Viscous displacement in Hele-Shaw flow-

a tale of unexpected singular perturbation

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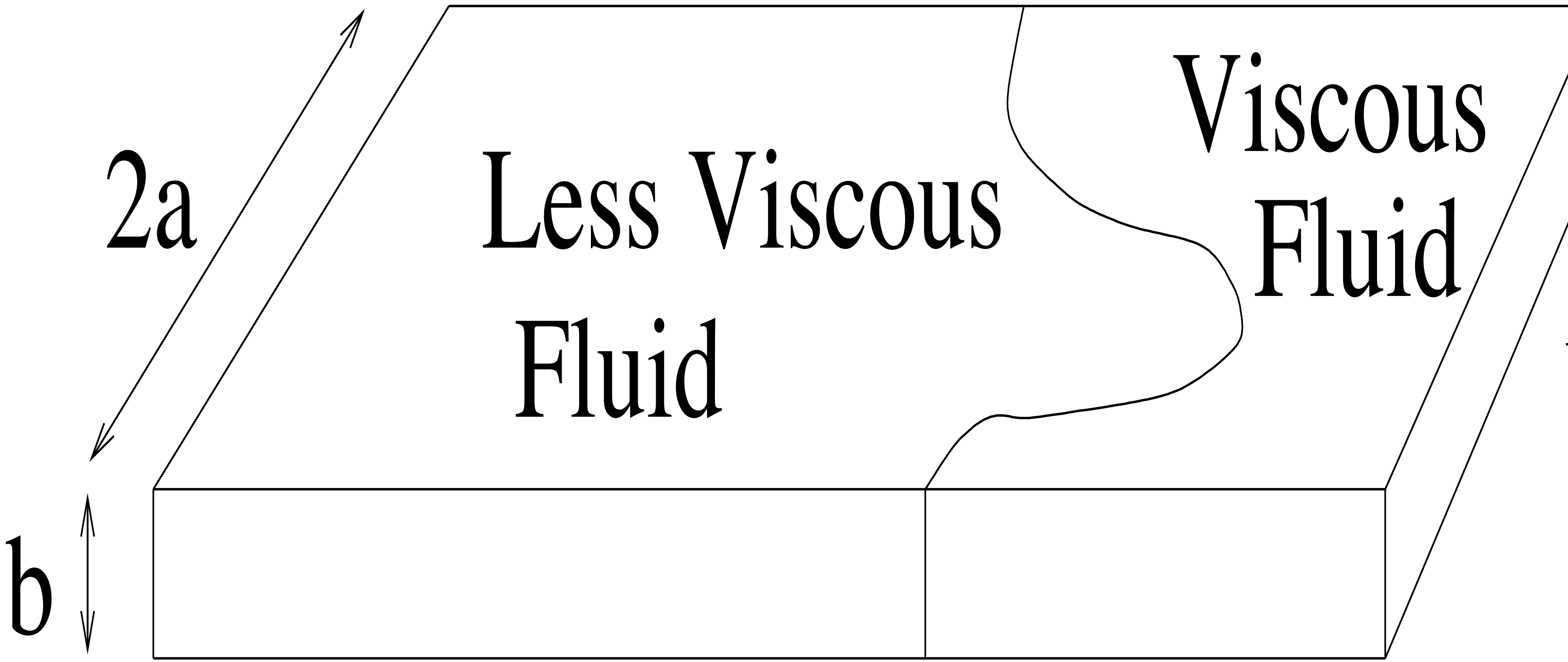
Hele-Shaw Cell

Concerned with displacement of more viscous by less viscous fluid in a Hele-Shaw cell. Note $b/a << 1$



