INTERFACE GROWTH IN TWO DIMENSIONS: A LOEWNER-EQUATION APPROACH

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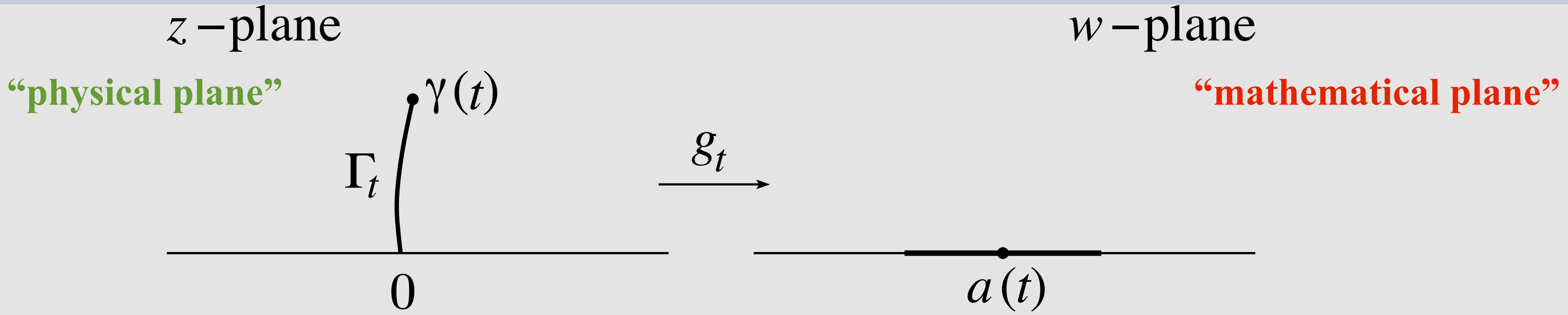
Outline

- The Loewner Equation
- Laplacian Growth as a Loewner Evolution
- Fingered Growth in the Upper Half-Plane and in the Channel Geometry
- Interface Growth Model in the Upper Half-Plane and in the Channel
- Future Directions and Conclusions

Loewner Equation



- Simple curve: $\gamma : (0, \infty) \rightarrow \mathbb{H}, \quad \Gamma_t = \gamma(0, t]$
- Loewner function: $g_t : \mathbb{H} \backslash \Gamma_t \rightarrow \mathbb{H}$
- Hydrodynamic b.c.: $g_t(z) = z + O(\frac{1}{|z|}), \quad z \rightarrow \infty$
- Initial condition: $g_0(z) = z$
- Chordal Loewner equation: $\dot{g}_t(z) = \frac{d(t)}{g_t(z) - a(t)}$

$a(t) = g(\gamma(t))$: driving function

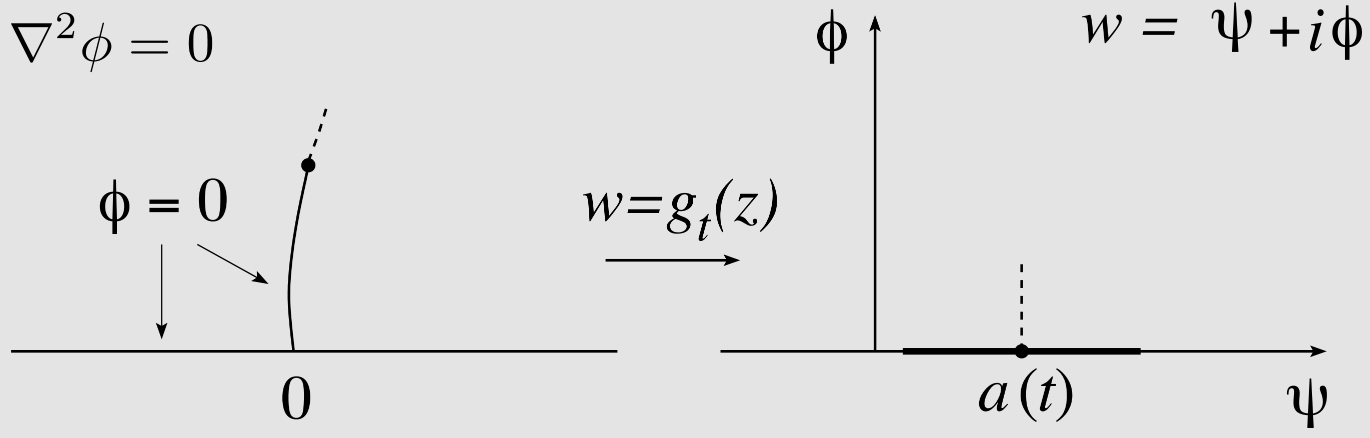
$d(t)$: growth factor (related to the hull capacity; $d(t) = 2$, *w.l.g.*)

