SDP Relaxations for the Grassmann Orbitope

@ Convex Algebraic Geometry Workshop, Banff

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joined work with R. Sanyal and B. Sturmfels

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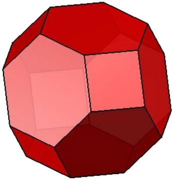
What are orbitopes?

We are interested in the following objects:

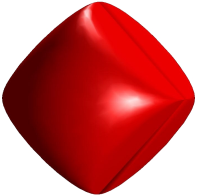
Def.: Orbitope

An orbitope $\mathcal{O}_v$ is the convex hull of an orbit of a compact algebraic group G acting on a real vector space V, i.e. fix $v \in V$ and consider the set

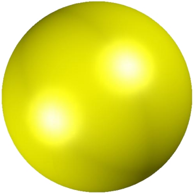
$$\mathcal{O}_v = \text{conv}\,\{g \cdot v \,|\, g \in G\}\,.$$



Permutahedron, orbitope for the symmetric group



Projection of the Grassmann Orbitope $\text{conv}(\mathrm{Gr}_{2,4})$



Orbitope $\text{conv}(\mathrm{Gr}_{2,3})$

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$$\mathcal{O}_v = \operatorname{conv}\{g \cdot v \mid g \in G\}.$$

- Orbits are highly symmetric objects
- Orbits are real algebraic varieties
- Orbitopes are convex semi-algebraic sets

... for more details see [Sanyal, Sottile, Sturmfels, '09].

Central object

Def.: Grassmann orbitope

The set $\mathcal{O}_v = \mathrm{conv}(\mathrm{Gr}_{k,n})$ is also the convex hull of the orbit of $v = e_1 \wedge e_2 \wedge \ldots \wedge e_k \in \bigwedge^k \mathbb{R}^n$ under the group $G = \mathrm{SO}(n)$

$$\mathrm{conv}(\mathrm{Gr}_{k,n}) = \mathrm{conv}\left(g \cdot e_1 \wedge e_2 \wedge \ldots \wedge e_k \,|\, g \in \mathrm{SO}(n)\right)$$

Elements $g \in \mathrm{SO}(n)$ of the special orthogonal group

$$\mathrm{SO}(n) = \left\{X \in \mathbb{R}^{n\times n} \mid X \cdot X^T = \mathrm{Id}_n,\, \det(X) = 1\right\}$$

act on $\bigwedge^k \mathbb{R}^n$ by $g \cdot (u_1 \wedge \ldots \wedge u_k) = (gu_1 \wedge \ldots \wedge gu_k)$

A bit of notation

We consider the vector space $V = \bigwedge^k \mathbb{R}^n \cong \mathbb{R}^{\binom{n}{k}}$ of all *skew-symmetric tensors* of order k over $\mathbb{R}^n$.

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If the vectors $\{e_1, \ldots, e_n\}$ form an ordered basis of $\mathbb{R}^n$ we can write

$$\xi = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1,\ldots,i_k} \underbrace{e_{i_1,\ldots,i_k}}_{e_{i_1} \wedge \cdots \wedge e_{i_k}} \qquad \text{for all } \xi \in \bigwedge^k \mathbb{R}^n.$$

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Decomposable tensor $\xi = u_1 \wedge \cdots \wedge u_k$ with $\|\xi\| = 1$

$\Leftrightarrow$ (oriented) k-dimensional plane $\mathrm{span}(u_1, u_2, \ldots, u_k) \subseteq \mathbb{R}^n$

$\Leftrightarrow \xi \in \mathrm{Gr}_{k,n}$ (i.e. ξ in the orbit of $v \in V$ under $\mathrm{SO}(n)$).

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Alternatively:

All vectors $p \in \mathbb{R}^{\binom{n}{k}}$, $\|p\| = 1$ where $p_{i_1,\ldots,i_k} = \det[U]_{i_1,\ldots,i_k}$ is the $k \times k$ subdeterminant of the matrix $U = \left[u_1, \ldots, u_k\right]$.

Central object

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We have

$$\mathrm{conv}(\mathrm{Gr}_{k,n}) = \mathrm{conv}(V_{\mathbb{R}}(I_{k,n}))$$

for the ideal

$$I_{k,n} = \underbrace{\langle \text{quad. Plücker rel's.} \rangle}_{\xi \text{ decomposable}} + \underbrace{\langle \sum_I p_I^2 - 1 \rangle}_{\|\xi\|^2 = 1} \subset \mathbb{R}[p_{i_1,\ldots,i_k}, \ldots]$$

(Oriented) Grassmann manifold

Def.: Grassmann manifold

The set $\mathrm{Gr}_{k,n}$ is the set of all (oriented) k-dimensional subspaces of $\mathbb{R}^n$.

