Approximation algorithms for SDPs with rank constraints

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Convex algebraic geometry at Banff
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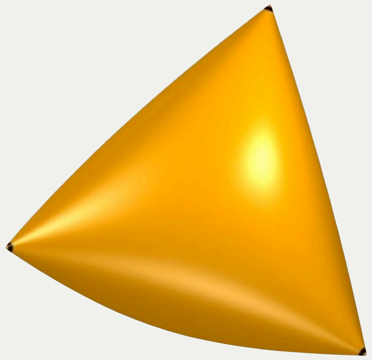
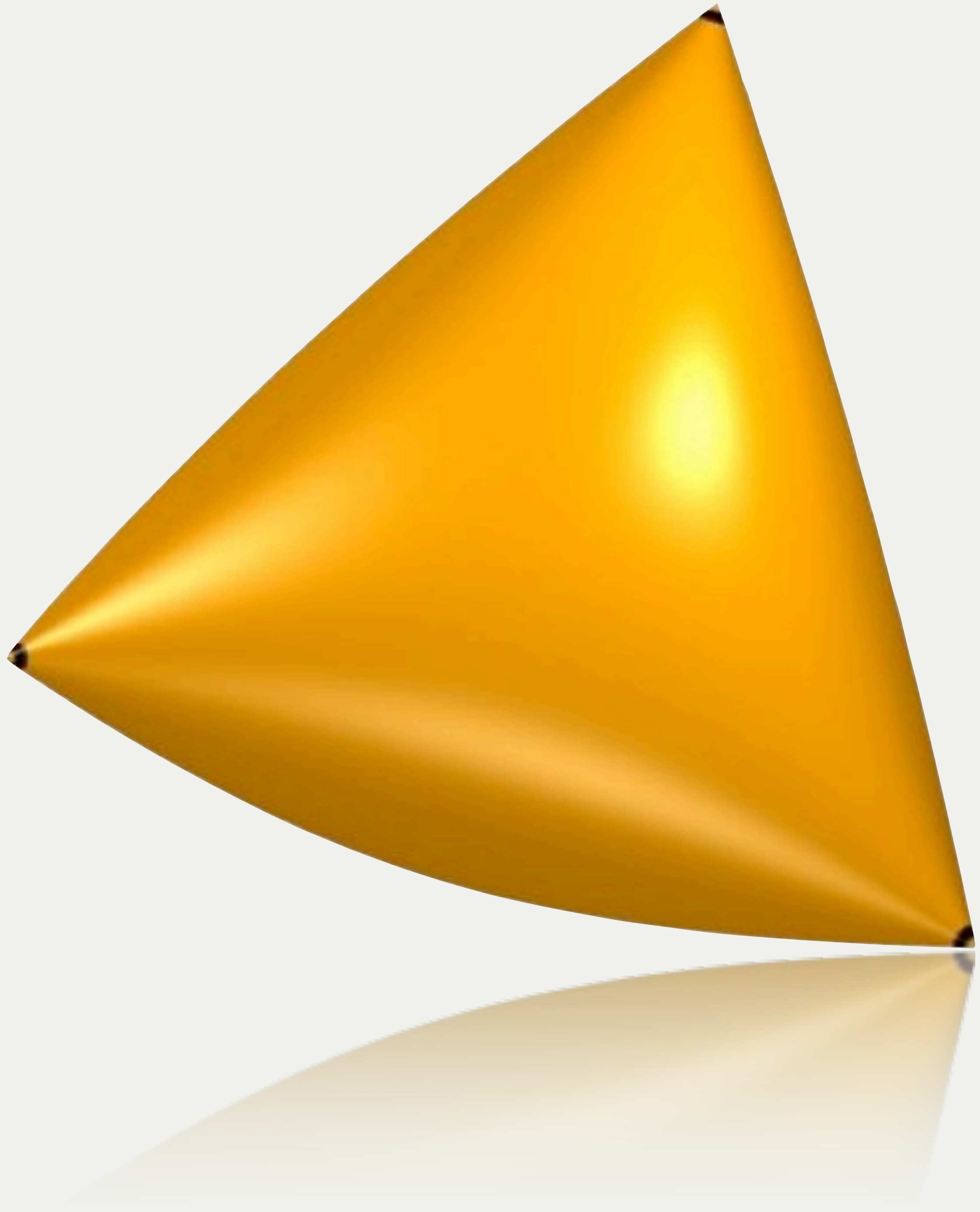