Stochastic Loewner Evolution

- Random driving function: $a(t) = \sqrt{\kappa} W(t)$
- Scaling limit of 2D statistical mechanics models
- Connections with conformal field theory

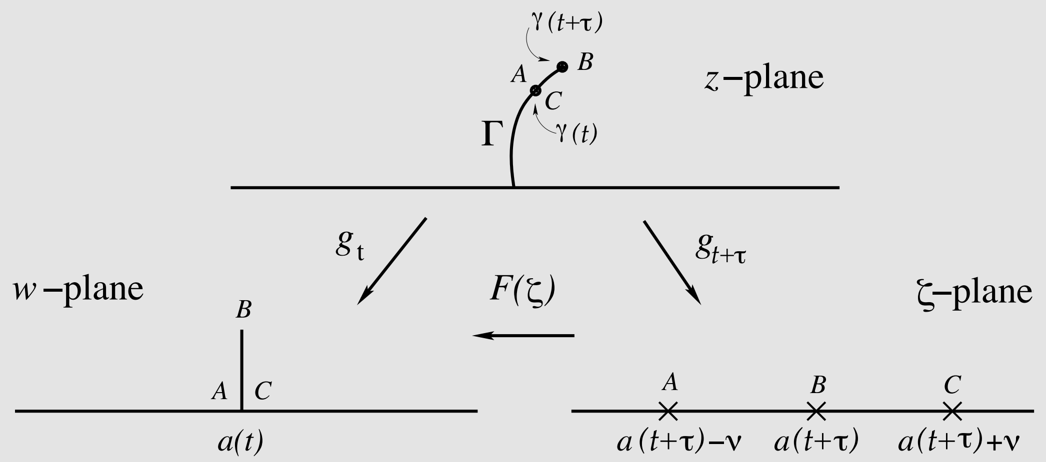
Schram (2000) and many others

Laplacian Growth as a Loewner Evolution



- Complex potential: $w = \psi + i\phi$
- Loewner function is the complex potential: $w = g_t(z)$
- Uniform 'flow' at infinity: $w(z) \approx z, \quad z \to \infty$
- Tip grows along gradient field lines: $v \sim |\vec{\nabla}\phi|^\eta$