Saffman-Taylor Experiment in 1958

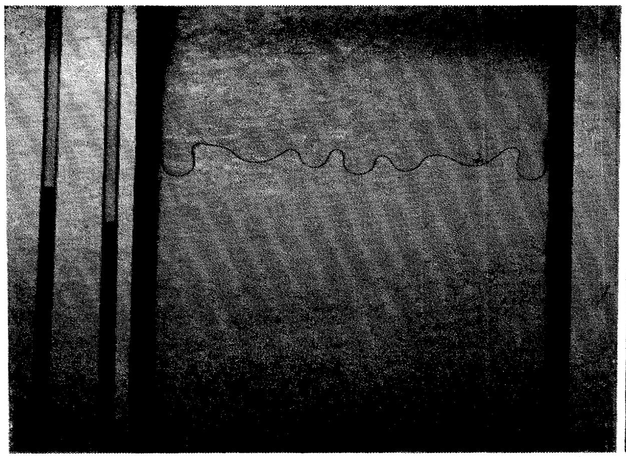


FIGURE 2

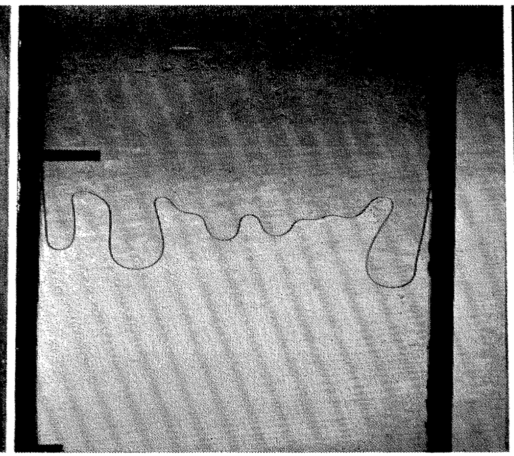


FIGURE 3

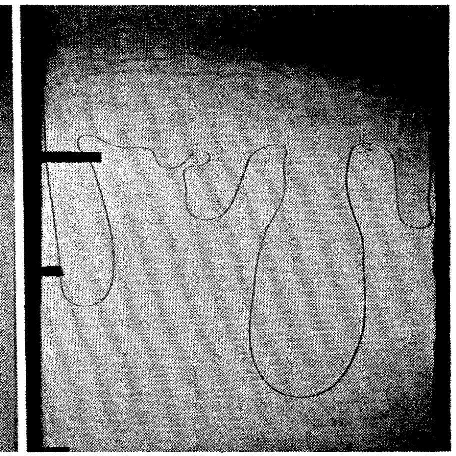


FIGURE 4

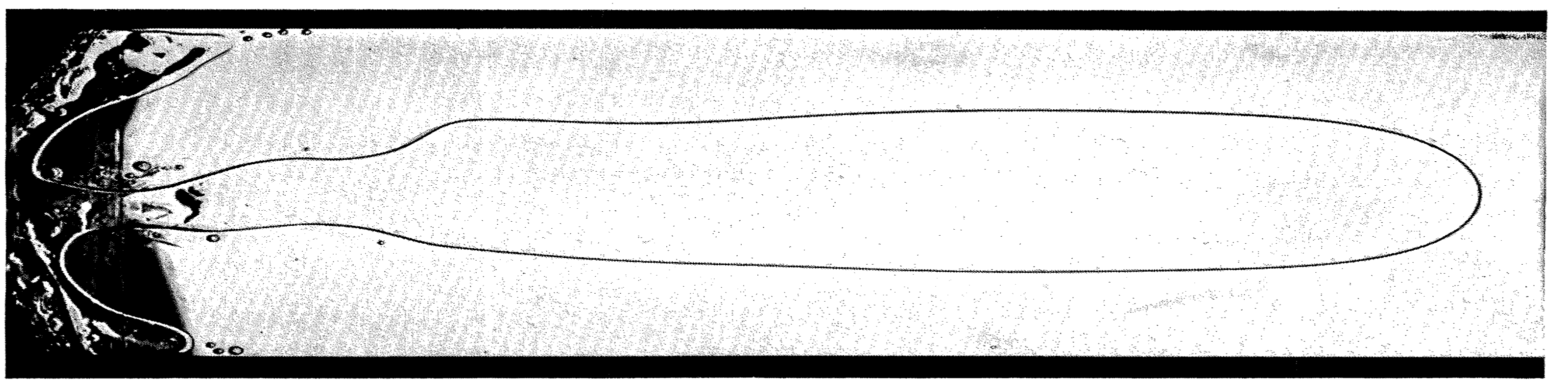
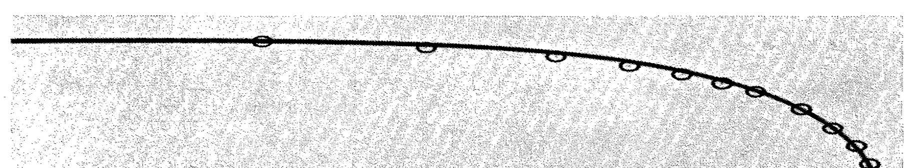


FIGURE 8



Idealized Theoretical model:

Averaging low-Reynolds number flow across the gap results in

$$\text{Darcy's Law} \; : \; \mathbf{u} = -\frac{b^2}{12\mu}\nabla p \; ; \; \text{continuity} : \; \nabla \cdot \mathbf{u} = 0$$