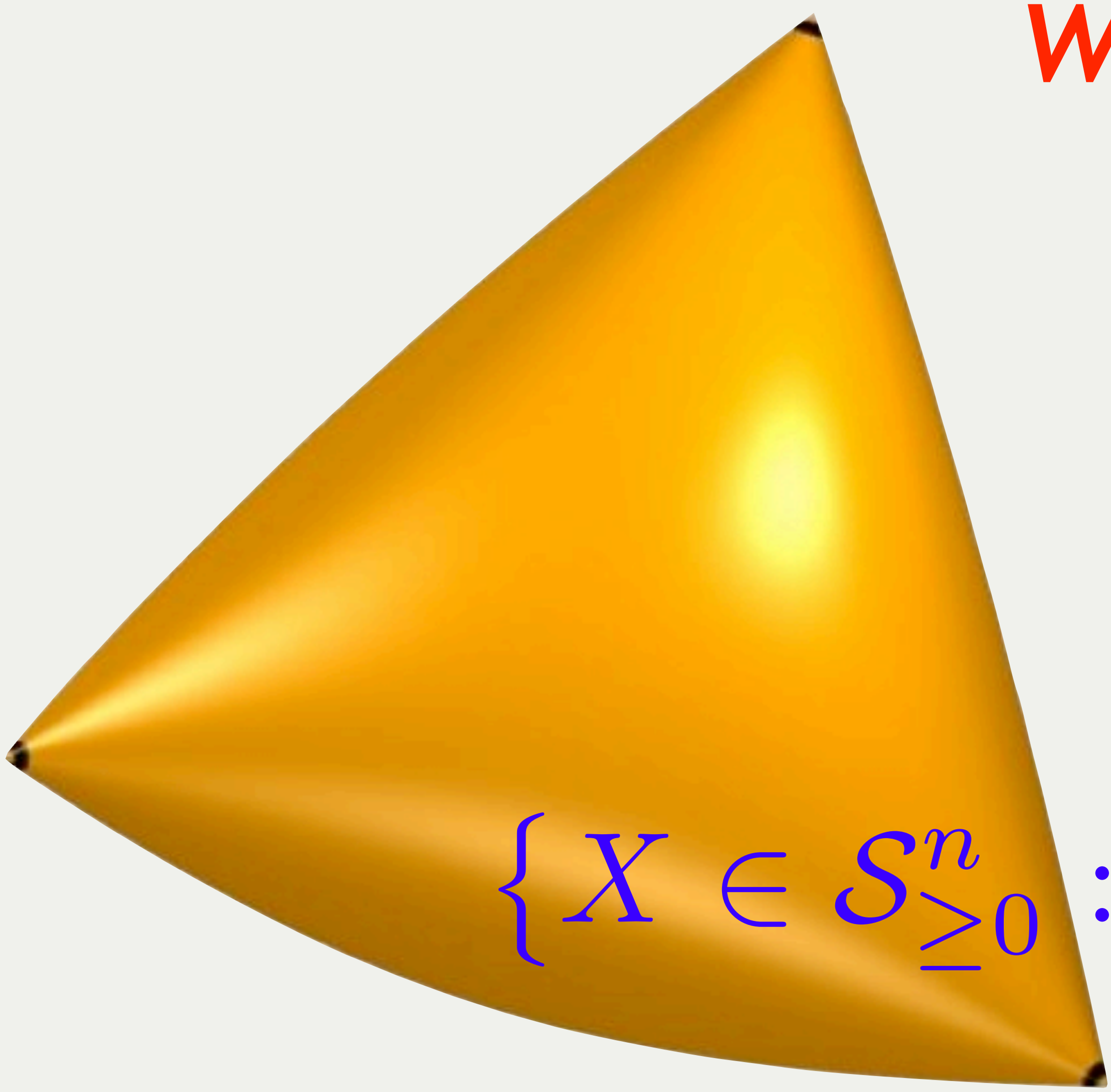
Overview

1. The problem
2. Classical Grothendieck inequalities
3. New Grothendieck inequalities
4. Approximation algorithm

1. The problem

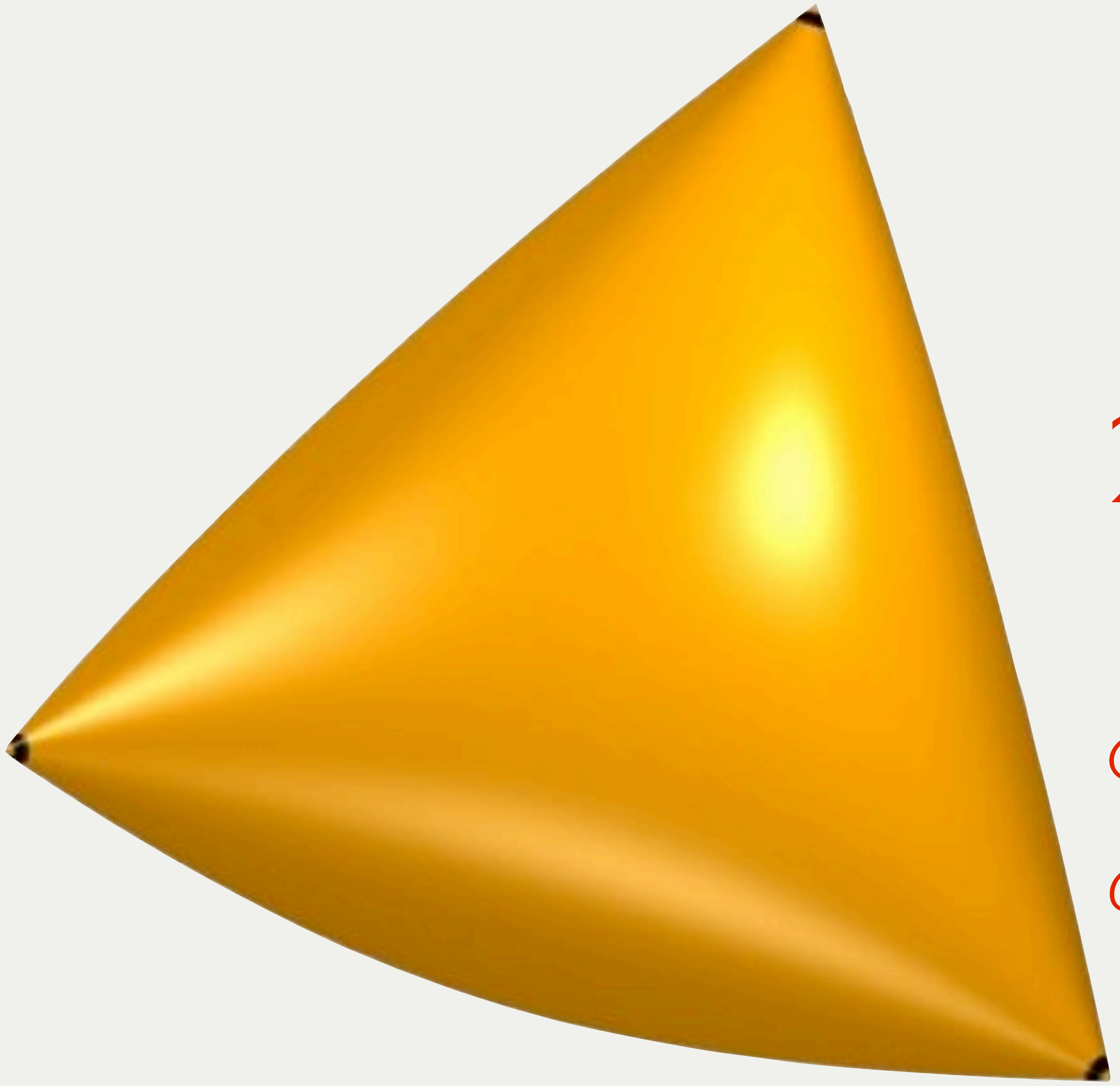


linear optimization
over elliptope
with rank constraint



$$\{X \in \mathcal{S}^n_{\geq 0} : X_{11} = \ldots = X_{nn} = 1\}$$

$$\max \{ \langle A, X \rangle : X \in \text{elliptope}, \text{rank } X \leq k \}$$



we care about:

1. hardness results,