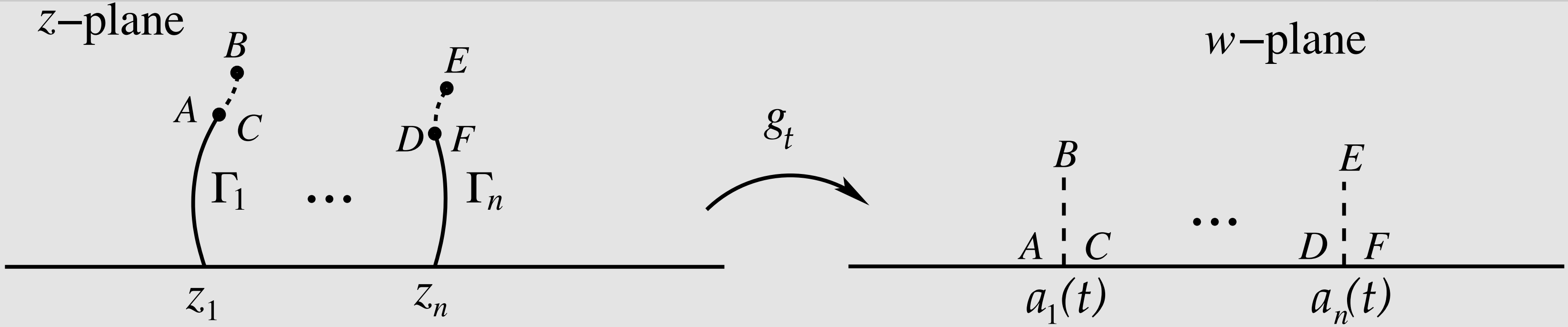
Loewner Equation for a Single Curve



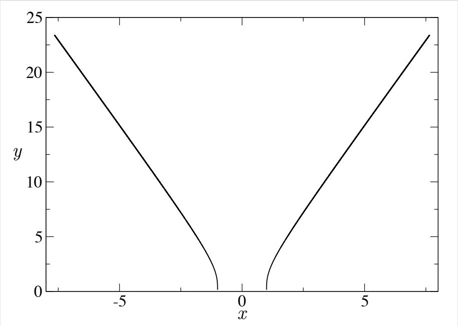
- Slit mapping: $w = F(\zeta)$
- Iterated maps: $g_t = F(g_{t+\tau})$
- In the limit $\tau \to 0$: $\dot{g}_t(z) = \dfrac{d(t)}{g_t(z) - a(t)}$, $\dot{a}(t) = 0$
- Growth factor: $d(t) = |f_t''(a(t))|^{-\eta/2-1}$, $f_t(w) = g_t^{-1}(w)$

Fingered Growth

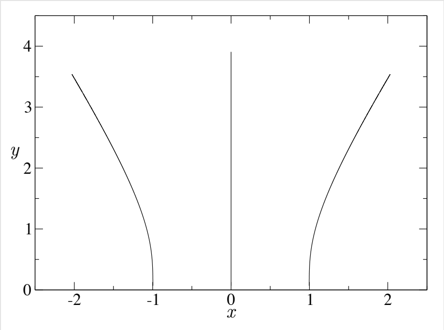
Carleson & Makarov 2002, Selander 1999



- Loewner equation: $\dot{g}_t = \sum_{i=1}^{n} \frac{d_i(t)}{g_t - a_i(t)}$
- Dynamics of singularities: $\dot{a}_i(t) = \sum_{j \neq i} \frac{d_j(t)}{a_i(t) - a_j(t)}$
- Exact solutions for 2 and 3 symmetric fingers:



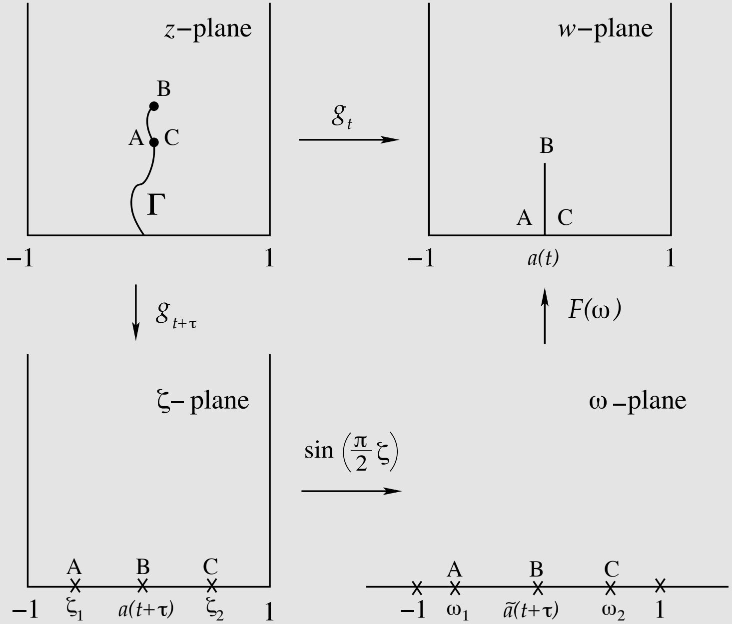
Gubiec & Szymczak, 2008



Durán & GLV, 2010

Fingered Growth in the Channel Geometry

Gubiec & Szymczak, 2008



- Loewner equation: $$\dot{\tilde{g}}_t = \frac{\pi^2}{4}(1-\tilde{g}_t^2)\sum_{i=1}^{n}\frac{d_i(t)}{\tilde{g}_t-\tilde{a}_i(t)}$$