In nondimensional form: $\Delta\phi = 0$ **in** Ω

Far-field and wall conditions:

$$\phi \sim x + O(1) \text{ as } x \to +\infty \; , \; \text{and} \; \frac{\partial \phi}{\partial y}(x, \pm 1) = 0$$

Interfacial conditions:

$$v_n = \frac{\partial \phi}{\partial n}, \; \text{and} \; \phi = \epsilon\kappa$$

Note above interfacial conditions are rough approximations

Selection problem

Saffman & Taylor experiment gave finger width $\lambda \approx \frac{1}{2}$

ST and Zhuravlev ('56) $\epsilon = 0$ solution gave shapes:

$$x = 2\frac{1-\lambda}{\pi} \log \cos\left(\frac{\pi y}{2\lambda}\right) \quad , \quad y \in (-\lambda, \lambda) \ , \ \lambda \in (0,1)$$

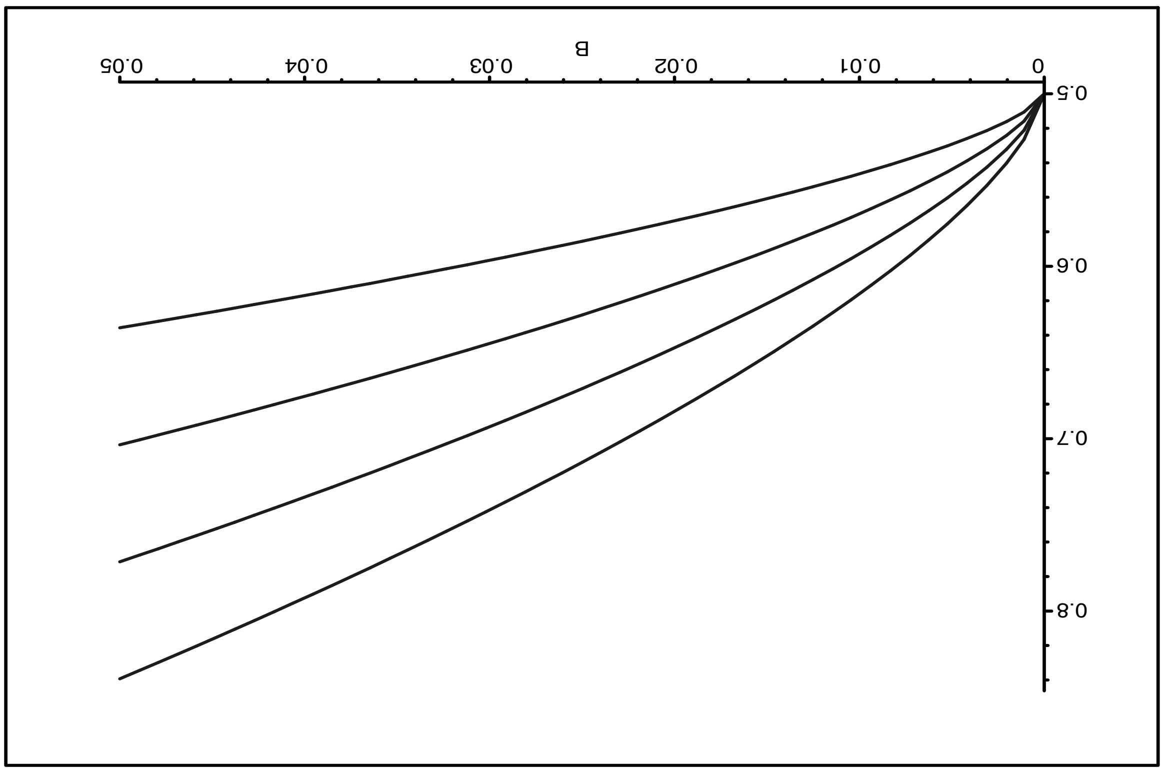
No λ prediction in $\epsilon = 0$ theory, though shapes for experimental λ matched.