2. approximation algorithms

depending on the rank

and structure of objective matrix

$$\mathrm{SDP}_k(A) = \max\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, x_i \cdot x_j \,:\, x_i \in S^{k-1} \right\}$$

Grothendieck problem with rank constraint

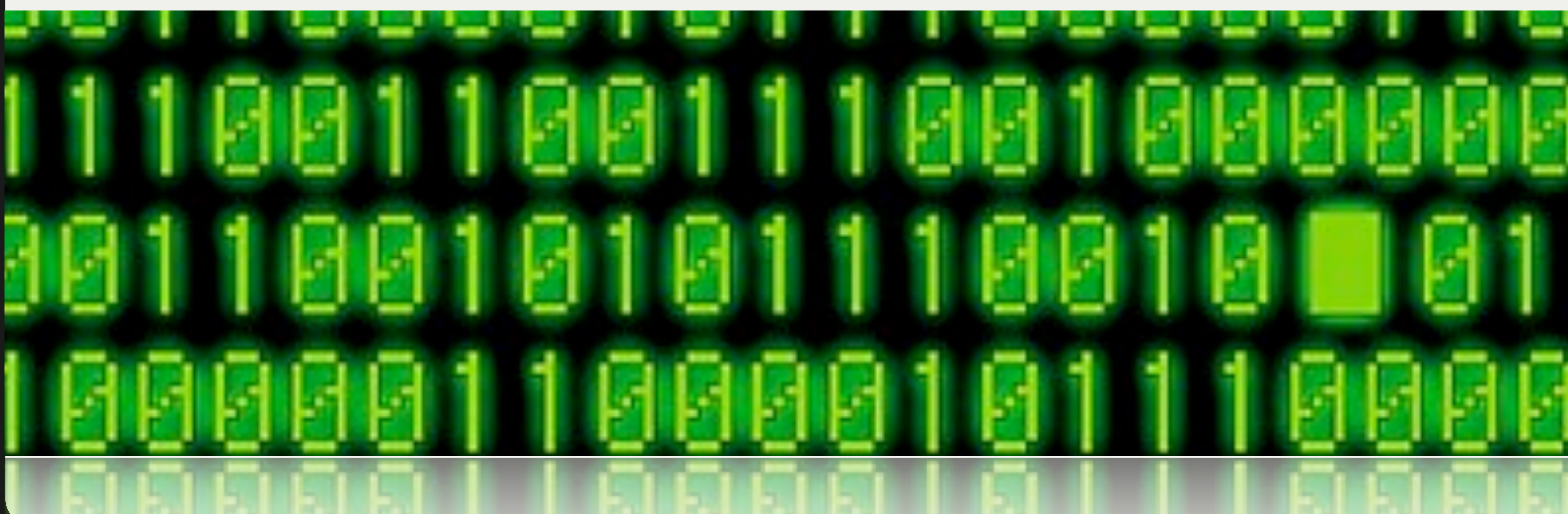
A lot of recent and beautiful work

fundamental and unifying problem in many areas:

optimization, functional analysis, complexity theory, combinatorics, quantum information