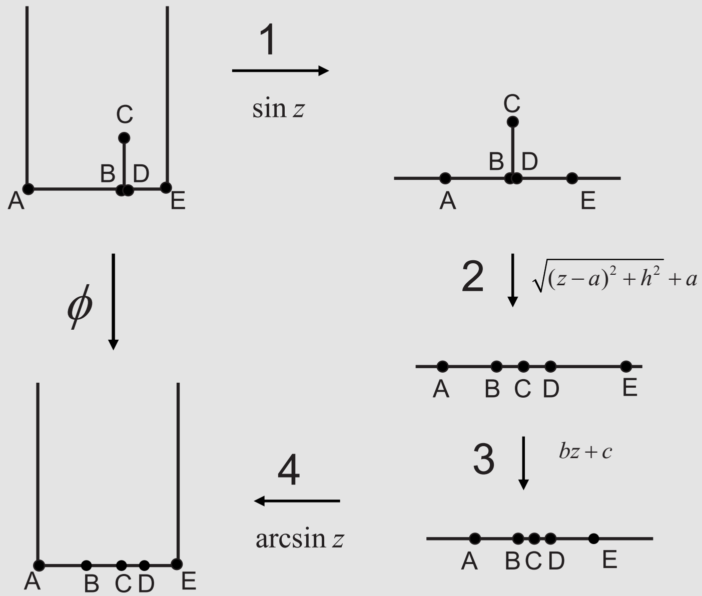
where

$$\dot{\tilde{a}}_i = -\frac{\pi^2}{8}d_i(t)\tilde{a}_i + \frac{\pi^2}{4}(1-\tilde{a}_i^2)\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{d_i(t)}{\tilde{a}_i-\tilde{a}_j}$$

$$\tilde{g}_t = \sin\left(\frac{\pi}{2}g_t\right)$$

Fingered Growth in the Channel Geometry

Gubiec & Szymczak, 2008



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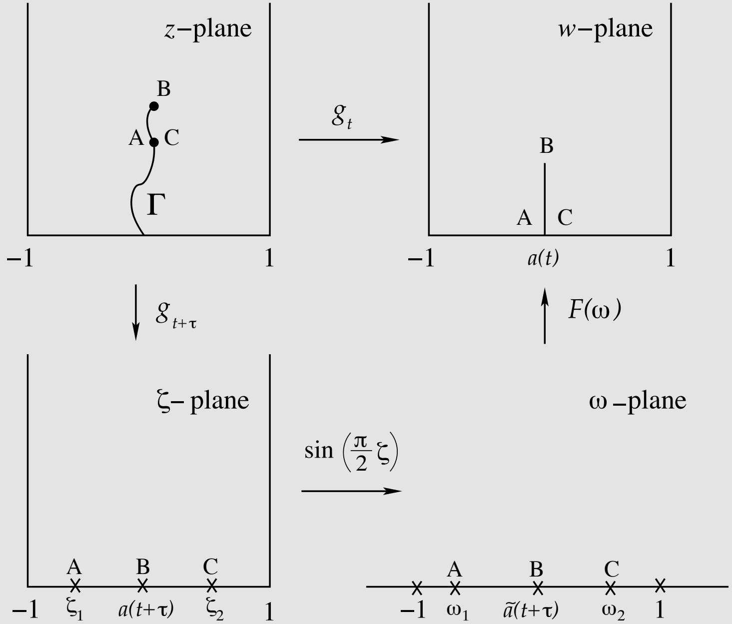
Fingered Growth in the Channel Geometry



Durán & GLV, 2010

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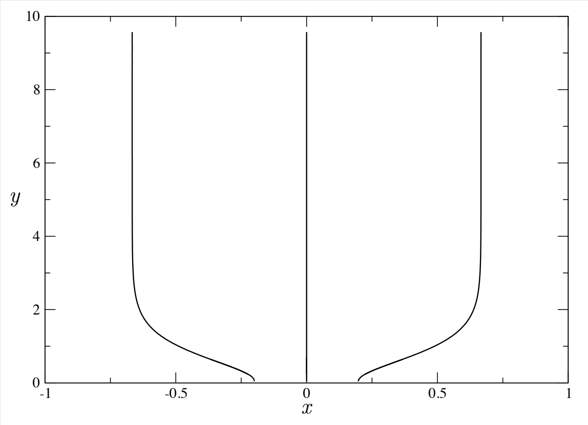
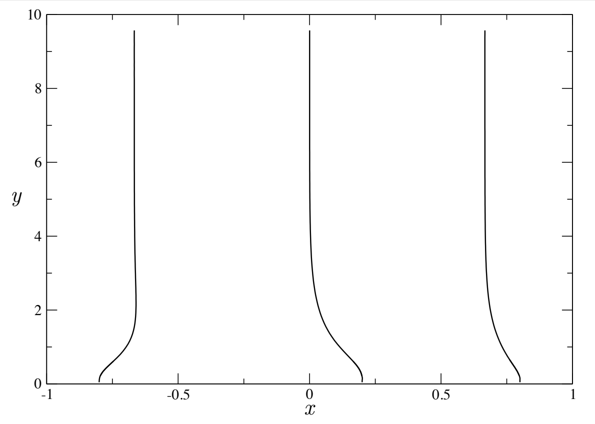
$$\tilde{g}_t = \sin\left(\frac{\pi}{2}g_t\right)$$

