Random normal matrices by Riemann-Hilbert Problem

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Normal matrices → Coulomb gas

The eigenvalues of $n \times n$ normal matrices with the probability distribution

$$\mathrm{Prob}(M)\mathrm{d}M = \frac{1}{\mathcal{Z}} \mathrm{e}^{-N\,\mathrm{Tr}\left[MM^{\dagger}+V(M)+V(M)^{\dagger}\right]}\mathrm{d}M,$$

distributes by the probability density

$$P(\lambda_1, ..., \lambda_n) = \frac{1}{\mathcal{Z}}\Big|\prod_{j<k}^{n}(\lambda_j - \lambda_k)\Big|^2 \exp\left(-N\sum_{j=1}^{n} Q(\lambda_j)\right),$$

where the potential is given by

$$Q(z) = |z|^2 + V(z) + \overline{V(z)},$$

i.e. **Gaussian** plus **harmonic** function when V is holomorphic.

Eigenvalues of random normal matrices are Coulomb gas in 2-dimension.

Continuum limit of Coulomb gas: $n, N \to \infty$

Define $t = \lim_{n,N\to\infty} n/N$.

For real analytic Q,

$$\frac{1}{n}\sum_j \delta(z-\lambda_j) \xrightarrow{\text{weak}} \rho(z) + \frac{1}{n}\rho_{1/2}(z) + (\text{fluctuation})$$

where [Wiegmann-Zabrodin, Ameur-Hedenmalm-Makarov, ...]

$$\rho = \frac{\Delta Q}{4\pi t}\mathbf{1}_S,$$

$$\rho_{1/2} = \tfrac{2-\beta}{8\pi\beta}\Delta\big((\log \Delta Q)^H + \mathbf{1}_S\big) = 0.$$

In our case, $\beta = 2$ and $\Delta Q = 4$.

The density is asymptotically constant on S, and $\text{Area}(S) = \pi t$.

S is determined by the support of σ that minimizes

$$\int_{\mathbb{C}} Q(w)\sigma(w)\mathrm{d}^2 w - \frac{n}{N}\iint_{\mathbb{C}^2}\sigma(z)\sigma(w)\log|z-w|\mathrm{d}^2 z\,\mathrm{d}^2 w.$$

Correspondence to Hele-Shaw flow: S is the domain of non-viscous fluid in ideal Hele-Shaw flow when t is the Hele-Shaw time.

Orthogonal polynomials on $\mathbb{C}$

(Joint) Probability densities are given by (for $k \le n$)

$$P(\lambda_1, ..., \lambda_k) \propto \det \left(K_n(\lambda_i, \lambda_j) \mathrm{e}^{-\frac{N}{2}(Q(\lambda_i)+Q(\lambda_j))} \right)_{i,j=1}^{k},$$

where the (reproducing) kernel K_n is defined by

$$K_n(z, w) = \sum_{j=0}^{n-1} \frac{p_j(z)\overline{p_j(w)}}{h_j},$$

and $p_j = x^j + ...$ is a polynomial of degree j defined by

$$h_j \delta_{ij} = \int_{\mathbb{C}} p_i(z)\, \overline{p_j(z)}\, \mathrm{e}^{-NQ(z)} \mathrm{d}A(z).$$

Therefore, the kernel K_n is **the most wanted** in our analysis. For example, the density is given by

$$\rho_n(z) := \frac{1}{N} K_n(z, z)\, \mathrm{e}^{-NQ(z)}.$$

Quantized Hele-Shaw flow

Hele-Shaw time t is quantized by n/N.
So it is natural to expect

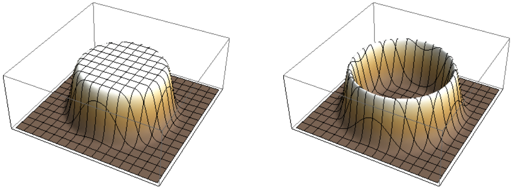
$$N(\rho_{n+1}(z) - \rho_n(z)) = \frac{|p_n(z)|^2}{h_n} \mathrm{e}^{-NQ(z)} \sim \mathbf{1}_{\delta S}$$

where δS is the growing part of S for a small time interval.