Similar problems for translating bubbles of specified area (Taylor & Saffman, 1959); also in other pattern formation problems

Selection quandary worse than popularly understood. There exists an entire family of non-symmetric fingers (Taylor-Saffman, 1959) and bubbles (T., 1987a)

Selection for $\epsilon \neq 0$

Numerical work (McLean-Saffman,'80, VandenBroeck, '82, Kessler & Levine, '86) suggested selection for $\epsilon \neq 0$ of a discrete family of solution branches:



McLean-Saffman numerical results did not agree with formal expansion $\phi = \phi^{(0)} + \epsilon \phi^{(1)} + ..$

Selection Through Exponential Asymptotics

Formal exponential asymptotics (Combescot et al ’86, Hong & Langer ’86, Shraiman, ’86, T. ’86, ’87b, Dorsey-Martin ’87) suggested selection and stability of one branch only (T., 87c, Bensimon-Pelce ’87), though differing conclusions by Xu ’91, Scwartz & DeGregoria ’87

Selection results proved rigorously (Xie & T. ’03, T. & Xie, 2003, Xie, ’08)

Formal exponential asymptotics with 3-D effects available (T., 1990); qualitatively similar results

Other $\lambda \neq \frac{1}{2}$ non-symmetric finger solutions accessible in experiment with etching of glasses (BenJacob *et al*, 1985), Needle Piercing the interface (Zocchi *et al*, ’87), small bubble in front of a finger (Couder *et al* ’86).

Time evolution for $\epsilon = 0$, $\epsilon \neq 0$

$\epsilon = 0$ **dynamics is rich (Polabarinova-Kochina, '46, Galin, '46, Richardson, Gustaffson, ...**

However, $\epsilon = 0$ **evolution problem has** *no continuity* **with respect to I.C. in a physically reasonable norm (say** H^1**) Howison ('86), Fokas & T. ('98).**

For $\epsilon \neq 0$**, local existence (Duchon & Roberts). Global existence in unforced case (Constantin & Pugh) and for pressure gradient causing a near circular bubble to translate (Ye & T.)**

Denote solution as $u(t; \epsilon)$ **for** $\epsilon \neq 0$**. Question: When is** $\lim_{\epsilon \to 0^+} u(t; \epsilon) = u(t; 0)$?**.**

Define $T_0 \leq \infty$ **to be the singularity time for** $u(t; 0)$**. Evidence shows that in some cases, there exists** $T_d < T_0$**, independent of** ϵ **so that** $\lim_{\epsilon \to 0} u(t; \epsilon) \neq u(t, \epsilon)$ **for** $t \in (T_d, T_s)$ **(Siegel** *et al* **'96)**

Structural stability and physical relevance

Any mathematical model can be described abstractly by

$$\mathcal{N}\,[\mathrm{u};\epsilon] = 0,$$

where operator $\mathcal{N}$ can describe arbitrary differential, integral or algebraic operator and ϵ describes parameters.

The above problem is *structurally stable* **if solution set $\{u\}$ depends continuously on ϵ.**

Consider initial value problem (IVP): $u_t = \mathcal{N}[u], \;\; u(0) = u_0$. The problem above is *well-posed* **if there exists unique solution to the IVP that depends continuously on u_0**

Models that are not structurally stable or well-posed are generally irrelevant physically since ϵ and u_0 not known exactly in experiment. Singular ϵ effects cannot be ignored.

Toy problem for exponential asymptotic selection

Consider the solution $\phi(x, y)$ to

$$\Delta\phi = 0 \ \text{ for } y > 0$$

On $y = 0$, require Boundary Condition

$$\epsilon\phi_{xxx}(x, 0) + (1 - x^2 + a)\phi_x(x, 0) - 2x\phi_y(x, 0) = 1,$$

where $a \in (-1, \infty)$ is real. Also require that as $x^2 + y^2 \ \to \ \infty$, $(x^2 + y^2)\ |\nabla\phi|$ bounded.

Can show $W(x + iy) \ = \ \phi_x(x, y) - i\phi_y(x, y)$ satisfes

$$\epsilon W'' + \left[-(z + i)^2 + a\right] W = 1$$

For $\epsilon \ = \ 0$, $W \ = \ W_0 \ \equiv \ \frac{1}{-(z+i)^2+a}$. *Ansatz* $W \ = \ W_0 + \epsilon W_1 + \ ..$ **consistent. Suggests no restriction on a. Yet, we will discover this conclusion to be incorrect !**

Toy problem for exponential asymptotics–II

With scaling of dependent and independent variable, obtain:

$$z + i = i2^{-1/2}\epsilon^{1/2} Z \ ; \ W \ = \ 2^{-1}\epsilon^{-1} G(Z) \ ; a \ = \ 2\epsilon\alpha$$

$$G'' - (\frac{1}{4} Z^2 + \alpha) G = -1$$

Using parabolic cylinder functions, the above problem has an explicit solution. Requiring $G \to 0$ as $Z \to \infty$ for $\arg Z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is possible if and only for integer $n \geq 0$

$$\alpha = (2n + \frac{3}{2}) \ , \ i.e. \ a = 2\epsilon\alpha = 2\epsilon \left(2n + \frac{3}{2}\right)$$

$\lim_{\epsilon \to 0+}$ solution not equal $\epsilon = 0$ solution, unless a is as above. Discontinuity of solution set at $\epsilon = 0$. So ϵ term cannot be discarded, despite consistency of regular perturbation series.

