Asymptotics of the interface of Laplacian growth with multiple point sources

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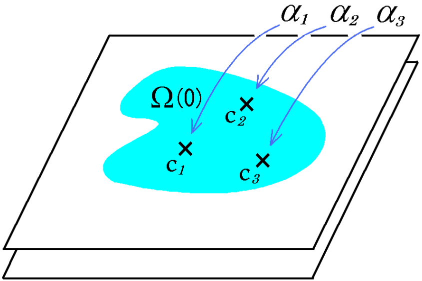
1. Asymptotic shape of the free boundary
   - Hele-Shaw flows
   - Weak solutions and Quadrature domains
   - Main result and its proof

2. Stability of the free boundary
   - Derivation of an evolution equation
   - Main result and its proof

Asymptotic shape of the free boundary

Hele-Shaw flows

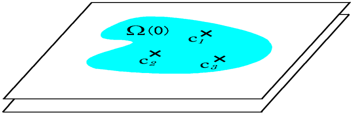
$\Omega(0)$: initial domain
$c_1, \ldots, c_l \in \Omega(0)$: injection points
$\alpha_1, \ldots, \alpha_l > 0$: injection rates



Hele-Shaw flows

$$\begin{cases} (1) & -\Delta p = \sum_{j=1}^{l} \alpha_j \delta_{c_j} & \text{in } \Omega(t) \\ (2) & p = 0 & \text{on } \partial\Omega(t) \\ (3) & -\partial_n p = v_n & \text{on } \partial\Omega(t) \end{cases}$$

$p(z,t)$: pressure of fluid
$\Omega(t)$: fluid domain at time $t \geq 0$
δ_{c_j}: the Dirac measure



$V = -\nabla p$: velocity field

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$$(1), (2) \Rightarrow p(z,t) = \sum_{j=1}^{l} \alpha_j G_{c_j, \Omega(t)}(z)$$

($G_{c_j,\Omega(t)}$: Green's function for $-\Delta$)

Hele-Shaw flows

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$$(1), (2) \Rightarrow p(z,t) = \sum_{j=1}^{l} \alpha_j G_{c_j,\Omega(t)}(z)$$

Find $\{\Omega(t)\}_{t>0}$ s.t.

$$-\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n \quad \text{on } \partial\Omega(t).$$

Weak solutions

Assume $-\sum_{j=1}^{l} \alpha_j \frac{\partial G_{c_j,\Omega(t)}}{\partial n} = v_n \quad \text{on } \partial\Omega(t).$

For any $s \in SL^1(\Omega(t))$ (subharmonic and L^1),

$$\int_{\Omega(t)\setminus\Omega(0)} s(z)\,dm$$

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Weak solutions

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(*) \begin{cases} \displaystyle\int_{\Omega(0)} s\,dm + t\sum_{j=1}^{l} \alpha_j s(c_j) \le \int_{\Omega(t)} s\,dm \\ \qquad\qquad\qquad\qquad\qquad \big(\forall s \in SL^1(\Omega(t))\big), \\ m(\Omega(t)) < \infty. \end{cases}
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Weak solutions

Weak solution

For each $t > 0$, find $\Omega(t)$ s.t.

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existence of weak solution [Sakai (1982)]

Let $\Omega(0)$ be a domain with $m(\Omega(0)) < \infty$. Then, there exists a domain $\Omega(t)$ satisfying $(*)$.

uniqueness of weak solution [Sakai (1982)]

If $\Omega(t)$ and $\Omega(t)'$ satisfy $(*)$, then $\chi_{\Omega(t)} = \chi_{\Omega(t)'}$ m-a.e.

Weak solutions

Weak solution

Quadrature Domain of $t\sum \alpha_j \delta_{c_j}$

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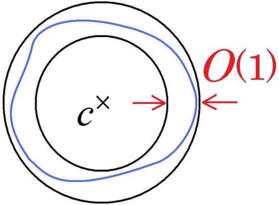
Known result [Sakai (1998)]

Let $\Omega(0) \subset D(c,r)$ *and* $m(\Omega(0)) + t\sum_{j=1}^{l} \alpha_j \geq 4\pi r^2$.
Then,

$$D(c, \tilde{R}(t) - r) \subset \Omega(t) \subset D(c, \tilde{R}(t) + r)$$

holds, where $\tilde{R}(t) := \sqrt{\frac{1}{\pi}\left(m(\Omega(0)) + t\sum_{j=1}^{l} \alpha_j\right)}$.



