Large-time asymptotic behavior of multi-cut solutions to Hele-Shaw flows

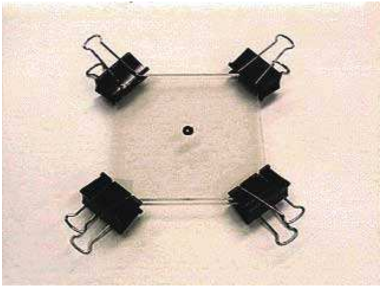
Yu-Lin Lin[1]

Joint work with B. Gustafsson

Integrable and stochastic Laplacian growth in modern mathematical physics

November 04, 2010

Experience by Hele-Shaw in 1898



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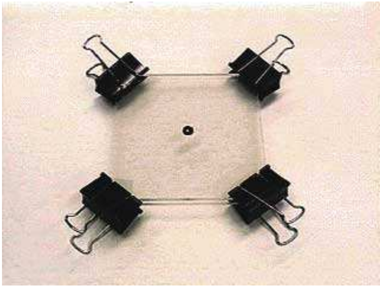

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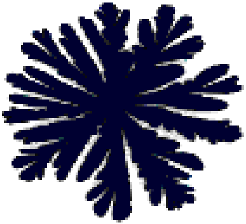

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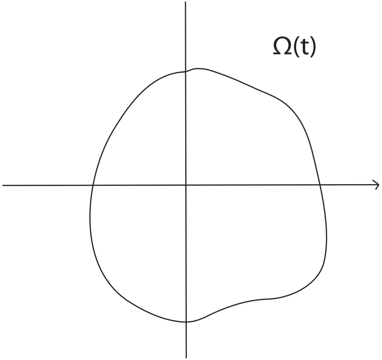
- Inject viscid fluid (glycerol) into inviscous fluid (colored water) slowly.
- The boundary is getting better.

The moving domain

- $p(z,t)$: pressure at $z \in \Omega(t)$.
- $\kappa(z,t)$: curvature at $z \in \partial\Omega(t)$.
- $v_n(z,t)$: normal velocity at z on $\partial\Omega(t)$.
- γ : surface tension.
- n : unit normal.
- Q: injection rate.



The free boundary problem

$$\begin{cases} \Delta p = -Q\delta_0 & \text{in } \Omega(t), \\ p = \gamma\kappa & \text{on } \partial\Omega(t), \\ v_n = -\frac{\partial p}{\partial n} & \text{on } \partial\Omega(t). \end{cases}$$

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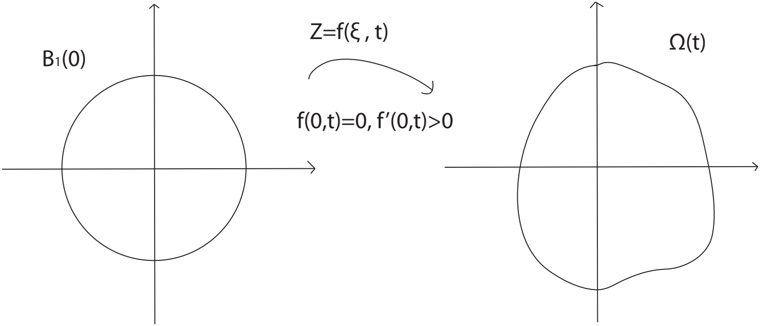
- Consider the problem $\gamma = 0$ now.
- Injection with speed Q; $\Omega(s) \subset \Omega(t)$ if $s < t$.

P. Ya. Polubarinova-Kochina and L. A. Galin (1945) gave a conformal formulation of the Hele-Shaw problem.

Reformulation by Riemann mapping



- $f(\zeta, t)$ is univalent in the closed unit disk $\overline{B_1(0)}$.

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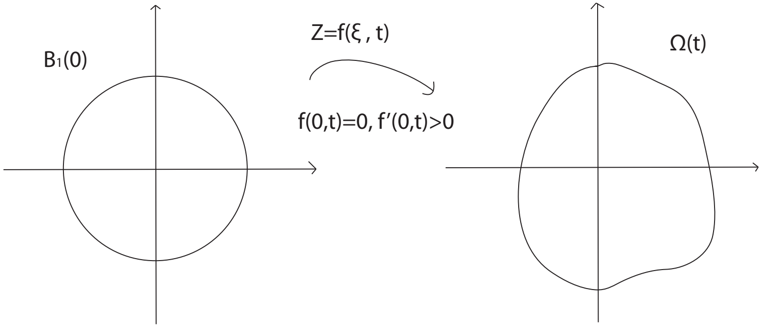

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  $O(E) = \{f(\zeta) \mid f(\zeta)$ is univalent in E , $f(0) = 0$ and $f'(0) > 0\}$.