Fingered Growth in the Channel Geometry

Durán & GLV, 2010

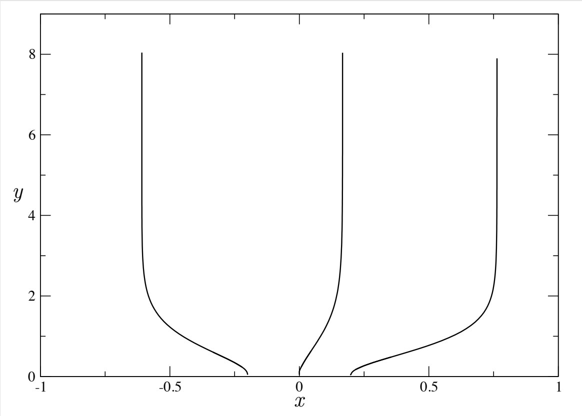
symmetrical configurations



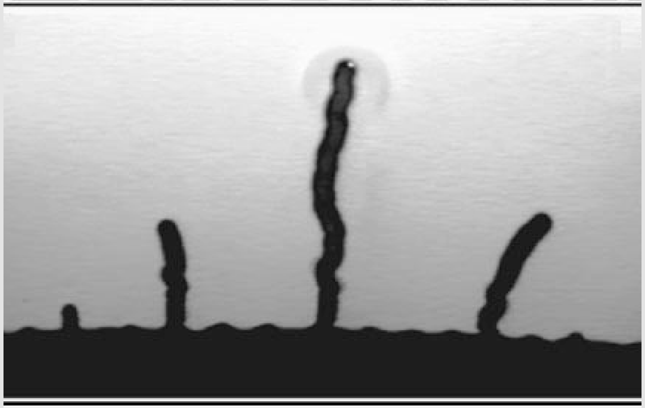


$$d_1 = d_2 = d_3 = 1$$

asymmetric configuration

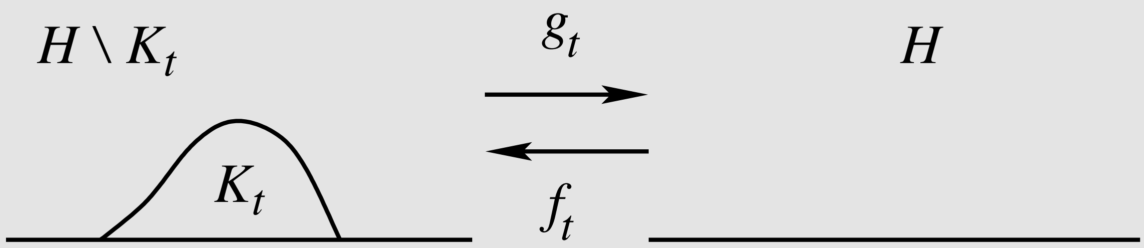


$$d_1 = d_2 = 1, \;\; d_3 = 0.5$$



Fingering in combustion, Zik & Moses, 2008

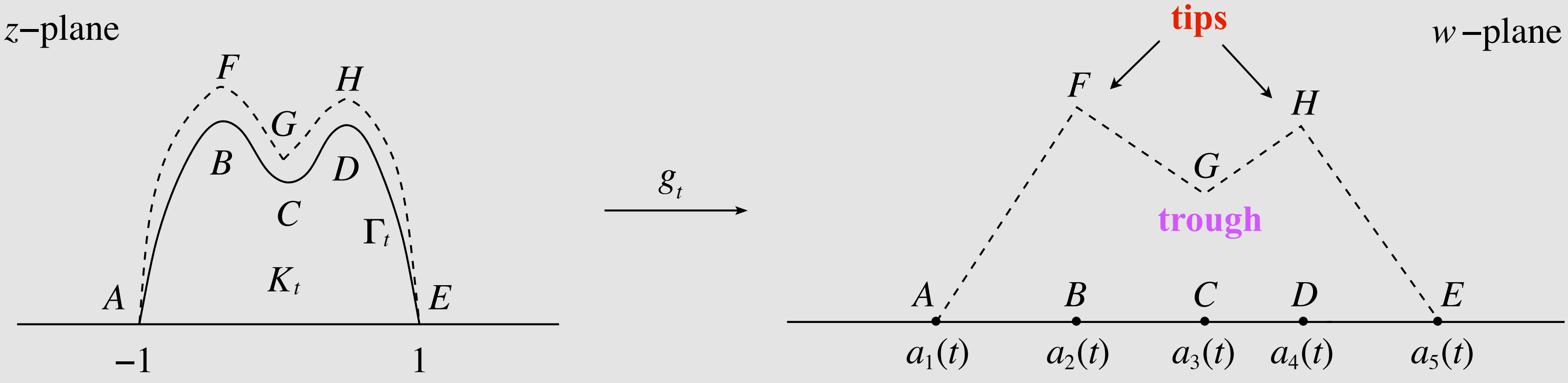
Loewner Chains



- K_t : family of growing hulls in $\mathbb{H}$
- Loewner function: $g_t : \mathbb{H} \backslash K_t \to \mathbb{H}$