Theorem [AHM]: $|p_n(z)|^2 \mathrm{e}^{-NQ(z)} \mathrm{d}A(z)$ converges to the harmonic measure at ∞ with respect to $\mathbb{C} \setminus S$.

The simplest case: Ginibre ensemble

When $V(z) = 0$ the orthogonal polynomials are $p_k(z) = z^k$.
The kernel is explicitly given in terms of Gamma functions.



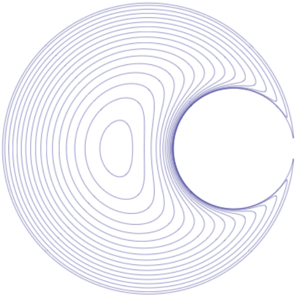
[Left] $K_n(z,z)\,\mathrm{e}^{-NQ(z)}$ for $n = N = 40$.

[Right] $|p_n(z)|^2\,\mathrm{e}^{-NQ(z)}$ for $n = N = 20$.

Goal (and some implications)

Taking $V(z) = -c\log(z-a)$ where $c > 0$, we will obtain the pointwise limit of $p_n(z)$ and $K_n(z,z)$ using Riemann-Hilbert method (DKMVZ '99).

- There is a topological transition.
- Similar technique may apply for any finite logarithmic singularities.
- RH method can, if necessary, give us an arbitrary order of accuracy in large N expansion.
- RH method effectively handles the singular region (for instance, where the merging transition occurs).



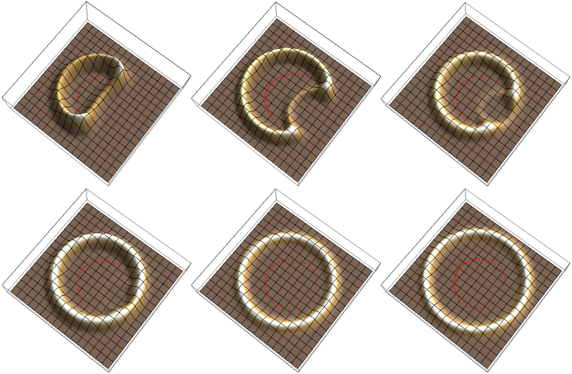
(Asymptotics of $p_n(z)$: Elbau-Felder)

Universality

- The kernel in the bulk is heat kernel.
- The kernel at the boundary is written in terms of erf.
- Merging transition is described, in the classical limit, by scaling solution such that $y^2 \sim x^2(x^2 + t)$.
- In the merging transition, orthogonal polynomial (and quantum Hele-Shaw) is given by PII parametrix (conjectured by Bettelheim-Lee-Wiegmann, Its-Bleher in 1-matrix model)
- The spectral edges in Hermitian matrix model correspond to the cusp-singularities.

  $\boxed{(4n+1,2) \text{ cusp}} \Leftrightarrow \boxed{\text{density that vanishes as } \sim x^{2n+1/2}}$.

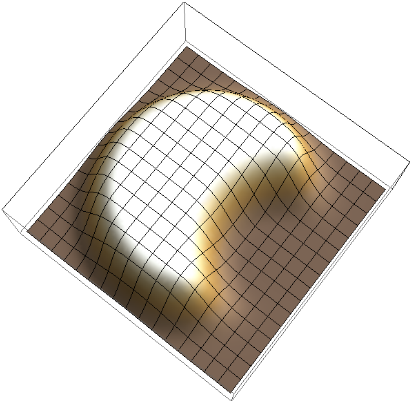
And the polynomials look like...



The plots of $|p_n(z)|^2\,\mathrm{e}^{-NQ(z)}$ for various n.

— The roots (red dots) are on 1 dimensional curve.
— The peak is on the growing part of the boundary.

The density looks like...



(Summing up to 12th polynomials...)