Surprising sensitivity to other small effects

Suppose $\epsilon_1 << \epsilon$ in the following variation of the toy problem:

$$\epsilon W'' + \left[-(z+i)^2 + a - \frac{\epsilon_1}{(z+i)^2} \right] W = 1 \;\; \text{for} \;\; y = \text{Im}\, z \geq 0$$

Question: Should we ignore ϵ_1 term ? Appears reasonable since a scales as ϵ without ϵ_1 term and $\frac{\epsilon_1}{(z+i)^2} << a$ for $y = \text{Im}\, z \geq 0$.

This reasoning is incorrect.

Explanation: what matters is the size of ϵ_1-term in an $\epsilon^{1/2}$ neighborhood of $z = -i$. It is $O(\epsilon)$ when $\epsilon_1 = O(\epsilon^2)$.

This explains the dramatic effect of small perturbation in experiment (BenJacob *et al*, Zocchi *et al*, Couder *et al*)

The toy problem illustrates disparate length (and time) scales can interact close to structural instability.

Toy problem for time evolution

Consider the following PDE for $\mathrm{Im}\,\xi \geq 0$:

$$G_t + iG_\xi = 1 \; + \; 2i\epsilon \left[G^{-1/2} \right]_{\xi\xi\xi} \qquad \text{with} \quad G(\xi, 0) = 1 - 2i\xi$$

Formal expansion $G \sim G^{(0)} + \epsilon G^{(1)} \; + \;$ .. gives:

$$G^0(\xi, t) \; = \; 2i \left(\xi_0(t) - \xi \right) \; , \quad \text{where} \;\; \xi_0(t) \; = \; -\frac{i}{2}(1 - t)$$

$$G_t^1 + iG_\xi^1 = 30(2i\xi_0(t) - 2i\xi)^{-7/2} \; , \;\; \text{where} \;\; G^1(\xi, 0) = 0$$

$$G^1(\xi, t) = -12 \left(2i\xi_0(t) - 2i\xi \right)^{-5/2} + 12 \left(2i\xi_d(t) - 2i\xi \right)^{-5/2},$$

$$\text{where} \quad \xi_d(t) = \xi_0(0) + it \; = \; -\frac{i}{2} + i\,t$$

Note $\xi_d(t)$ moves faster than $\xi_0(t)$ towards real axis

Inner scale and singular effects on real axis

When $\xi - \xi_d(t) \; = \; O(\mathcal{B}^{1/3}), t = O_s(1),$

$$G(\xi, t) \; \sim \; t \, M^{-2} \left\{ \mathcal{B}^{-1/3} [-i(\xi - \xi_d(t))] t^{1/6} \right\},$$

where $M(\eta)$ satisfies

$$-\frac{1}{2} M + \frac{1}{6} \eta M' = \left[-\frac{1}{2} + M''' \right] M^3 \;\; \text{with matching condition}$$

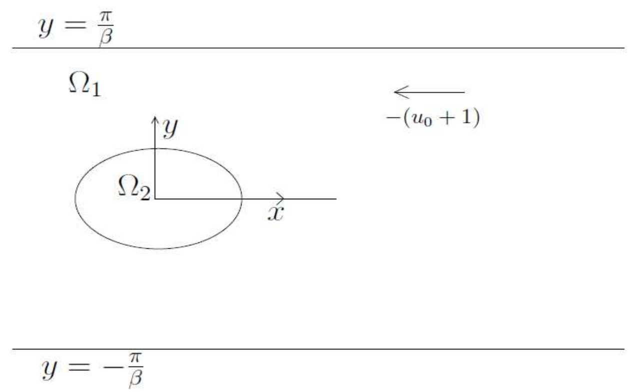
The inner ODE admits $(\eta - \eta_s)^{2/3}$ singularities; corresponding to $(\xi - \xi_s)^{-4/3}$ singularity for G, clustered near $\xi = \xi_d$

These singularities affect evolution on real ξ axis before $\xi_0(t)$ reaches real axis !