- Loewner equation: $\dot{g}_t(z) = \int \frac{\rho_t(x)dx}{g_t(z)-x}$
- Density $\rho_t(x)$: 'local growth rate'
- Laplacian growth: $\rho(x)_t = |f_t'(x)|^{-\eta-1}, \quad f_t(w) = g_t^{-1}(w)$

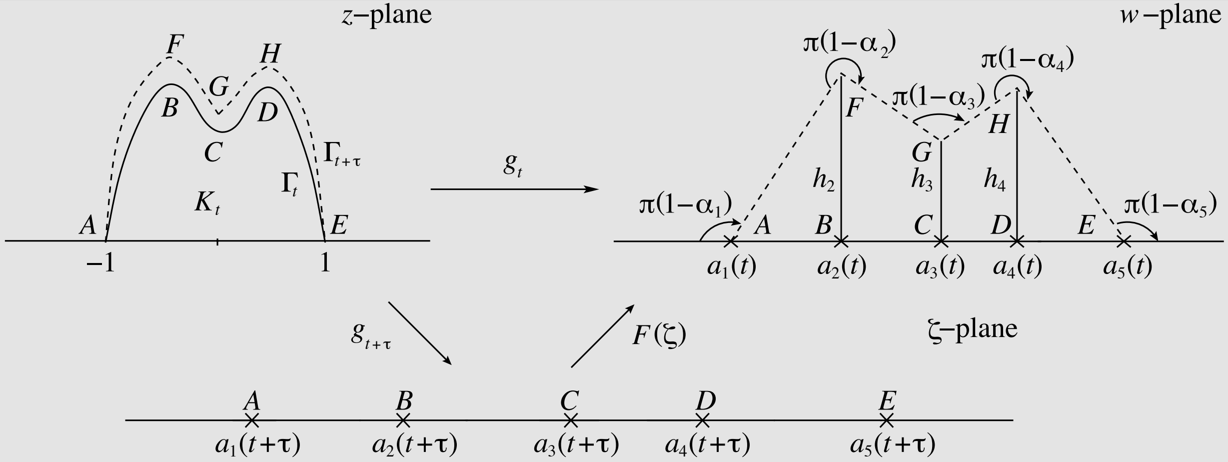
Interface Growth Model

Durán & GLV, PRE, 2010



- Growing domain (hull) K_t delimited by interface Γ_t
- Loewner function: $g_t : \mathbb{H}\backslash K_t \rightarrow \mathbb{H}$
- Endpoints, $z = \pm 1$, remain fixed
- Tips and troughs grow along gradient field lines
- Infinitesimal accrued domain mapped to a polygon