1st step: Area integral into Contour integral

$$\lim_{R\to\infty}\int_{|z|<R} p_j(z)(\bar{z}-a)^k|z-a|^{2Nc}\mathrm{e}^{-Nz\bar{z}}\mathrm{d}^2z$$

$$=\lim_{R\to\infty}\int_{|z|<R} p_j(z)(z-a)^{Nc}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\left(\int_a^{\bar{z}}(s-a)^{Nc+k}\mathrm{e}^{-Nzs}ds\right)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2\mathrm{i}}$$

$$=\lim_{R\to\infty}\oint_{|z|=R} p_j(z)(z-a)^{Nc}\left(\int_a^{\bar{z}}(s-a)^{Nc+k}\mathrm{e}^{-Nzs}ds\right)\frac{\mathrm{d}z}{2\mathrm{i}}$$

$$\Rightarrow\lim_{R\to\infty}\frac{\Gamma(Nc+k+1)}{2\mathrm{i}}\oint_{|z|=R}\frac{p_j(z)(z-a)^{Nc}}{(Nz)^{Nc+k+1}}\mathrm{e}^{-Naz}dz.$$

In the last line, $\lim_{R\to\infty}$ can be dropped.

$$0=\oint p_j(z)\frac{(z-a)^{Nc}\mathrm{e}^{-Naz}}{z^{Nc+j}}z^{j-k-1}\mathrm{d}z\quad\text{for } k=0,1,...,j-1.$$

$$=\oint p_j(z)w_n(z)z^s dz\quad\text{for } s=0,1,...,j-1,\quad w_j(z)=\frac{(z-a)^{Nc}\mathrm{e}^{-Naz}}{z^{Nc+j}}.$$

$p_j(z)$ is the orthogonal polynomial with respect to $w_j(z)\mathrm{d}z$!

Riemann-Hilbert problem

$$Y(z) = \begin{pmatrix} p_n(z) & \frac{1}{2\pi i}\oint \frac{p_n(z')}{z'-z} w_n(z') \mathrm{d}z' \\ -2\pi i Q_{n-1}(z) & -\oint \frac{Q_{n-1}(z')}{z'-z} w_n(z') \mathrm{d}z' \end{pmatrix} .$$

Here $Q_{n-1}(z)$ is the orthogonal polynomial of degree $n-1$ with respect to the measure $w_n(z)\mathrm{d}z$.

Then Y satisfies the Riemann-Hilbert problem:

$Y(z)$ is holomorphic in $\mathbb{C} \setminus \Gamma$.

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_n(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma .$$

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3} \quad z \to \infty .$$

Above, Γ is the contour that goes around the origin.

Eigenvalue support: S

... can be defined by the following step.

Write a general rational function that maps ∞ and α to ∞.

$$f(v) := \rho\, v + \frac{\kappa}{v - \alpha} + z_0.$$
$$S(v) := f(1/v).$$

Require that

- $S(v) \sim c/(f(v) - a)$ as $v \to 1/\alpha$,
- $S(v) \to (c + t)/f(v)$ as $v \to \infty$.

This determines $\rho, \kappa, \alpha, z_0$.

Now S is given by $f(\partial\mathbb{D})$.

The “inverse” function f^{-1} maps $\mathbb{C} \setminus S$ to the outside of **unit disk**.
$\Rightarrow$ $\log|f^{-1}(z)| = G(\infty, z)$. ($f^{-1}$ can be extended inside S.)

RH method gives...

The first column of Y: (in terms of geometric quantity)

$$p_n(z) = \sqrt{\rho\left(f^{-1}(z)\right)'}\, \mathrm{e}^{ng(z)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

$$Q_{n-1}(z) = c_n \frac{\sqrt{\rho\left(f^{-1}(z)\right)'}}{\rho(f^{-1}(z) - \alpha)}\, \mathrm{e}^{ng(z)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

- $\mathrm{Re}\, g(z)$ is the logarithmic potential of $\mathbf{1}_S$, i.e.

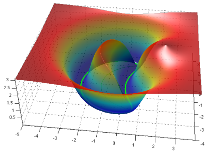
$$g(z) := \frac{1}{\pi t} \int_{\mathbf{1}_S} \log(z - \zeta) \mathrm{d}A(\zeta).$$

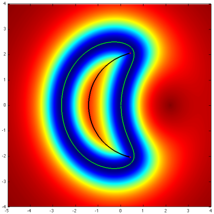
  (g can be analytically extended inside S.)
- $|(f^{-1}(z))'|$ gives the harmonic measure.