Similar singular effects occur for Hele-Shaw cell for small ϵ. Other regularizations cause similar effect

Global existence results for Hele-Shaw Problem:

Define harmonic ϕ_1, ϕ_2 in Ω_1, $\Omega_2 \subset \mathbb{R}^2$,



$$\phi_1 \sim -(u_0+1)x + O(1), \text{ as } x \to \infty, \ \frac{\partial \phi_1}{\partial y}(x, \pm\frac{\pi}{\beta}) = 0, \qquad (1)$$

On $\partial\Omega_1 \cap \partial\Omega_2$:

$$(2+u_0)x + \phi_1 - \frac{\mu_2}{\mu_1}\phi_2 = \epsilon\kappa \text{ and } \frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} = v_n \qquad (2)$$

ϵ, n, v_n, $\frac{\mu_2}{\mu_1}$, $2+u_0$ denote surface tension, inwards normal, interface speed, viscosity ratio and steady bubble speed

Results using Hou-Lowengrub-Shelley formulation

Introduce $x + iy = z(\alpha, t)$, $\alpha \in (0, 2\pi)$ interface parametrization proportional to arclength so that $z_\alpha = L e^{i\pi/2 + i\alpha + i\theta}$

We use Hou-Lowengrub-Shelley formulation of interface evolution in terms θ and L

Definition: For $\theta(\alpha, t) = \sum_{k \in \mathbb{Z}} \hat{\theta}(k; t) e^{ik\alpha}$*, define* $\|.\|_r$ *so that*

$$\|\theta(., t)\|_r^2 = \sum_{k \in \mathbb{Z}} \left(1 + |k|^{2r}\right) |\hat{\theta}(k, t)|^2$$

Theorem: For $\beta = 0$ *and any surface tension* $\epsilon > 0$ *and* $r \geq 3$*, there exists* $\delta > 0$ *such that if* $\|\theta_0\|_r < \delta$ *and* $|L_0 - 2\pi| < \delta < \frac{1}{2}$*, then there exists a unique solution* $(\theta, L) \in C\left([0, \infty), H_r \times \mathbb{R}\right)$ *to the Hele-Shaw problem. Further,* $\theta - \hat{\theta}(0; t)$ *approaches* 0 *exponentially,* $\hat{\theta}(0; t)$ *remains finite, while* L *approaches* $2\sqrt{\pi A}$ *exponentially.*

Similar results for $\beta \neq 0$ small, but with symmetric IC

Conclusion

The classic zero surface tension model is structurally unstable to small regularizing effects such as surface tension.

This structural instability for steady shapes implies that most zero surface tension shapes are physically irrelevant.

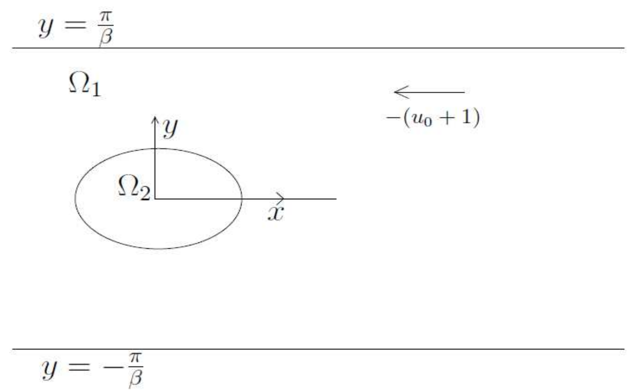
Near structurally unstable system are unpredictably sensitive to other small effects. No universality independent of regularization

Structural instability in time evolution problems occur: $\lim_{\epsilon \to 0^+} u(t;\epsilon) \neq u(t;0)$ even for $t < T_s$, the singular time for $u(t;0)$. Strong evidence that this is the case, though mathematical proof of this part is an open problem.

Steadily translating bubbles in a channel under the action of a pressure gradient are nonlinearly stable with a shrinking basin of attraction as $\epsilon \to 0$ for large sidewall distances.

Streamer Propagation Problem

Define harmonic ϕ in $\Omega \subset \mathbb{R}^2$,



$$\phi \sim -(u_0 + 1)x + O(1), \text{ as } x \to \pm\infty \tag{3}$$

On $\partial\Omega$:

$$\phi - \epsilon v_n \text{ and } \frac{\partial \phi}{\partial n} = v_n \tag{4}$$

ϵ accounts for thickness of charge layer at the interface

Global Existence Results Ye & T.