In its (Plücker) embedding in the unit sphere of $\bigwedge^k \mathbb{R}^n \cong \mathbb{R}^{\binom{n}{k}}$ it yields an important object in several applications...

...e.g. for *area minimizing surfaces*.

Excursion: Area minimizing surfaces

Theorem (Harvey and Lawson, '82)

If all the tangent planes to a manifold $\mathcal{M}$ lie in the same face of $\mathrm{conv}(\mathrm{Gr}_{k,n})$*, then $\mathcal{M}$ is area-minimizing among all oriented surfaces with the same boundary.*

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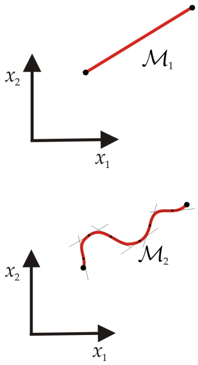
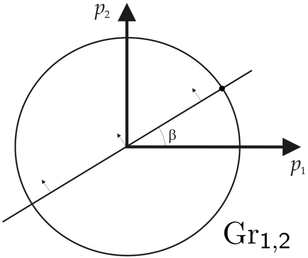
Quiz: Which one is area minimizing?





Grassmann manifold $\mathrm{Gr}_{1,2}$.

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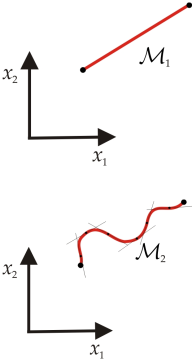

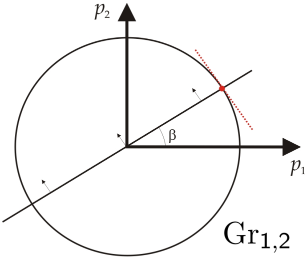

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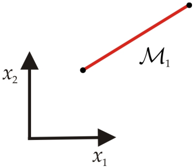

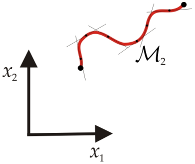

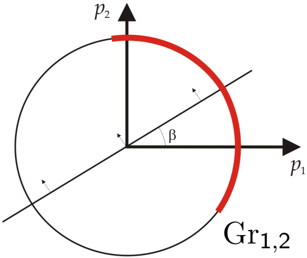

Grassmann manifold $\mathrm{Gr}_{1,2}$.

Example: $\mathrm{Gr}_{2,4}$

The oriented Grassmann variety $\mathrm{Gr}_{2,4}$ is defined by

$$I_{2,4} = \langle\, p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}\,,\; p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{23}^2 + p_{24}^2 + p_{34}^2 - 1 \,\rangle.$$

This is the highest weight orbit of $G = SO(4)$ acting on $\bigwedge^2 \mathbb{R}^4$.

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A linear change of coordinates

$$u = \tfrac{1}{\sqrt{2}}(p_{12} + p_{34}), \quad v = \tfrac{1}{\sqrt{2}}(p_{13} - p_{24}), \quad w = \tfrac{1}{\sqrt{2}}(p_{14} + p_{23}),$$
$$x = \tfrac{1}{\sqrt{2}}(p_{12} - p_{34}), \quad y = \tfrac{1}{\sqrt{2}}(p_{13} + p_{24}), \quad z = \tfrac{1}{\sqrt{2}}(p_{14} - p_{23}).$$

yields

$$I_{2,4} = \langle\, u^2 + v^2 + w^2 - \frac{1}{2}\,,\; x^2 + y^2 + z^2 - \frac{1}{2} \,\rangle.$$

The orbitope $\mathrm{conv}(\mathrm{Gr}_{2,4})$ is the direct product of two 3-balls of radius $1/\sqrt{2}$.

What about higher Grassmannians?

Central objects: spectrahedra and projections

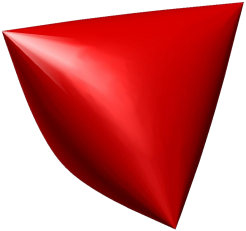
- Which other Grassmann orbitopes are **spectrahedra**, i.e.:

$$\mathcal{C} = \{x \in \mathbb{R}^n \,|\, A(x) \succeq 0\}$$

for some $A(x) = A_0 + \sum_{i=1}^{n} A_i x_i$ with symmetric $A_i \in \mathbb{R}^{N \times N}$?