$\mathbf{1}_S$ was defined by a potential problem such that

$$Q(z) - 2\,t\,\mathrm{Re}\,g(z) = |z|^2 + 2\,\mathrm{Re}\Big(- c\log(z-a) - t\,g(z)\Big)$$

gets minimized on S (or on ∂S).

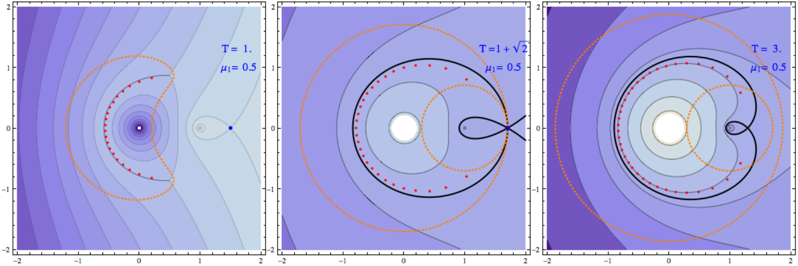




$$\Rightarrow \quad |p_n(z)|^2 \mathrm{e}^{-nQ(z)} = \rho |(f^{-1}(z))'| \mathrm{e}^{-n(Q(z) - 2\,t\,\mathrm{Re}\,\mathrm{g(z)})} \left(1 + \mathcal{O}\left(n^{-1}\right)\right)$$

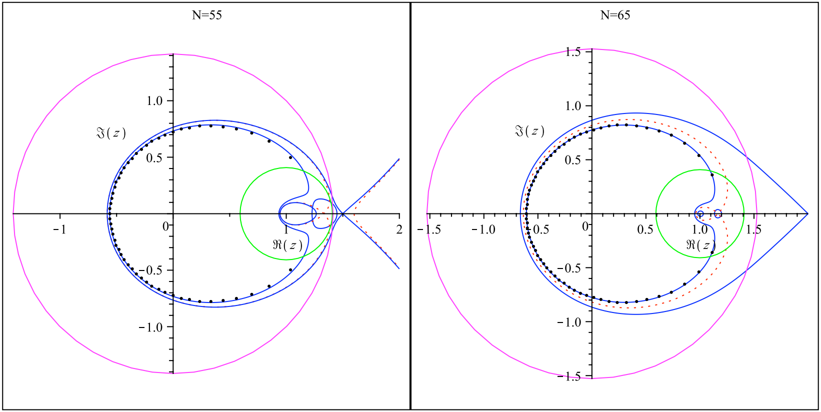
is peaked on ∂S and the line density proportional to the harmonic measure.

Zero locus and S (orange dotted line).



⇒ At the edge of the zero locus, special functions are used to describe the strong asymptotics (Airy function, Painlevé II function, Parabolic cylinder function).

Better approximation



— Parabolic cylinder function.

Christoffel-Darboux identity

When applying "Riemann-Hilbert technique" to 1 matrix model, one obtains the asymptotic expansion of $p_j(z)$'s and obtains the kernel by using the identity

$$\sum_{j=0}^{n-1} \frac{p_j(z)p_j(w)}{h_j} \propto \frac{p_n(z)p_{n-1}(w) - p_n(w)p_{n-1}(z)}{z-w}.$$

This relates the large sum into a few highest degree polynomials.
In normal matrix model ($\in$ 2 matrix model) CD identity exists but does not relate to $K_n(z,w)$ directly.

CD identity for the bi-orthogonal polynomial

Let us consider the following (normalized) BOP with polynomial potentials V and W. (Our case: $W(w) \to \overline{V(\overline{w})}; w \to \overline{z}$)

$$\delta_{nm} = \iint \mathrm{d}z\,\mathrm{d}w\, p_n(z) q_m(w) \mathrm{e}^{-N\left(zw+V(z)+W(w)\right)}.$$

Define $\widetilde{p}_m(z) := p_m(z)\mathrm{e}^{-NV(z)}$; $\widetilde{q}_m(w) := q_m(w)\mathrm{e}^{-NW(w)}$.