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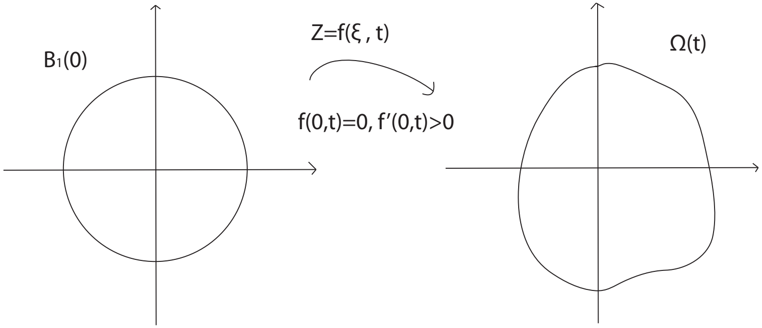

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- Definition:
  $O(E) = \{f(\zeta) \mid f(\zeta)$ is univalent in E , $f(0) = 0$ and $f'(0) > 0\}$.
- $f(\zeta, t) \in O(\overline{B_1(0)})$.

The Polubarinova-Galin equation

$$Re\Big[\frac{d}{dt}f(\zeta,t)\overline{f'(\zeta,t)\zeta}\Big]=\frac{Q}{2\pi},\zeta\in\partial B_1(0),f(\zeta,t)\in O(\overline{B_1(0)}).$$

A reformulation of the P-G equation

B. Gustafsson (1984) gave a new formulation of the P-G equation to be a Löwner-Kufarev type equation.
For $f(\zeta, t) \in O(\overline{B_1(0)})$

$$f_t(\zeta, t) = \frac{Q}{2\pi} \frac{f'(\zeta, t)\zeta}{2\pi i} \int_{\partial B_1(0)} \frac{1}{|f'(z, t)|^2} \frac{z+\zeta}{z-\zeta} \frac{dz}{z}, |\zeta| < 1.$$

Definition of a strong solution to the P-G equation

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  A solution $f(\zeta,t)$ is a strong solution of the P-G equation if $f(\zeta,t) \in O(\overline{B_1(0)})$ is continuously differentiable with respect to t in $[0,\epsilon)$.

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- **Definition**

  If a strong solution $f(\zeta,t)$ fails to exist at $t = T_0$, we say the strong solution $f(\zeta,t)$ blows up at $t = T_0$.

Two special general solutions

- B. Gustafsson (1984) found a general set of solutions

$$f(\zeta,t)=\sum_{j=1}^{m} d_j(t)\zeta^j + d_0(t) + \sum_{l=1}^{n}\sum_{k=1}^{s_l}\frac{a_{l,k}(t)}{(\zeta-\zeta_l(t))^k}$$

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- Abanov, Ar. and Mineev-Weinstein, M. and Zabrodin, A. (2009) found multi-cut solutions

$$f(\zeta,t)=\sum_{j=1}^{m}d_j(t)\zeta^j+d_0(t)+\sum_{l=1}^{n}\sum_{k=1}^{s_l}\frac{a_{l,k}(t)}{(\zeta-\zeta_l(t))^k}+\sum_{l=1}^{m_0}e_l\ln(\zeta-\zeta_{-l}(t)),$$

where e_l are constant.

The Richardson complex moments

- Given $\Omega(t)$ which solves the problem, then the Richardson complex moments are

$$M_k(t) = \frac{1}{\pi}\int_{\Omega(t)} z^k dxdy, z = x + iy, k \geq 0.$$

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$$(a) M_0(t) = M_0(0) + \frac{Q}{\pi}t, \qquad (b) M_k(t) = M_k(0), k \geq 1.$$

- If $\Omega(t) = f_{k_0}(B_1(0), t)$ where $f_{k_0}(\zeta, t) = a_1(t)\zeta + \cdots + a_{k_0}(t)\zeta^{k_0}$ is a polynomial strong solution,

$$\begin{aligned} M_k(f_{k_0}(\zeta, t)) &= \frac{1}{2\pi i}\int_{\partial B_1(0)} f_{k_0}^k(\zeta, t) f_{k_0}'(\zeta, t)\overline{f_{k_0}(\zeta, t)} d\zeta \\ &= \sum_{i_1, \cdots, i_{k+1}} i_1 a_{i_1}(t) a_{i_2}(t) \cdots a_{i_{k+1}}(t)\overline{a_{i_1 + \cdots + i_{k+1}}(t)}. \quad (1) \end{aligned}$$

$M_{k_0}, M_{k_0+1}, \cdots = 0.$

The strong global solutions

- Some solutions blow up

The strong global solutions

- Some solutions blow up
- Some solutions are global.