Example: The spectrahedron defined by

$$\mathcal{C} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0\right\}.$$



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Only a few cases are known, see [Sanyal, Sottile, Sturmfels, '09]:

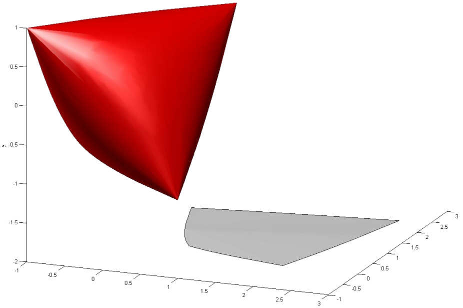
- $\text{conv}(\mathrm{Gr}_{2,n})$ is a spectrahedron.
- $\text{conv}(\mathrm{Gr}_{3,6})$ is *not* a spectrahedron.

Central objects: spectrahedra and projections

- Which Grassmann orbitopes are **projections of spectrahedra**, i.e.:

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m \text{ with } A(x,y) \succeq 0\}$$

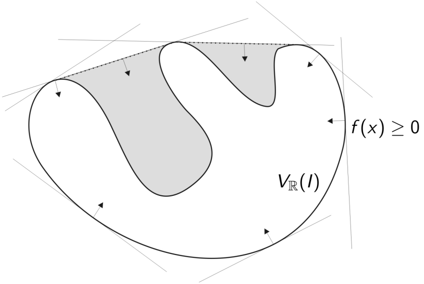
  for some $A(x,y) = A_0 + \sum_i A_i x_i + \sum_j B_j y_j$ with symmetric $A_i, B_j \in \mathbb{R}^{N \times N}$?
- How to construct such a *lifted SDP representation* for $\mathrm{conv}(\mathrm{Gr}_{k,n})$?



Special case: Convex hull of a real algebraic variety

Given an real radical ideal I with real variety $V_{\mathbb{R}}(I)$, we define the set of all supporting hyperplanes

$$F_{\text{supp}} = \left\{ f(x) = a^T x - b \mid f(x) \geq 0 \, \forall \, x \in V_{\mathbb{R}}(I) \right\}.$$



Real Variety with some supporting hyperplanes.

We obtain: $\overline{\text{conv}(V_{\mathbb{R}}(I))} = \{x \in \mathbb{R}^n \mid f(x) \geq 0 \, \forall f \in F_{\text{supp}}\}$

Special case: Convex hull of a real algebraic variety

Sequence of approximations called Moment relaxation (Lasserre/Laurent) or Theta bodies (Gouveia, Parrilo, Thomas) for an ideal I:

$$\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \ldots \supseteq \overline{\mathrm{conv}(V_{\mathbb{R}}(I))}$$

with

Def.: Theta body

The set of all points

$$\mathrm{TH}_k(I) = \{x \in \mathbb{R}^n \,|\, f \geq 0 \;\forall\, f \in F_{I,k}\}$$

where $F_{I,k}$ contains all affine polynomials $f = a^T x - b$ such that

$$f \equiv \sum \sigma_i^2 \mod I \text{ with } \deg(\sigma_i) \leq k.$$

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We call an ideal TH_k-exact if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(V_{\mathbb{R}}(I))}$.

Why do we care?

The set $\mathrm{TH}_k(I)$ is (the closure of) the projection of a spectrahedron

$$\mathrm{TH}_k(I) = \overline{\{x \in \mathbb{R}^n \,|\, \exists y \in \mathbb{R}^m \text{ with } A(x,y) \succeq 0\}}$$

...obtainable by a SOS/Moment construction!

Particularly interesting:

The first Theta body $\mathrm{TH}_1(I)$:

Theorem (Gouveia, Parrilo and Thomas, '08)

$$\mathrm{TH}_1(I) = \bigcap_{q \in \{\text{convex quadrics in } I\}} \operatorname{conv} V_{\mathbb{R}}(q)$$

Moderate size SDP representation:

- Semidefinite cone of size $(n+1) \times (n+1)$
- Number of variables bounded by $\binom{n+2}{2}$

The case $\operatorname{conv}(\mathrm{Gr}_{2,n})$

Facial structure of $\operatorname{conv}(\mathrm{Gr}_{2,n})$ is well known:

- Only $\lfloor n/2 \rfloor$ face orbits
- (Up to symmetry) only one inclusion maximal face

Theorem

All Grassmann orbitopes $\operatorname{conv}(\mathrm{Gr}_{2,n})$ *are* TH_1*-exact.*

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We have $\mathrm{TH}_1(I_{2,n}) \supseteq \mathrm{conv}(\mathrm{Gr}_{2,n})$. To show: Every inclusion maximal face $\mathcal{F} = \{x \in \mathbb{R}^n \mid f(x) = 0\}$ of $\mathrm{conv}(\mathrm{Gr}_{2,n})$ is also a face of $\mathrm{TH}_1(I_{2,n})$.