Differentiating inside the integral,

$$\begin{aligned} 0 &= \iint \mathrm{d}z\,\mathrm{d}w \frac{\mathrm{d}}{N\mathrm{d}w}\left(\widetilde{p}_n(z)\widetilde{q}_m(w)\mathrm{e}^{-Nzw}\right) \\ &= \iint \widetilde{p}_n(z)\frac{\mathrm{d}\widetilde{q}_m(w)}{N\mathrm{d}w}\mathrm{e}^{-Nzw} - \iint \widetilde{p}_n(z)\, z\, \widetilde{q}_m(w)\mathrm{e}^{-Nzw}. \end{aligned}$$

- Obviously, $z\,\widetilde{p}_n(z)$ is spanned by $\{\widetilde{p}_0(z), ..., \widetilde{p}_{n+1}(z)\}$.
- $\frac{\mathrm{d}}{\mathrm{d}z}\widetilde{q}_m(z)$ is a linear combination of $\{\widetilde{q}_{m+d}(z), ..., \widetilde{q}_0(z)\}$ where d is the degree of $W'(z)$.

  (This is not true if $W'(w)$ is not polynomial.)
  (See, "Biorthogonal polynomials for two-matrix models with semiclassical potentials" by Bertola)

We consider the truncated vectors:

$$\widetilde{\mathbf{p}}_n(z) := (\widetilde{p}_{n-1}(z), ..., \widetilde{p}_0(z))^T,$$
$$\widetilde{\mathbf{q}}_n(w) := (\widetilde{q}_{n-1}(w), ..., \widetilde{q}_0(w))^T.$$

The kernel is given by $\widetilde{\mathbf{q}}_n(w)^T\widetilde{\mathbf{p}}_n(z)$.

We have,

$$z\,\widetilde{\mathbf{p}}_n(z) = \left(\iint \mathrm{d}\zeta\,\mathrm{d}w\,\mathrm{e}^{-N\zeta w}\widetilde{\mathbf{p}}_n(\zeta)\,\zeta\,\widetilde{\mathbf{q}}_n(w)^T\right)\widetilde{\mathbf{p}}_n(z) + a_n\left(\widetilde{p}_n(z), 0, \ldots, 0\right)^T$$

where the red part is the projection operation into $\{\widetilde{p}_0, \ldots, \widetilde{p}_{n-1}\}$
(The big braket is an $n \times n$ matrix).
Similarly,

$$\frac{\mathrm{d}}{N\mathrm{d}w}\widetilde{\mathbf{q}}_n(w)^T = \widetilde{\mathbf{q}}_n(w)^T\left(\iint \mathrm{d}z\,\mathrm{d}\zeta\mathrm{e}^{-Nz\zeta}\widetilde{\mathbf{p}}_n(z)\,\frac{\mathrm{d}\widetilde{\mathbf{q}}_n(\zeta)^T}{N\mathrm{d}\zeta}\right) + (\text{mostly zero vector}).$$

Note that the same $n \times n$ matrix appears.

$\widetilde{\mathbf{q}}_n(w)^T \times (\text{1st eq.}) - (\text{2nd eq.}) \times \widetilde{\mathbf{p}}_n(z)$ gives

$$\left(z - \frac{\mathrm{d}}{N\mathrm{d}w}\right)\widetilde{\mathbf{q}}_n(w)^T\widetilde{\mathbf{p}}_n(z) = -\frac{\mathrm{e}^{Nzw}}{N}\frac{\mathrm{d}}{\mathrm{d}w}\left(\mathrm{e}^{-Nzw}\widetilde{\mathbf{q}}_n(w)^T\widetilde{\mathbf{p}}_n(z)\right)$$

$$= (\text{involving a few highest degree polynomials}).$$

Let's work... Remind $w_n(z) = (z-a)^{Nc} \mathrm{e}^{-Naz} / z^{Nc+n}$

We have defined $Q_{n-1}(z) = c_n z^{n-1} + \ldots$ such that

$$\oint Q_{n-1}(z) w_n(z) z^k \mathrm{d}z = \delta_{k,n-1} \quad \text{for} \quad k = 0, \ldots, n-1.$$

We define ($p_n(z) = z^n + b_n z^{n-1} + \ldots$ is monic polynomial)

$$\widetilde{h}_j := \oint p_n(z) p_n(z) w_n(z) \mathrm{d}z.$$

For $k \leq n-1$, we have

$$\int \big(p_n(z) - z p_{n-1}(z)\big) w_n(z) z^k \mathrm{d}z$$
$$= \int p_n(z) w_n(z) z^k \mathrm{d}z - \int p_{n-1}(z) w_{n-1}(z) z^k \mathrm{d}z = -\widetilde{h}_{n-1} \delta_{k,n-1}$$