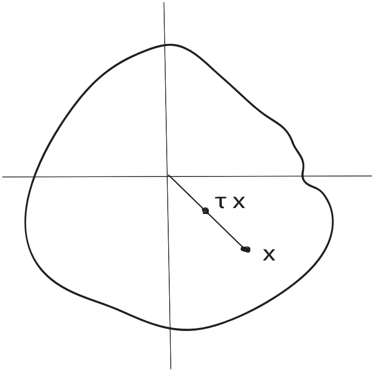
Now assume $Q = 2\pi$

The global dynamics

B. Gustafsson, D. Prokhorov, and A. Vasilev(2004)

Theorem

If $f(\zeta,0)$ is a starlike function and $f(\zeta,0)\in O(\overline{B_1(0)})$, then the strong solution is global and $f(\zeta,t)$ is starlike forever.



M.Sakai (1998)

Theorem

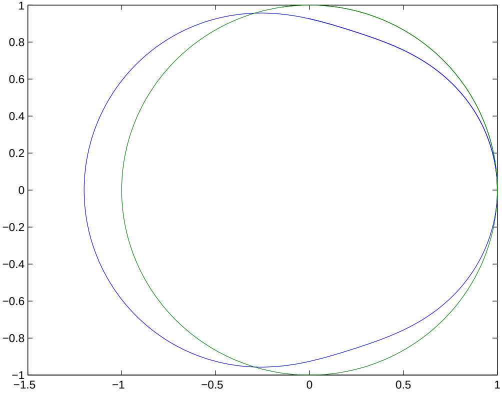
If $\Omega(0) \subset B_R(0)$, *and* t *is large, then*

$$B_{\sqrt{(|\Omega(0)|/\pi+2t)}-R} \subset \Omega(t) \subset B_{\sqrt{(|\Omega(0)|/\pi+2t)}+R}.$$

Small data rescaling behavior-Past work 2

E. Vondenhoff(2008):
$\Omega(0)$ is a small perrturbation of $B_R(0)$ where $|\Omega(0)| = |B_R(0)|$.
Then

- The solution $\Omega(t)$ is global.



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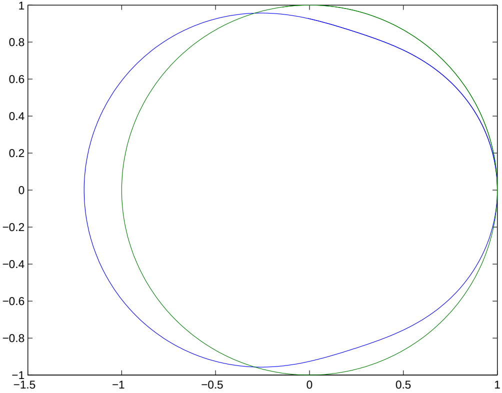
- A rescaling behavior is described in terms of moments.



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- But I only assume solutions are global and more details result about coefficients of solutions are obtained.

outline of our work-1

(Yu-Lin Lin)Large-time rescaling behaviors of Stokes and Hele-Shaw flows driven by injection

- Assume $f_{k_0}(t)$ is a global polynomial solution.

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(Yu-Lin Lin)Large-time rescaling behaviors of Stokes and Hele-Shaw flows driven by injection

- Assume $f_{k_0}(t)$ is a global polynomial solution.
- Understand how each coefficient decays and grows in terms of moments.
- Obtain precise large-time rescaling behavior in terms of moments.

outline of our work-2

(B. Gustafsson and Yu-Lin Lin, in preparation)On the dynamics of roots and poles for solutions of the Polubarinova-Galin equation.

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The decay of polynomial coefficients

Let $f_{k_0} = a_1(t)\zeta + \cdots + a_{k_0}(t)\zeta^{k_0}$ be a global polynomial solution.