$$k = 1$$

2. Classical Grothendieck inequalities

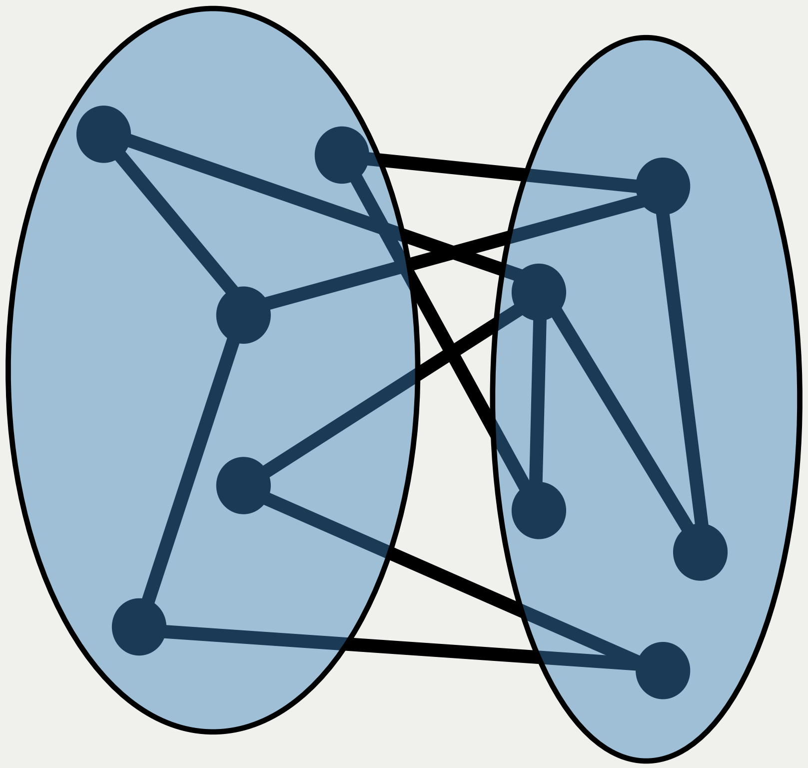


k equals one: difficult

$$\mathrm{SDP}_1(A) = \max\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, x_i \cdot x_j : x_i \in S^0\right\}$$

$$x_i \in \{-1,+1\}$$

is NP-hard (MAXCUT is special case).



$$\mathrm{MAXCUT(G)} = \mathrm{SDP}_1(L_G)$$

L_G — Laplacian matrix of graph G

k large: easy

$$\mathrm{SDP}_\infty(A) = \max\left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, x_i \cdot x_j : x_i \in S^\infty \right\}$$

is an SDP without rank constraint.

How big is the gap?

want to prove theorems like:

Given a property P there is a smallest constant $K_{P,k}$ so that:

For all matrices A having property P:

$$\mathrm{SDP}_k(A) \leq \mathrm{SDP}_\infty(A) \leq K_{P,k}\, \mathrm{SDP}_k(A)$$

Grothendieck inequality

A (randomized) polytime ap...

Assuming UGC: no polytime algorithm can do better.

Given a property P there is a smallest constant $K_{P,k}$ so that:
For all matrices A having property P:

$$\mathrm{SDP}_k(A) \leq \mathrm{SDP}_\infty(A) \leq K_{P,k}\ \mathrm{SDP}_k(A)$$

A (randomized) polytime approximation algorithm achieves K_P.
Assuming UGC: no polytime algorithm can do better.

problems which have been studied:

1. $K_{G,k}$: A is of the form $\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$
2. $K_{\succeq 0,k}$: A is positive semidefinite
3. $K_{mc,k}$: A is Laplacian matrix of a graph
4. $K_{n,k}$: A is of size n and $A_{ii} = 0$
5. $K_{\Gamma,k}$: support of A gives adjacency matrix of graph Γ

relations

1. $K_{G,k}$: A is of the form $\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$

inequality: Krivine 1978, Reeds 1993

$$1.67\ldots \leq K_{G,1} \leq \frac{\pi}{2\log(1+\sqrt{2})} = 1.78\ldots$$

algorithm: Alon, Naor 2006

UGC hardness: Raghavendra, Steurer 2009

No polytime algorithm attaining $K_{G,1} - \varepsilon$

2. $K_{\succeq 0,k}$: A is positive semidefinite

inequality: Rietz 1974, Grothendieck 1953

$$K_{\succeq 0,1} = \frac{\pi}{2} = 1.57\ldots$$

algorithm: Nesterov 1997

UGC hardness: Khot, Naor 2008

No polytime algorithm attaining $K_{\succeq 0,1} - \varepsilon$

3. $K_{mc,k}$: A is Laplacian matrix of a graph

inequality: Goemans, Williamson 1996, Feige, Schechtman 2002

$$K_{mc,1} = 1.13\ldots$$

algorithm: Goemans, Williamson 1996

UGC hardness: Khot, Kindler, Mossel, O'Donnell 2007

No polytime algorithm attaining $K_{mc,1} - \varepsilon$

4. $K_{n,k}$: A is of size n and $A_{ii} = 0$

inequality: Nemirovski, Roos, Terlaky 1999
Charikar, Wirth 2004,
Alon, Naor, Makarychev, Makarychev 2006

$$K_{n,1} = \Theta(\log n)$$

algorithm: Nemirovski, Roos, Terlaky 1999
Charikar, Wirth 2004,

UGC hardness: not completely settled (but almost)

Arora, Berger, Hazan, Kindler, Safra 2005

5. $K_{\Gamma,k}$: support of A gives adjacency matrix of graph Γ

inequality: Alon, Naor, Makarychev, Makarychev 2006

$$\Omega(\log \omega(\Gamma)) \leq K_{\Gamma,1} \leq O(\log \vartheta(\overline{\Gamma}))$$

algorithm: Alon, Naor, Makarychev, Makarychev 2006

UGC hardness: nothing specific known

given:

M — set of labels

$G = (V \cup W, E)$ — bipartite graph

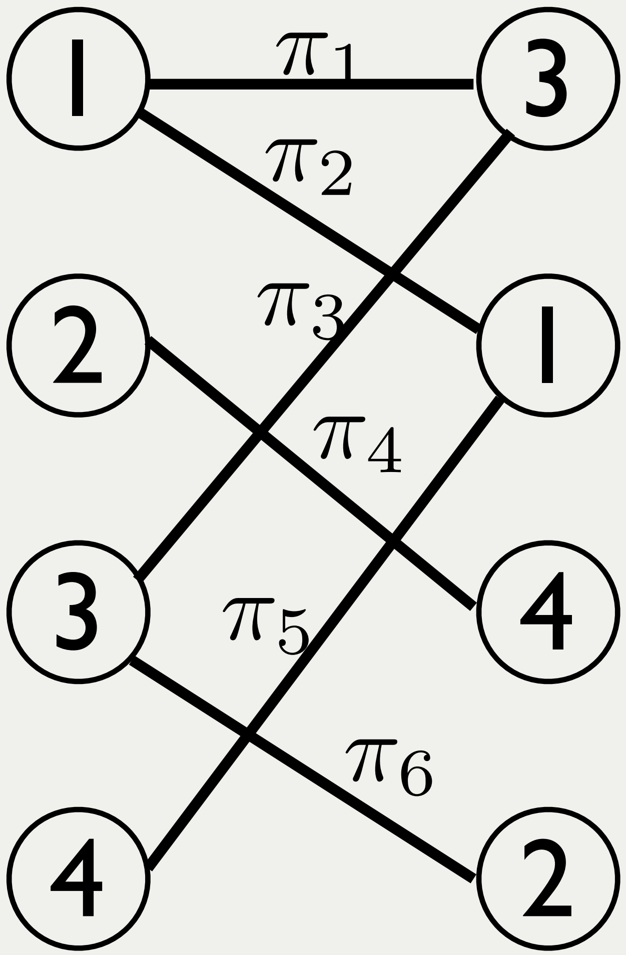
$\pi_e : M \to M$ — permutation for every edge $e \in E$.

find:

$f : V \cup W \to M$ — labeling of vertices

satisfying as many permutations as possible:

$$\pi_{(v,w)}(f(v)) = f(w)$$



Unique games conjecture (Khot 2002):

There is no polynomial time algorithm which distinguishes between instances where almost all or almost none permutations are satisified.

$$k > 1$$

3. New Grothendieck inequalities

$$\mathrm{SDP}_k(A) = \max\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, x_i \cdot x_j : x_i \in S^{k-1} \right\}$$