Theorem

Let $\Omega(0) \subset D(c, r)$. *Then, for sufficiently large* $t > 0$,

$$D\left(w_l, R(t) - \varepsilon_l^-(t)\right) \subset \Omega(t) \subset D\left(w_l, R(t) + \varepsilon_l^+(t)\right)$$

holds, where

$$w_l := \frac{\sum_{j=1}^{l} \alpha_j c_j}{\sum_{j=1}^{l} \alpha_j}, \; R(t) := \sqrt{\frac{t}{\pi} \sum_{j=1}^{l} \alpha_j},$$

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$$\varepsilon_l^-(t) := \sqrt{\frac{\pi}{\sum_{k=1}^l \alpha_k}} \left(\sum_{j=2}^l \frac{\alpha_j \sum_{k=1}^{j-1} \alpha_k}{\left(\sum_{k=1}^j \alpha_k\right)^2} \left|w_{j-1} - c_j\right|^2\right) t^{-1/2} + O(t^{-1}),$$

$$\varepsilon_l^+(t) := \sqrt{\frac{\pi}{\sum_{k=1}^l \alpha_k}} \left(\sum_{j=2}^l \frac{\alpha_j \sum_{k=1}^{j-1} \alpha_k}{\left(\sum_{k=1}^j \alpha_k\right)^2} \left|w_{j-1} - c_j\right|^2 + \frac{r^2}{2}\right) t^{-1/2} + O(t^{-1}).$$

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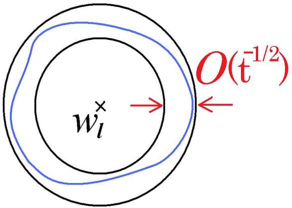

Outline of Proof

Recall the Hele-Shaw problem: Find $\Omega(t)$ s.t.

$$\int_{\Omega(0)} s\,dm + t\sum_{j=1}^{l} \alpha_j s(c_j) \le \int_{\Omega(t)} s\,dm \quad \left(\forall s \in SL^1(\Omega(t))\right).$$

Outline of Proof

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Outline of Proof

the Schwarz function the Schwarz function of $\partial\Omega_0(t)$.

If we have a holomorphic func. in a nb'd of $\partial\Omega_0(t)$ satisfying

(i) $S(z) = \overline{z}$ on $\partial\Omega_0(t)$;

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(ii) S is meromorphic in $\Omega_0(t)$ and has simple poles at $c_1, \ldots, c_l$ with respective residues $t\alpha_1/\pi, \ldots, t\alpha_l/\pi$,

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then, $\forall f$: holomorphic in a nb'd of $\overline{\Omega_0(t)}$,

$$\int_{\Omega_0(t)} f\,dm = \frac{1}{2i}\int_{\partial\Omega_0(t)} f(z)\overline{z}\,dz$$
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Hence, $\Omega_0(t)$ is expected to be the desired q. d.

Outline of Proof

1. Case $l = 2$
   - **(a)** construction of $\Omega_0(t)$ and its estimate
   - **(b)** holomorphic class $\to SL^1$
2. Case $l \geq 3$
   - **(c)** semi-group property of q. d. :

$$\Omega\left(\sum_{j=1}^{l} \nu_j\right) = \Omega\left(\chi_{\Omega\left(\sum_{j=1}^{l-1} \nu_j\right)} + \nu_l\right)$$

   - **(d)** induction on l

Outline of Proof

1. Case $l = 2 \quad \Rightarrow \quad c_1 = i,\ c_2 = -i,\ \alpha_1 = 1$
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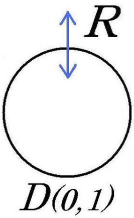
(a) construction of $\Omega_0(t)$ and its estimate

Example. (the Schwarz function of $\partial D(0,1)$)

$$S(z) := \frac{1}{z}.$$

Note that $R(z) := \overline{S(z)} = z/|z|^2$ is the reflection associated to the unit circle $\partial D(0,1)$.



(a) construction of $\Omega_0(t)$ and its estimate

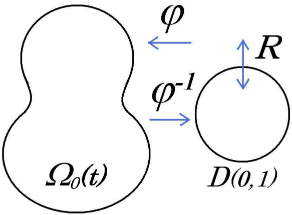
We define

$$\Omega_0(t) := \varphi\left(D(0,1)\right),$$

where

$$\varphi(z) := \frac{ab(z-ic)}{z^2+b^2} + ibc, \quad R(z) = 1/\overline{z}.$$

$$(a(t),\ b(t),\ c(t):\ \text{parameters})$$



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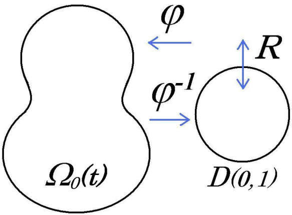

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$\Rightarrow$ • S is the Schwarz function of $\partial\Omega_0(t)$, i.e., S satisfies (i).