- $\bullet M_{k-1}(t) = M_{k-1}(0), k \geq 2$

$$\Leftrightarrow \sum_{i_1,\cdots,i_k} i_1 a_{i_1}(t)\cdots a_{i_k}(t)\overline{a_{i_1+\cdots+i_k}(t)} = M_{k-1}(0), k \geq 2.$$

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$$M_{k-1} = a_1^k(t)\overline{a_k(t)} + \sum_{(i_1,\cdots,i_k)\neq(1,\cdots,1)} i_1 a_{i_1}(t)\cdots a_{i_k}(t)\overline{a_{i_1+\cdots+i_k}(t)}$$

The decay of polynomial coefficients

- Then by induction, we can get

$$a_k(t)\Big(a_1^k(t)\Big) = \overline{M_{k-1}} + O(\frac{1}{a_1^4(t)}), 2 \leq k \leq k_0.$$

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- Let $n_0 = \min\{k \geq 1 \mid M_k(0) \neq 0\}$.

$$a_k(t)\Big(a_1^{(n_0+1)}(t)\Big) = o(1), 2 \leq k \leq n_0.$$

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$$\lim_{t\to\infty}\Big[f(\zeta,t) - \sqrt{2t + M_0(0)\zeta}\Big]\big(\sqrt{2t}\big)^{n_0+1} = \overline{M_{n_0}}\zeta^{n_0+1} \neq 0 \qquad (2)$$

Multi-cut solutions

Theorem

Let $f(\zeta,t)=\frac{\sum_{j=1}^{m}b_j\zeta^j}{\prod_{l=1}^{n}(\zeta-\zeta_l)^{s_l}}+\sum_{l=1}^{m_0}e_l\ln(\zeta-\zeta_{-l})+b_0=\sum_{j=1}^{\infty}a_j\zeta^j$ *be a global solution. Assume* $n_0=\min\{k\geq 1|M_k\neq 0\}$. *Then we have*

1. $$a_k(t)\left(a_1^k(t)\right)=\overline{M_{k-1}}+O\left(\frac{1}{a_1^4(t)}\right),2\leq k<\infty.$$

   $$a_k(t)\left(a_1^{(n_0+1)}(t)\right)=o(1),2\leq k\leq n_0.$$

2. $$\lim_{t\to\infty}\max_{\zeta\in\partial B_1(0)}\left|\left[f(\zeta,t)-\sqrt{2t+M_0(0)}\zeta\right]\left(\sqrt{2t}\right)^{n_0+1}-\overline{M_{n_0}}\zeta^{n_0+1}\right|=0 \tag{3}$$

3. *Let* $n_1=\{\zeta_j|\zeta_j$ *singularity of* $f(\zeta,0)\}$. *Then* $n_0\leq m+n_1-1$.

Key of the proof

Denote $f = \sum_{j=1}^{\infty} a_j \zeta^j$ and $f_k = \sum_{j=1}^{k} a_j \zeta^j$.

$$\max_{\zeta \in \overline{B_1(0)}} \left| f_k(\zeta, t) - f(\zeta, t) \right| = O(a_1(t)^{-i_k}) \tag{4}$$

and

$$\max_{\zeta \in \overline{B_1(0)}} \left| f_k'(\zeta, t) - f'(\zeta, t) \right| = O(a_1(t)^{-i_k}) \tag{5}$$

where $i_k \to \infty$ as $k \to \infty$.

In this case, there exists f_{l_k} such that

$$\begin{aligned}
M_{k-1} =& \frac{1}{2\pi i}\int_{\partial B_1(0)} f^k(\zeta,t)f'(\zeta,t)\overline{f(\zeta,t)}d\zeta \\
=& \frac{1}{2\pi i}\int_{\partial B_1(0)} f_{l_k}^k(\zeta,t)f_{l_k}'(\zeta,t)\overline{f_{l_k}(\zeta,t)}d\zeta + O(\frac{1}{a_1^4}) \\
=& \sum_{i_1,\cdots,i_k}^{\sum_{j=1}^k l_j \le l_k} i_1 a_{i_1}(t)a_{i_2}(t)\cdots a_{i_k}(t)\overline{a_{i_1+\cdots+i_k}(t)} + O(\frac{1}{a_1^4}).
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- $$\lim_{t\to\infty} \max_{\zeta\in\partial B_1(0)} \left| \left[f(\zeta,t) - \sqrt{2t + M_0(0)}\zeta \right] \left(\sqrt{2t}\right)^{n_0+1} - \overline{M_{n_0}}\zeta^{n_0+1} \right| = 0 \tag{7}$$

Why Multi-cut solutions behave like polynomial

- Understand precise large-time behavior of singularity ζ_l, ζ_{-l}.

$$f(\zeta,t) = \frac{\sum_{j=1}^{m} b_j \zeta^j}{\prod_{l=1}^{n} (\zeta - \zeta_l)^{s_l}} + \sum_{l=1}^{m_0} e_l \ln(\zeta - \zeta_{-l}) + b_0 = \sum_{j=1}^{\infty} a_j \zeta^j,$$

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- Understand large-time behavior of b_j.