introduced in the context of quantum nonlocality

A generalized Grothendieck inequality and entanglement in XOR games

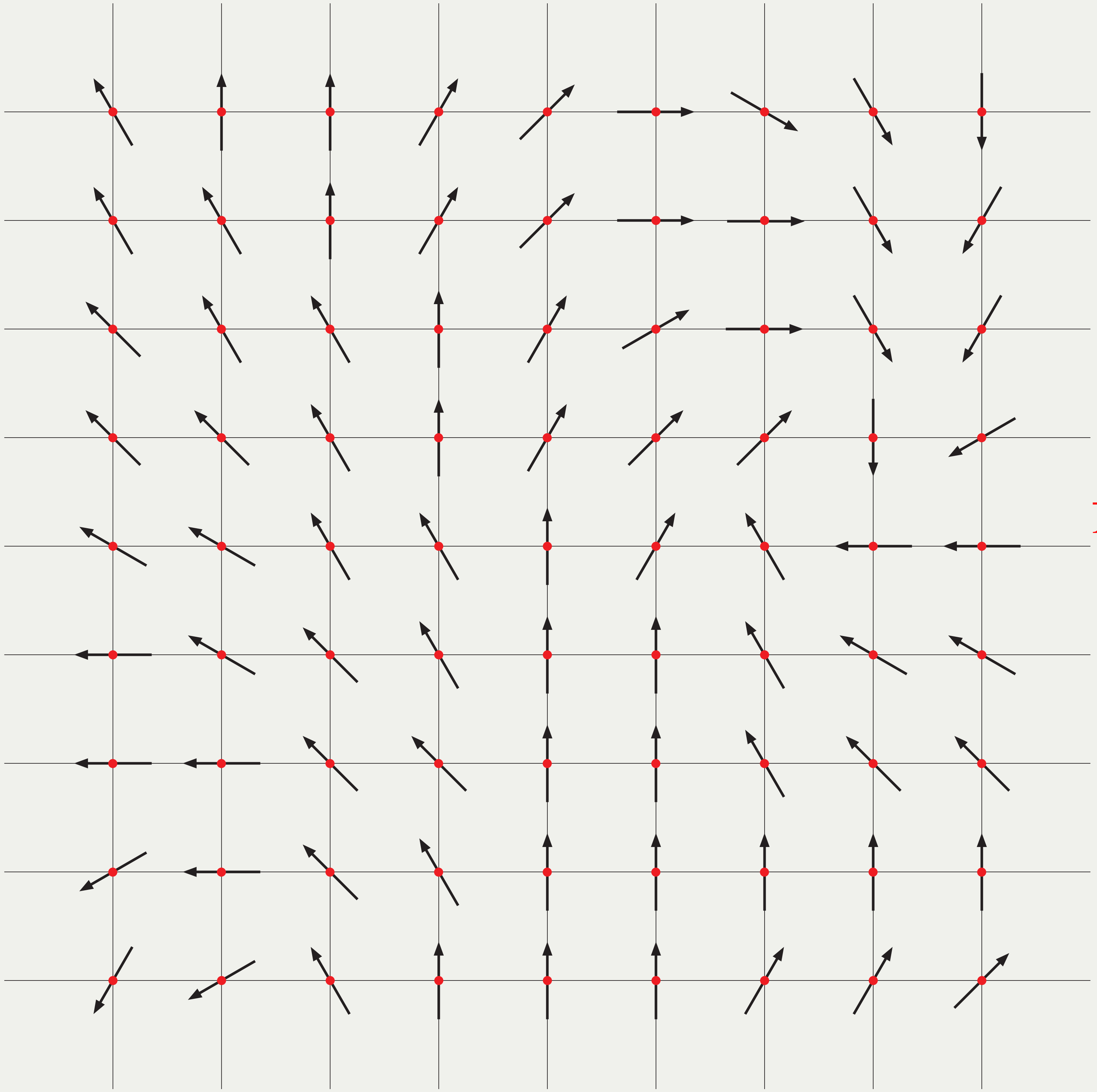
Jop Briët* Harry Buhrman* Ben Toner*

January 14, 2009

Abstract

Suppose Alice and Bob make local two-outcome measurements on a shared entangled state. For any d, we show that there are correlations that can only be reproduced if the local dimension is at least d. This resolves a conjecture of Brunner et al. [*Phys. Rev. Lett.* 100, 210503 (2008)] and establishes that the amount of entanglement required to maximally violate a Bell inequality must depend on the number of measurement settings, not just the number of measurement outcomes. We prove this result by establishing the first

XY-model



A_{ij} — potential between i and j

$u_1, \ldots, u_n \in S^1$ — spins

find ground state = minimize

$$H = -\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, u_i \cdot u_j$$

total energy

1. $K_{G,k}$: A is of the form $\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$

inequality: Haagerup 1987, Briet, Buhrman, Toner 2009

$$1.27\ldots \leq K_{G,2} \leq 1.40\ldots$$

algorithm: Haagerup's argument is algorithmic

UGC hardness: nothing specific known

2. $K_{\succeq 0,k}$: A is positive semidefinite

inequality: BOV 2009, Briet, Buhrman, Toner 2009

$$K_{\succeq 0,k} = \frac{k}{2}\left(\frac{\Gamma(k/2)}{\Gamma((k+1)/2)}\right)^2 = 1 + \Theta(1/k)$$

$$K_{\succeq 0,1} = \pi/2 = 1.57\ldots$$

$$K_{\succeq 0,2} = 4/\pi = 1.27\ldots$$

$$K_{\succeq 0,3} = (3\pi)/8 = 1.17\ldots$$

algorithm: BOV 2009

UGC hardness: BOV 2009

No polytime algorithm attaining $K_{\succeq 0,k} - \varepsilon$

3. $K_{mc,k}$: A is Laplacian matrix of a graph

inequality: BOV 2009

$$K_{mc,1} = 1.13\ldots$$
$$K_{mc,2} \leq 1.06\ldots$$
$$K_{mc,3} \leq 1.04\ldots$$

algorithm: BOV 2009

UGC hardness: nothing specific known

4. $K_{n,k}$: A is of size n and $A_{ii} = 0$

inequality: nothing specific known

algorithm: nothing specific known

UGC hardness: nothing specific known

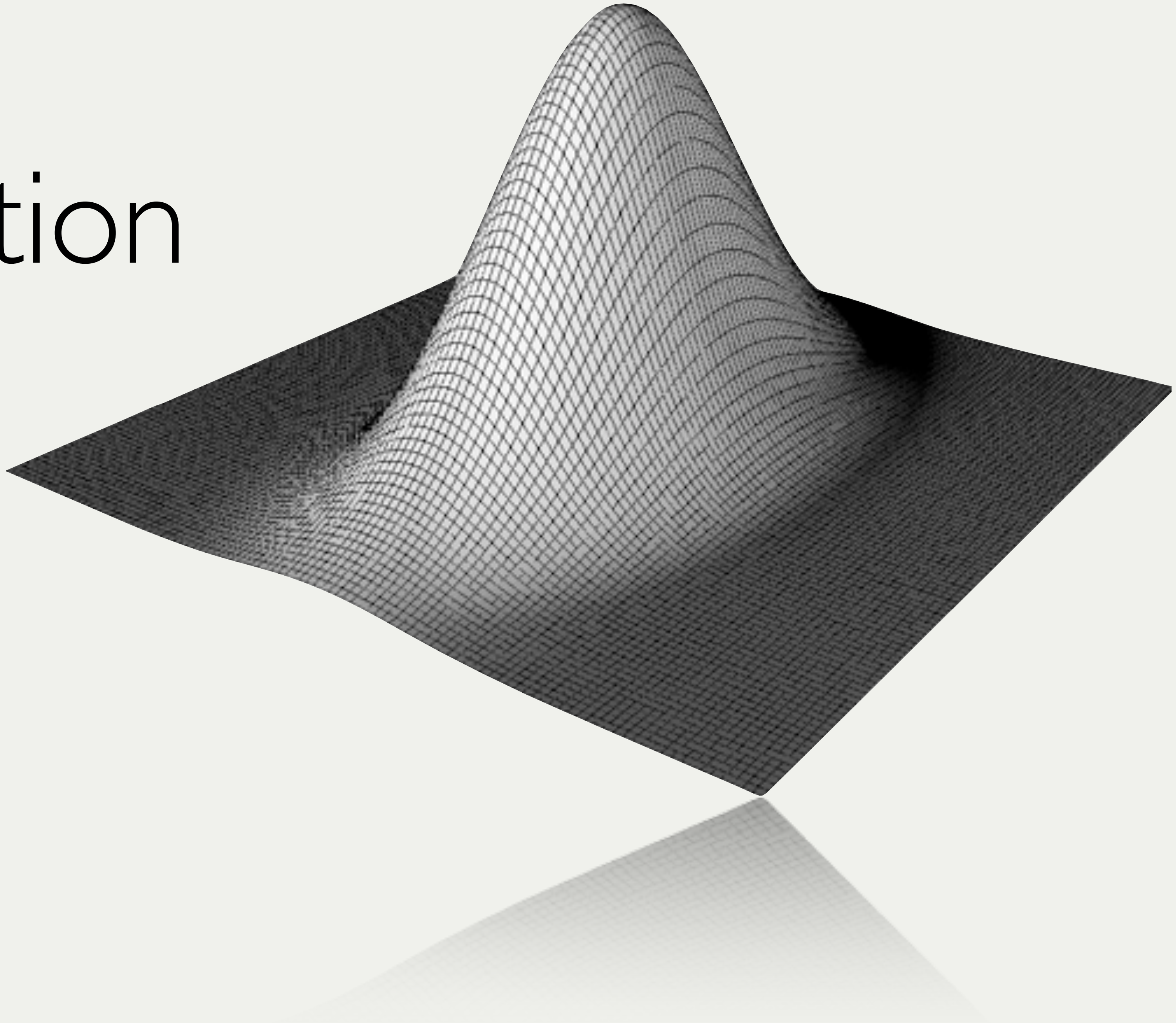
5. $K_{\Gamma,k}$: support of A gives adjacency matrix of graph Γ