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Well, there is essentially only one... take e.g. $f(x) = 1 - \sum_i p_{2i-1,2i}$ and we can explicitly compute $f(x) \equiv \sum \sigma_i^2 \mod I_{2,n}$ with σ_i affine (thus $\mathcal{F}$ is also a face of $\mathrm{TH}_1(I_{2,n})$).

General Grassmann orbitopes

More general $\operatorname{conv}(\mathrm{Gr}_{k,n})$: *Face lattice* is more complicated (and only understood in certain cases).

E.g. for $\operatorname{conv}(\mathrm{Gr}_{3,6})$ we have, [Morgan, '85]:

- Four types of faces: vertices, edges, complex faces, special Lagrangian faces
- Inclusion maximal faces: Special Lagrangians and edges
- (Up to symmetry) only one special Lagrangian face orbit but *infinitely many* edge orbits

Showing TH_1-exactness requires infinitely many (or a parametrized) SOS decomposition or new techniques.

Linear optimization over the Grassmannian

Experimental evidence: Optimizing linear functionals $\phi(\xi)$ over $\mathrm{TH}_1(I_{k,n})$ and comparing with optimal value over $\mathrm{conv}(\mathrm{Gr}_{k,n})$ (where it is known).

That is, compare

$$\begin{array}{ll} \underset{\xi}{\text{maximize}} & \phi(\xi) \\ \text{subject to} & \xi \in \mathrm{conv}(\mathrm{Gr}_{k,n}). \end{array}$$

with

$$\begin{array}{ll} \underset{\lambda,\sigma_i}{\text{minimize}} & \lambda \\ \text{subject to} & \lambda - \phi \equiv \sum_i \sigma_i^2 \mod I_{k,n}, \\ & \lambda \in \mathbb{R}, \sigma_i \text{ affine poly's.} \end{array}$$

Linear optimization over the Grassmannian

Experimental evidence: Optimizing linear functionals $\phi(\xi)$ over $\mathrm{TH}_1(I_{k,n})$ and comparing with optimal value over $\mathrm{conv}(\mathrm{Gr}_{k,n})$ (where it is known).

Result:

- All tested cost functions for $\mathrm{conv}(\mathrm{Gr}_{3,6})$, $\mathrm{conv}(\mathrm{Gr}_{3,7}), \ldots,$ $\mathrm{conv}(\mathrm{Gr}_{4,8})$ lead to correct results.
- Corresponding optimal faces have correct dimension.
- Reasonable computation time even for relatively large orbitopes.

Computation times

Grassmannian	#vars	# Plücker rel's	Avg. time[1] [s]
conv($\mathrm{Gr}_{2,4}$)	6	1	<0.5
conv($\mathrm{Gr}_{2,6}$)	15	15	0.5
conv($\mathrm{Gr}_{2,8}$)	28	70	1
conv($\mathrm{Gr}_{2,10}$)	45	210	6
conv($\mathrm{Gr}_{2,12}$)	66	495	60
conv($\mathrm{Gr}_{2,13}$)	78	715	200
conv($\mathrm{Gr}_{3,6}$)	20	35	0.6
conv($\mathrm{Gr}_{3,7}$)	35	140	2
conv($\mathrm{Gr}_{3,8}$)	56	420	40
conv($\mathrm{Gr}_{3,9}$)	84	1050	570
conv($\mathrm{Gr}_{4,8}$)	70	721	180

Table: Average computation time for optimizing a generic cost function.

[1] @Lenovo T60, 2GHz, 1GB RAM

Example: conv($\mathrm{Gr}_{3,7}$)

Face type	Dim. in $\mathrm{TH}_1(I_{k,n})$	Dim.in conv($\mathrm{Gr}_{3,7}$)
Associative	27	27
Special Lagrangian	12	12
CP2	8	8
CP1	3	3
Double CP1	3	3
Vertex	0	0
Edge	1	1
S3	13	13
S2	8	8
S1	4	4

Table: All types of faces of conv($\mathrm{Gr}_{3,7}$), [Harvey, Morgan, '86] can be found in $\mathrm{TH}_1(I_{3,7})$!