Tool: $|a_1(t)| \approx \sqrt{2t}$, $|a_k(t)|, k \geq 2$ are uniformly bounded

Derive the equation for singularity

$$g = f'(\zeta, t) = \frac{P_1(\zeta, t)}{\prod(\zeta - \theta_j)^{l_j}}, \theta_j \text{ are distinct singularity of } f(\zeta, t).$$

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- $$\frac{\dot{P}_1}{P_1} + \sum \frac{l_j \dot{\theta}_j}{\zeta - \theta_j} = \left[(\zeta P)^{'} + \frac{1}{i}(\partial_\alpha \ln g) \cdot P\right].$$

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- $$\frac{\dot{P}_1}{P_1} + \sum \frac{l_j \dot{\theta}_j}{\zeta-\theta_j} = \left[(\zeta P)^{'} + \frac{1}{i}(\partial_\alpha \ln g)\cdot P\right].$$
- Multiply $(\zeta-\theta_j)$ and let $\zeta-\theta_j \to 0$. We can obtain $\dot{\theta}_j$.

How to understand the behavior of singularity

- $$f(\zeta,t)=\frac{\sum_{j=1}^{m}b_j\zeta^j}{\prod_{l=1}^{n}(\zeta-\zeta_l)^{s_l}}+\sum_{l=1}^{m_0}e_l\ln(\zeta-\zeta_{-l})+b_0,$$

$$\frac{d}{dt}(\ln|\zeta_k|)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\alpha},t)|^2}\frac{|\zeta_k|^2-1}{|\zeta_k-e^{i\alpha}|^2}d\alpha>0.$$

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- There exist $c_1,c_2>0$ such that

$$c_1\le\frac{|\zeta_k(t)|}{a_1(t)}\le c_2,\quad t\ge 0$$

Geometric meaning of the rescaling behavior

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- Let $\Omega(t) = f(B_1(0), t)$ and $n_0 = \min\{k \geq 1 \mid M_k(0) \neq 0\}$.
- Rescaling $\Omega(t)$ by $\sqrt{|\,\Omega(t)\,|\,/\pi}$, we have the new domain $\Omega'(t)$ with area π.

Geometric meaning of the rescaling behavior

- Let $f(\zeta,t)$ be a global strong multi-cut solution.
- Let $\Omega(t)=f(B_1(0),t)$ and $n_0=\min\{k\geq 1 \mid M_k(0)\neq 0\}$.
- Rescaling $\Omega(t)$ by $\sqrt{|\,\Omega(t)\,|/\pi}$, we have the new domain $\Omega'(t)$ with area π.
- Let $\kappa(z,t)$ be the curvature for $z\in\partial\Omega'(t)$. Then

$$\max_{z\in\partial\Omega'(t)} \|\,z\,|-1\,| = O\left(\frac{1}{t^{1+\frac{n_0}{2}}}\right)$$

$$\max_{z\in\partial\Omega'(t)} |\,\kappa(z,t)-1\,| = O\left(\frac{1}{t^{1+\frac{n_0}{2}}}\right).$$

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- The value $1 + \frac{n_0}{2}$ is the best rate we can get.

$$\limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} \|\, z \,| - 1 \,| (2t)^{1+\frac{n_0}{2}} = |\, M_{n_0} \,|\,.$$

$$\limsup_{t\to\infty} \max_{z\in\partial\Omega'(t)} |\, \kappa(z,t) - 1 \,| (2t)^{1+\frac{n_0}{2}} = (n_0 - 1)(n_0 + 1)\,|\, M_{n_0} \,|\,.$$

Why $n_0 \leq m + n_1 - 1$, n_1 =total singularity ?

- $$f(\zeta, t) = \frac{\sum_{j=1}^{m} b_j \zeta^j}{\prod_{l=1}^{n} (\zeta - \zeta_l)^{s_l}} + \sum_{l=1}^{m_0} e_l \ln(\zeta - \zeta_{-l}) + b_0 = \sum_{j=1}^{\infty} a_j \zeta^j.$$

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$$\tilde{b}_s = 0, \quad \tilde{c}_s = 0, \quad s \geq m + n_1$$

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$$\tilde{c}_0(n_0+1)a_{n_0+1} = -\sum_{j=1}^{n_0-1} \tilde{c}_j a_{n_0+1-j}(n_0+1-j) \tag{9}$$

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- If $f(\zeta, t)$ is rational, $n_0 \leq m-1$.