inequality: nothing specific known

algorithm: nothing specific known

UGC hardness: nothing specific known

4. Approximation algorithm



Approximation algorithm

1. Solve $\text{SDP}_\infty(A)$. Gives vectors $v_1, \ldots, v_n \in S^{n-1}$.
2. Take random $k \times n$ Gaussian matrix $Z = (Z_{ij})$, $Z_{ij} \sim N(0,1)$.
3. Round vectors $x_i = \frac{Zv_i}{\|Zv_i\|} \in S^{k-1}$.
4. Expected approximation of SDP_k is

$$\text{SDP}_k(A) \geq \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i \cdot x_j\right] = \sum_{i=1}^n \sum_{j=1}^n A_{ij} \mathbb{E}\left[x_i \cdot x_j\right]$$

$$\geq \gamma(k) \sum_{i=1}^n \sum_{j=1}^n A_{ij} v_i \cdot v_j = \gamma(k)\text{SDP}_\infty(A)$$

$$\gamma(k) = \frac{2}{k}\left(\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}\right)^2$$

2 important properties of

$$E_k(v_i, v_j) = \mathbb{E}\left[x_i \cdot x_j\right] = \mathbb{E}\left[\frac{Zv_i}{\|Zv_i\|} \cdot \frac{Zv_j}{\|Zv_j\|}\right]$$

1. $E_k(v_i, v_j)$ only depends on the inner product $v_i \cdot v_j \in [-1, 1]$

2. $E_k : [-1, 1] \to \mathbb{R}$ is of *positive type*, i.e.

$$\begin{pmatrix} E_k(u_1 \cdot u_1) & \dots & E_k(u_1 \cdot u_m) \\ \vdots & \vdots & \vdots \\ E_k(u_m \cdot u_1) & \dots & E_k(u_m \cdot u_m) \end{pmatrix} \in \mathcal{S}^m_{\geq 0}$$

for all choices of $u_1, \dots, u_m \in S^{n-1}$

Schoenberg's characterization (1942)

A continuous function $f : [-1, 1] \to \mathbb{R}$ is of positive type

$\iff$ it can be represented as

$$f(z) = \sum_{i=0}^{\infty} f_i z^i \qquad f_0, f_1, f_2, \ldots \geq 0 \qquad \sum_{i=0}^{\infty} f_i < \infty$$

$\Longleftarrow$ follows from Schur product

if $X \in \mathcal{S}^n_{\geq 0}$

$$f(X) = \sum_{i=0}^{\infty} f_i \underbrace{(X \circ \ldots \circ X)}_{i \text{ times}} \in \mathcal{S}^n_{\geq 0}$$

subtracting the linear term

$$E_k(z) = \sum_{i=0}^{\infty} f_i z^i \qquad f_0, f_1, f_2, \ldots \geq 0$$

Hence,

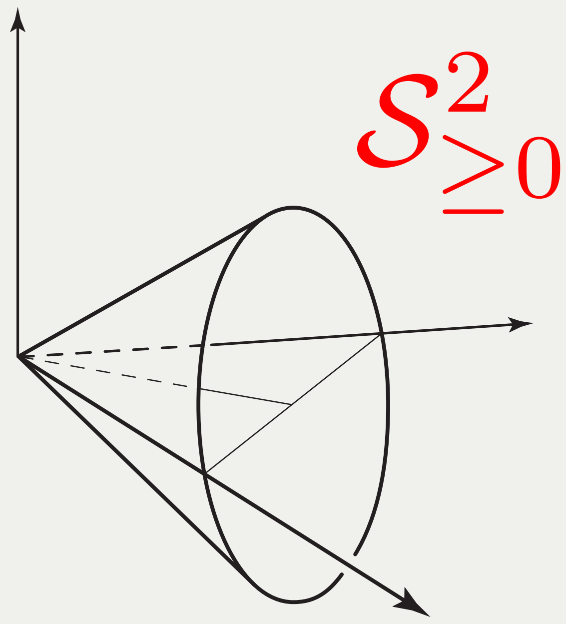
$$z \mapsto E_k(z) - f_1 z \qquad \text{is of positive type}$$

Hence,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} E_k(v_i \cdot v_j) = \langle A, (E_k(v_i \cdot v_j))_{ij} \rangle$$

$$\geq \langle A, f_1 (v_i \cdot v_j)_{ij} \rangle = f_1 \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} v_i \cdot v_j$$

What's f_1?