The Harvey Lawson Conjecture

In [Harvey and Lawson, '82] it is conjectured that

$$\begin{aligned} \underset{\xi}{\text{maximize}} \quad & \phi(\xi) \\ \text{subject to} \quad & \xi \in \text{conv}(\mathrm{Gr}_{k,n}). \end{aligned}$$

is equivalent to

$$\begin{aligned} \underset{\lambda,\sigma_i}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \lambda^2 \sum_I p_I^2 - \phi^2 \equiv \sum_i \sigma_i^2 \mod J_{k,n} \\ & \lambda \in \mathbb{R}, \sigma_i \text{ linear poly's.} \end{aligned}$$

where $J_{k,n} = \langle \text{quad. Plücker rel's} \rangle$ ~~$+ \langle \sum_I p_I^2 - 1 \rangle$~~.

Conclusion

- (Lifted-) Spectrahedral descriptions for orbitopes are desirable
- TH_k-bodies/moment relaxations often generate good approximations
- Grassmann orbitopes $\mathrm{conv}(\mathrm{Gr}_{2,n})$ are TH_1-exact
- Strong numerical evidence for higher Grassmannians to be TH_1-exact

What other Grassmann orbitopes $\mathrm{conv}(\mathrm{Gr}_{k,n})$ are TH_1-exact? All?

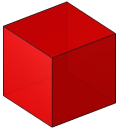
How about other orbitopes?

Recall from Frank's talk: Sections of Orbitopes

Tautological orbitope:

$$\text{conv}(\mathrm{O}(n)) = \text{conv}\{g \cdot \mathrm{Id}_n \mid g \in \mathrm{O}(n)\}$$

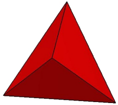
face orbits characterizable by a *cube*.



Cube for $\text{conv}(\mathrm{O}(n))$.

$$\text{conv}(\mathrm{SO}(n)) = \text{conv}\{g \cdot \mathrm{Id}_n \mid g \in \mathrm{SO}(n)\}$$

face orbits characterizable by a *halfcube*.

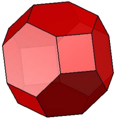


Halfcube for $\text{conv}(\mathrm{SO}(n))$.

Schur-Horn orbitopes:

$$\mathcal{O}_M = \text{conv}\left\{g \cdot M \cdot g^T \mid g \in \mathrm{SO}(n)\right\}$$

face orbits characterizable by a *permutahedron*.



Permutahedron for $\mathcal{O}_M$.

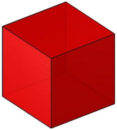
Results on other orbitopes

Theorem

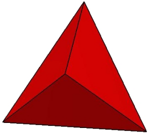
Each of the previous orbitopes is TH_k*-exact if the underlying polytope is* TH_k*-exact. (For* $k = 1$ *also the reverse implication holds.)*

More precisely:

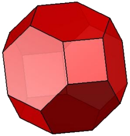
- The $\mathrm{O}(n)$-orbitopes are TH_1-exact.
- The $\mathrm{SO}(n)$-orbitopes are TH_1-exact only for $n = 1, 2, 3, 4$.
- The symmetric/skew symmetric Schur-Horn orbitopes are usually not TH_1-exact.



Cube for $\mathrm{conv}(\mathrm{O}(n))$.



Halfcube for $\mathrm{conv}(\mathrm{SO}(n))$.



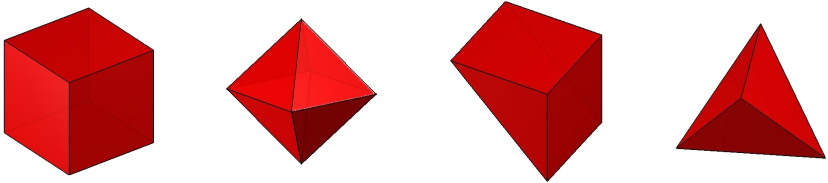
Permutahedron for $\mathcal{O}_M$.

TH_1-exact polytopes

Theorem (Gouveia, Parrilo and Thomas, '08)

For a finite set $S \subset \mathbb{R}^n$, the vanishing ideal $I(S)$ is TH_k-exact if $P = \text{conv}(S)$ is a $(k+1)$-level[a] polytope. (For $k = 1$ also the reverse implication holds.)

[a] i.e. $P = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0\}$ s.t. all g_i take at most $k+1$ different values on S.



Different 2-level polytopes.

Questions?

Thank you very much for your attention!

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