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- $a(t) = a_1 t + a_2 + \cdots$,
  $b(t) = b_1 t^{1/2} + b_2 t^{-1/2} + \cdots$,
  $c(t) = c_1 t^{1/2} + c_2 t^{-1/2} + \cdots$.

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  $c(t) = c_1 t^{1/2} + c_2 t^{-1/2} + \cdots$.
- Estimate $|\varphi(z) - w_l|, \quad z \in \partial D(0,1)$.

Outline of Proof

1. Case $l = 2$
   - (a) construction of $\Omega_0(t)$ and its estimate
   - (b) holomorphic class $\to SL^1$
2. Case $l \geq 3$
   - (c) semi-group property of q. d. :

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(d) induction on l

$$\begin{aligned} &\Omega\left(t\sum_{j=1}^{3}\alpha_j\delta_{c_j}\right) \\ &= \Omega\left(\chi_{\Omega\left(t(\alpha_1\delta_{c_1}+\alpha_2\delta_{c_2})\right)} + t\alpha_3\delta_{c_3}\right) \end{aligned}$$

(d) induction on l

$$\Omega\left(t\sum_{j=1}^{3}\alpha_j\delta_{c_j}\right)$$
$$=\Omega\left(\chi_{\Omega\left(t(\alpha_1\delta_{c_1}+\alpha_2\delta_{c_2})\right)}+t\alpha_3\delta_{c_3}\right)$$
$$\approx\Omega\left(\chi_{D\left(\frac{\alpha_1c_1+\alpha_2c_2}{\alpha_1+\alpha_2},\sqrt{\frac{t}{\pi}(\alpha_1+\alpha_2)}\right)}+t\alpha_3\delta_{c_3}\right)$$

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Stability of the free boundary

Derivation of an evolution equation

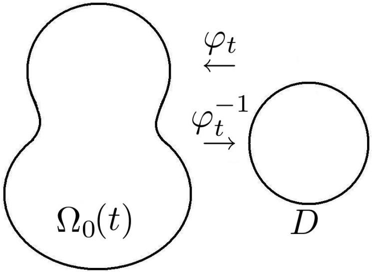
$$\begin{cases} (1) \quad -\Delta p = \alpha_1 \delta_i + \alpha_2 \delta_{-i} & \text{in } \Omega(t) \\ (2) \quad p = 0 & \text{on } \partial\Omega(t) \\ (3) \quad -\partial_n p = v_n & \text{on } \partial\Omega(t) \end{cases}$$

Exact solution

$$\varphi_t(z) := \frac{ab(z - ic)}{z^2 + b^2} + ibc,$$

$\Omega_0(t) := \varphi_t(D)$,

$\{\Omega_0(t)\}_{t>t_0}$: classical solution.



Q. What if the initial domain $\Omega(t_0)$ is close enough to $\Omega_0(t_0)$?

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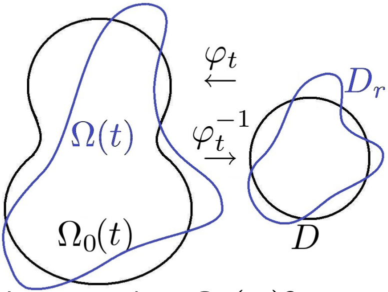

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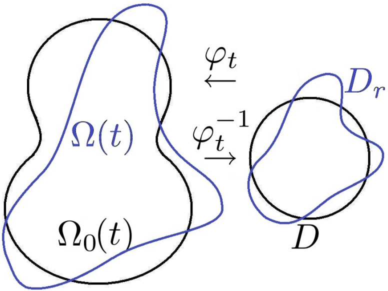
Framework

For $r \in C(\partial D)$, set
$\partial D_r := \{(1 + r(\xi))\xi \mid \xi \in \partial D\}$.



Derivation of an evolution equation

$$\begin{cases} (1) & -\Delta p = \alpha_1 \delta_i + \alpha_2 \delta_{-i} & \text{in } \Omega(t) \\ (2) & p = 0 & \text{on } \partial\Omega(t) \\ (3) & -\partial_n p = v_n & \text{on } \partial\Omega(t) \end{cases}$$

Framework

For $r \in C(\partial D)$, set
$\partial D_r := \{(1 + r(\xi))\xi \mid \xi \in \partial D\}$.