Now the real work starts…

$$E_k(z) = \frac{1}{2^k \Gamma_2(k/2)} \int_{\mathcal{S}^2_{\geq 0}} \frac{x^\mathsf{T} Y y}{\sqrt{(x^\mathsf{T} Y x)(y^\mathsf{T} Y y)}} e^{\mathrm{Tr}(Y)/2} (\det Y)^{(k-3)/2} dY$$

$$x = (1,0)^\mathsf{T}, \quad y = (z, \sqrt{1-z^2})^\mathsf{T}$$

$Y \in \mathcal{S}^2_{\geq 0}$ — distributed according to Wishart distribution

$$Y = Z^\mathsf{T} Z,$$

$$Z = (Z_{ij}) \in \mathbb{R}^{k \times 2}, \quad Z_{ij} \sim N(0,1)$$

$$
\begin{aligned}
f_1 &= \frac{\partial E_k}{\partial z}(0) \\
&= \dots \\
&= \frac{k-1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{r(1-r^2)^{(k-1)/2}}{(1-r^2(\sin\phi)^2)^{3/2}} d\phi dr \\
&= \dots \\
&= \frac{2}{k} \left(\frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \right)^2 \\
&= \gamma(k)
\end{aligned}
$$

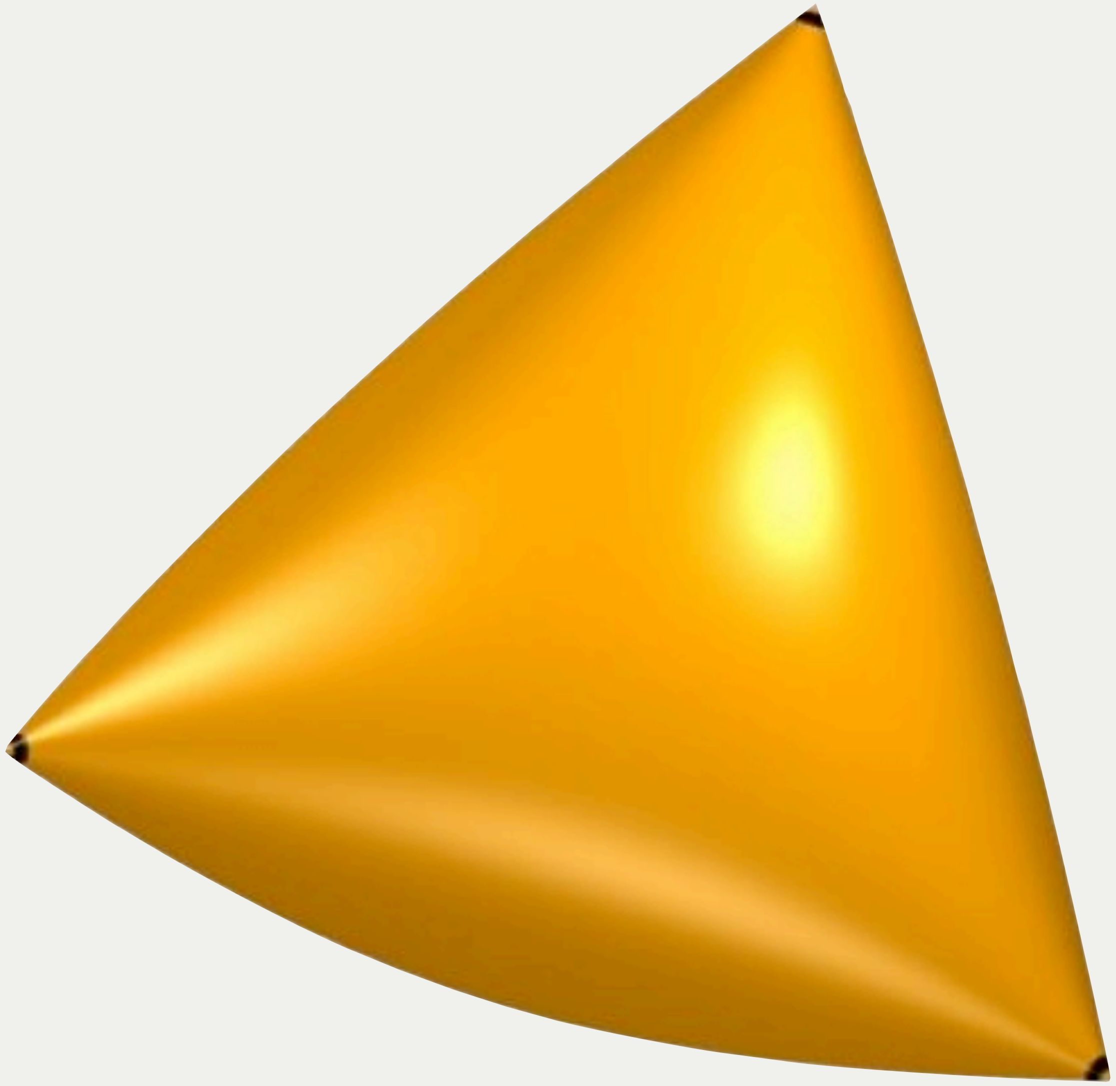
Reference

THE POSITIVE SEMIDEFINITE GROTHENDIECK PROBLEM WITH RANK CONSTRAINT

JOP BRIËT, FERNANDO MÁRIO DE OLIVEIRA FILHO, AND FRANK VALLENTIN

ABSTRACT. Given a positive integer n and a positive semidefinite matrix $A = (A_{ij}) \in \mathbb{R}^{m \times m}$ the positive semidefinite Grothendieck problem with rank-n-constraint is

$$(\mathrm{SDP}_n) \quad \text{maximize} \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij}\, x_i \cdot x_j, \text{ where } x_1, \ldots, x_m \in S^{n-1}.$$



Thank you