^2y^2+6x^2z^2+6y^2z^2 \in K_{3,4}\setminus Q_{3,4}$. This was in my 1979 paper. My thesis proved the existence of such an element by a 20 page perturbation argument that I am frankly afraid to look at now. I apologized to the committee when I improved the argument in the paper.

17. Another blender I've worked on for a long time without reaching a complete understanding is $\hat{W} := \{\sum_k (a_kx^2+b_kxy+c_ky^2)^4\}$, which could be thought of as a projection of $Q_{3,4}$, which we know about, under $(x_1,x_2,x_3)\mapsto(x^2,xy,y^2)$. I guess it's a composition of Veronese's. The best I can tell you is that $g_\alpha(x,y) := x^8+\alpha x^4y^4+y^8 \in \hat{W}$ if and only if $\alpha \ge -\frac{14}{9}$. I have a more general conjecture. Suppose $p(x,y)\in F_{2,4k}$. As a special case of Becker's theory of higher orders, we know that p is a sum of fourth powers of rational functions if and only if it is psd and all zeros occur to order a multiple of 4. (So for $\alpha>-2$, g_α is a sum of fourth powers with denominators.) I have a conjecture (wrote "theorem" on the board, in a burst of confidence):

CONJ: The form $p(x,y)$ is a sum of 4th powers if and only if there are psd forms f,g so that $p=f^2+g^2$.

This conjecture is true when $\deg p = 4$ and for forms of the special kind $x^8+\gamma x^6y^2+\alpha x^4y^4+\gamma x^2y^6+y^8$. One direction is easy, since $f^2=(q_1^2+q_2^2)^2$ can be written as a sum of three fourth powers. See (with M.D. Choi, T.Y. Lam, and A. Prestel) *Sums of 2m-th powers of rational functions in one variable over real closed* *fields*, Math. Z., 221 (1996), 93-112 (MR 96k:12003), which will be on the web by Summer 2010.

18. Inspired by Greg Blekherman's talk, let's look at ternary sextics. Thanks to a joint paper (with M. D. Choi and T. Y. Lam) *E*ven symmetric sextics, Math. Z., 195 (1987), 559-580 (MR 88j.11019), on the web by Summer 2010, one can completely say when an even ternary sextic is psd and sos. I will omit the details, and give two candidates, both known to be psd, and the second is not sos.

$$M_2^3 = (x^2 + y^2 + z^2)^3 = x^6 + y^6 + z^6 + 3(x^4y^2 + \cdots) + 6x^2y62z^2; \\ R = x^6 + y^6 + z^6 - (x^4y^2 + \cdots) + 3x^2y^2z^2. \tag{32}$$

On the unit sphere, the average value of M_2^3 is, well 1, with, well, maximum 1 and minimum 1. By my (not contractual) computation, the average value of R is 2/7, with maximum 1 and minimum 0, because it has zeros. Thus, if $\alpha M_2^3 + \beta R$ is psd, then $\alpha \geq 0$. Scale so that $\alpha = 1$. Evaluation at (1,0,0) shows that $M_2^3 + \beta R$ psd implies $\beta \geq -1$ and $M_2^3 - R$ is evidently a sum of squares of monomials. Thus $M_2^3 + \beta R$ is psd iff $\beta \geq -1$. There is a theorem in this paper which gives necessary and sufficient conditions to be sos; in this case, strangely enough, it's $\beta \in [-1, 48]$. The big question now is when is it convex? Since the coefficient of x^4y^2 is $3 - \beta$, we must have $\beta \leq 3$, and so it is impossible that such a form is convex and not sos. My computations on the range on which $M_2^3 + \beta R$ is convex are not, shall we say, converging at this writing. A very queasy upper limit for the interval is, improbably, $[-\frac{3}{4}, \frac{72}{29}]$. I told some people some other bounds during the conference. I was wrong.

19. Thanks to Greg Blekherman and John D'Angelo for helpful conversations and to Peter Kuchment, a classmate of V. I. Dmitriev, for trying to get in touch with him for me.

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