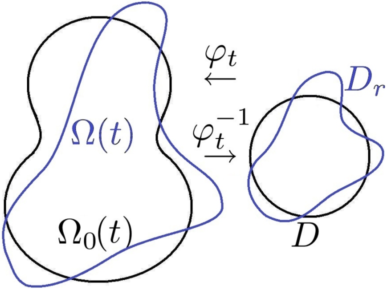
- $(p, \{\Omega(t)\})$

$\downarrow$

Assume $\Omega(t) = \varphi_t(D_{r(\cdot,t)})$.

$\downarrow$

- r : a non-local evolution equation



Derivation of an evolution equation

$$\partial_t r = \frac{-\langle \nabla p \circ \varphi, (D_w \varphi) n \rangle - \langle \partial_t \varphi, (D_w \varphi) n \rangle}{\det(D_w \varphi) \cdot \langle \xi, n \rangle}$$
$$=: \mathcal{F}(r(t), t)$$

$\boxed{\text{Framework}}$

For $r \in C(\partial D)$, set
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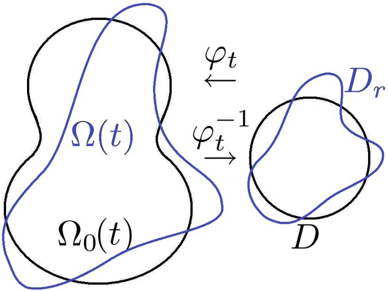

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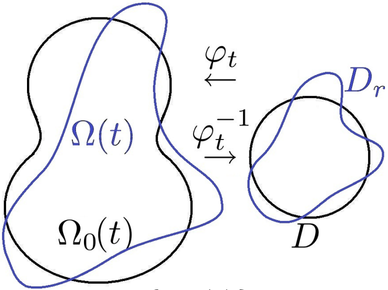

- $r \equiv 0 \Leftrightarrow \{\Omega_0(t)\}_{t > t_0}$.

Theorem

Evolution equation in $h^{1,\gamma}(\partial D)$

$$\begin{cases} r' = \mathcal{F}(r,t), \quad t > t_0, \\ r(t_0) = r_0 \in h^{2,\gamma}(\partial D), \end{cases}$$

where $\mathcal{F}(\cdot,t) : h^{2,\gamma}(\partial D) \to h^{1,\gamma}(\partial D)$, $\mathcal{F}(0,t) = 0$.

Theorem

$\boxed{\text{Evolution equation in } h^{1,\gamma}(\partial D)}$

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The little Hölder space

$$h^{k,\gamma}(\partial D) := \overline{C^\infty(\partial D)}^{C^{k,\gamma}(\partial D)}.$$

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Theorem

Suppose t_0 is sufficiently large. For $\varepsilon > 0$, there are $\delta, M > 0$ s.t. if $\|r_0\|_{h^{2,\gamma}} < \delta$, then there exists a unique solution $r \in C([t_0,\infty); h^{2,\gamma}) \cap C^1([t_0,\infty); h^{1,\gamma})$ satisfying

$$\|r(t)\|_{h^{2,\gamma}} + t\|r'(t)\|_{h^{1,\gamma}} \leq Mt^{-1+\varepsilon}\|r_0\|_{h^{2,\gamma}}.$$

Outline of Proof

Evolution equation in $h^{1,\gamma}$

$$\begin{cases} r' = \mathcal{F}(r,t), \quad t > t_0, \\ r(t_0) = r_0 \in h^{2,\gamma}, \end{cases} \qquad \begin{array}{l} \mathcal{F}(\cdot,t) : h^{2,\gamma} \to h^{1,\gamma}, \\ \mathcal{F}(0,t) = 0. \end{array}$$

Lemma 1 $t \cdot D_r\mathcal{F}(r,t) \to \mathcal{A}$ as $t \to \infty$, $r \to 0$.

Outline of Proof

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('97 Escher & Simonett, '09 Vondenhoff)

Outline of Proof

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Set $\tau := \log(t/t_0)$ and $\tilde{r}(\tau) := r(t)$, then

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$\tilde{r} \in C([0,\infty); h^{2,\gamma})$

$\Rightarrow \tilde{r} \in C([0,\infty); h^{2,\gamma})$?

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($\because$ maximal regularity in $h^{k,\gamma}$ ('79 Da Prato & Grisvard)).

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$\|\tilde{r}(\tau)\|_{h^{2,\gamma}} + \|\tilde{r}'(\tau)\|_{h^{1,\gamma}} \leq M'e^{(-1+\varepsilon)\tau}\|r_0\|_{h^{2,\gamma}}$.

Outline of Proof

Evolution equation in $h^{1,\gamma}$

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$\therefore \ \|r(t)\|_{h^{2,\gamma}} + t\|r'(t)\|_{h^{1,\gamma}} \leq Mt^{-1+\varepsilon}\|r_0\|_{h^{2,\gamma}}$.