Therefore, we have $c_n = (b_{n-1} - b_n)/\widetilde{h}_{n-1}$ and

$$\boxed{Q_{n-1}(z) = \frac{z\, p_{n-1}(z) - p_n(z)}{\widetilde{h}_{n-1}}.}$$

Similarly, consider the following polynomial of degree $n-1$ for $k \leq n-1$:

$$\begin{aligned}
&\int w_n(z) z^k \left(p_n(z) - c_{n+1}^{-1} Q_n(z)\right) \mathrm{d}z \\
&= \int w_n(z) z^k p_n(z) \mathrm{d}z - c_{n+1}^{-1} \int w_n(z) z^k Q_n(z) \mathrm{d}z \\
&= \frac{-1}{c_{n+1} \widetilde{h}_n} \int w_n(z) z^k \left(z\, p_n(z) - p_{n+1}(z)\right) \mathrm{d}z \\
&= \frac{-1}{c_{n+1} \widetilde{h}_n} \left(\int w_n(z) z^{k+1} p_n(z) \mathrm{d}z - \int w_{n+1}(z) z^{k+1} p_{n+1}(z) \mathrm{d}z \right) \\
&= -c_{n+1}^{-1} \delta_{k,n-1}
\end{aligned}$$

Therefore,

$$\boxed{p_n(z) - c_{n+1}^{-1} Q_n(z) = -c_{n+1}^{-1} Q_{n-1}(z).}$$

Combined with the previous equation $\Rightarrow$ Three-term recurrence.

Differential relation

Note that $\frac{\mathrm{d}}{\mathrm{d}z}\widetilde{p}_n(z)$ is not a linear combination of $\{\widetilde{p}_j\}$, but

$$\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}z}\left(\widetilde{p}_n(z) - \frac{p_n(a)}{p_{n+1}(a)}\widetilde{p}_{n+1}(z)\right) \text{ is.}$$

Even more, this is orthogonal to $\{\widetilde{p}_0, \dots, \widetilde{p}_{n-2}\}$ with respect to the area integral, hence

$$\boxed{\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}z}\left(\widetilde{p}_n(z) - \frac{p_n(a)}{p_{n+1}(a)}\widetilde{p}_{n+1}(z)\right) = \star\,\widetilde{p}_{n-1}(z) - \frac{p_n(a)}{p_{n+1}(a)}\left(c + \frac{n+1}{N}\right)\widetilde{p}_n(z).}$$

$\star$ is obtained after some algebra to be

$$\star = \frac{P_n(a)}{P_{n-1}(a)}\frac{c_n}{c_{n+1}}.$$

Kernel

$$e^{-Nzw}\left(z-\frac{1}{N}\frac{d}{dw}\right)\sum_{j=0}^{n-1}\frac{\widetilde{p}_j(w)\widetilde{p}_j(z)}{h_j} = -\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}w}\left(e^{-Nzw}\sum_{j=0}^{n-1}\frac{\widetilde{p}_j(w)\widetilde{p}_j(z)}{h_j}\right)$$
$$= \frac{e^{-Nzw}}{c_{n+1}h_n}\left(-\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}w}\widetilde{p}_n(w)\widetilde{Q}_{n-1}(z) + \star\,\widetilde{p}_{n-1}(w)\widetilde{Q}_n(z)\right).$$

The kernel is obtained by the antiderivative. Especially the density $\rho_n(z) = e^{-N|z|^2}K_n(z,z)$ can be obtained by integrating

$$\boxed{\partial_z\rho_n(z) = \frac{e^{-N|z|^2}}{c_{n+1}h_n}\left(-\frac{1}{N}\partial_z\widetilde{p}_n(z)\widetilde{Q}_{n-1}(\overline{z}) + \star\,\widetilde{p}_{n-1}(z)\widetilde{Q}_n(\overline{z})\right).}$$

This quantity converges to "$\partial_z\mathbf{1}_S$".


THANKS FOR THE ATTENTION.