Loewner Equation for Interface Growth



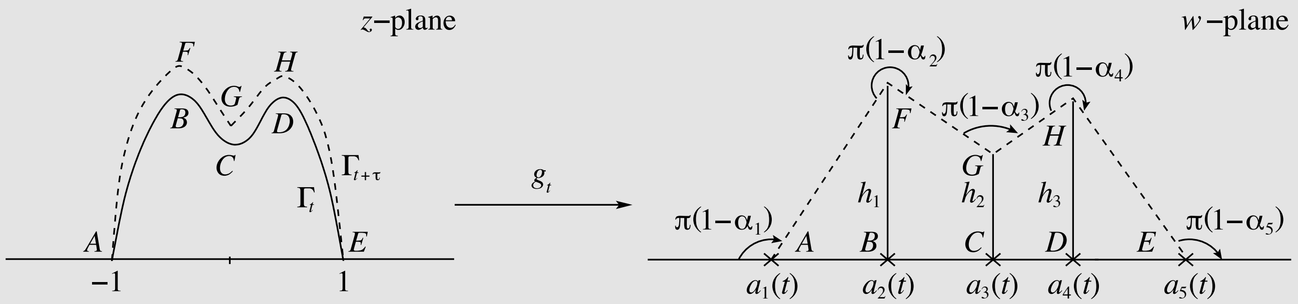
- Iterated maps: $g_t = F(g_{t+\tau})$
- Schwarz-Christoffel formula:

$$g_t = \int_{a_i(t+\tau)}^{g_{t+\tau}} \prod_{i=1}^{N} [\zeta - a_i(t+\tau)]^{-\alpha_i}\, d\zeta + a_i(t) + ih_i$$

- Expand integrand in first order of α_i :

$$g_t \approx \int_{a_i(t+\tau)}^{g_{t+\tau}} \left\{ 1 - \sum_{i=1}^{N} \alpha_i \ln[\zeta - a_i(t+\tau)] \right\} d\zeta + a_i(t) + ih_i$$

Loewner Equation for Interface Growth



$$\sum_{i=1}^{N} \alpha_i = 0$$

$$\sum_{i=1}^{N} a_i \alpha_i = 0$$

- Loewner equation:

$$\dot{g}_t(z) = \sum_{i=1}^{N} d_i(t)[g_t - a_i(t)] \ln[g_t - a_i(t)]$$

- Dynamics of singularities:

$$\dot{a}_i = \sum_{j \neq i} d_j(t)(a_i - a_j) \ln |a_i - a_j|$$

- Growth factors: $d_i(t) = \lim_{\tau \to 0} \frac{\alpha_i}{\tau}$

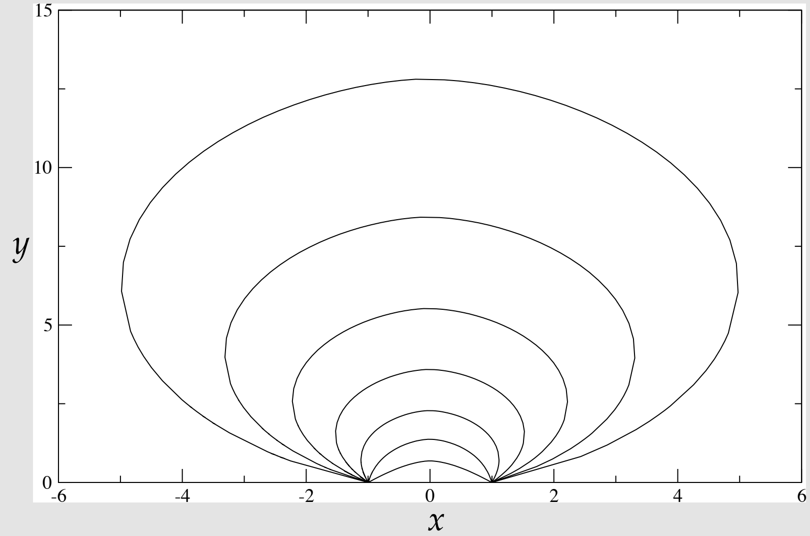
$$\sum_{i=1}^{N} d_i = 0, \quad \sum_{i=1}^{N} a_i d_i = 0$$

Examples: Single Tip

- Symmetrical interface: $a_2(t) = 0, \quad a_2(t) = -a_1(t) = a(t)$
- Loewner equation:

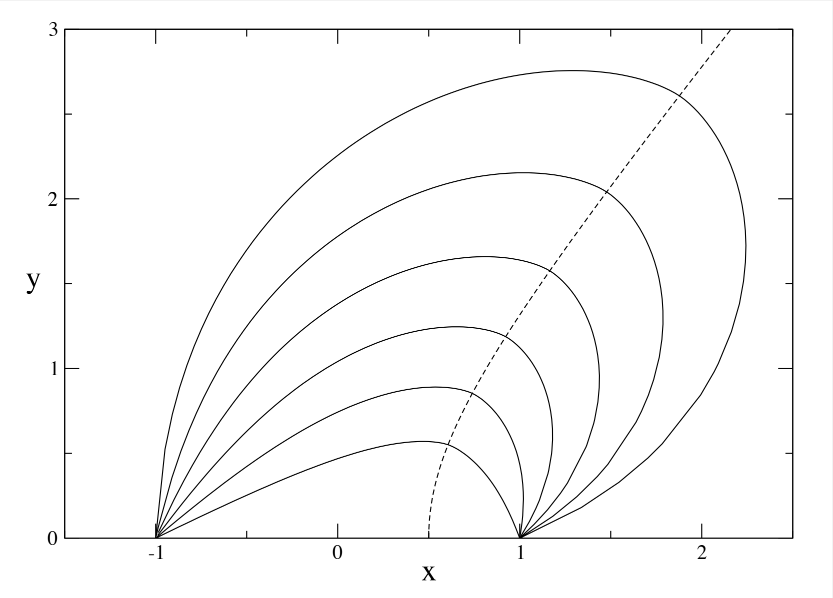
$$\dot{g}_t(z) = d(t)\left\{[g_t + a(t)]\ln[g_t + a(t)] + [g_t - a(t)]\ln[g_t - a(t)] - 2g_t \ln g_t\right\}$$

- Evolution of *a*(*t*): $\dot{a}(t) = (\ln 4)d(t)a(t) \;\Rightarrow\; a(t) = a_0 4^{\int_0^t d(t')t'}$



Examples: Single Tip

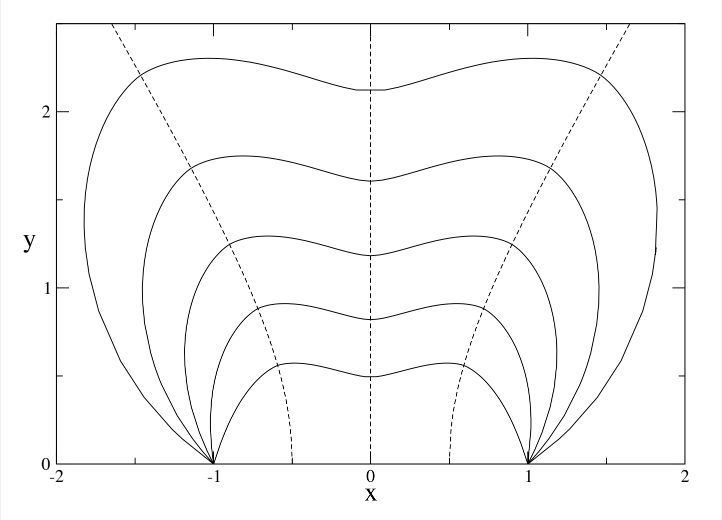
- Asymmetric interface:



- Asymmetry persists: tip approaches inclined straight line

Examples: Two Tips

- Symmetrical interface:

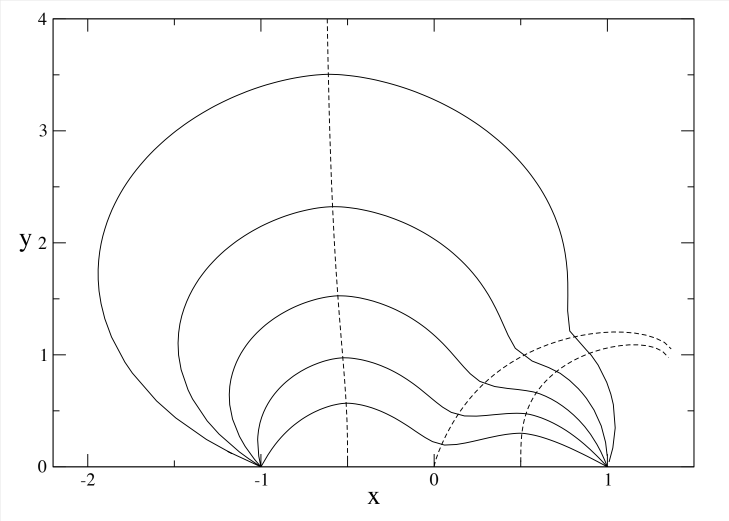


$$d_2 = d_4 = -1,\ d_3 = 0.5$$

- Trajectories of tips and trough resemble three-finger case

Examples: Two Tips

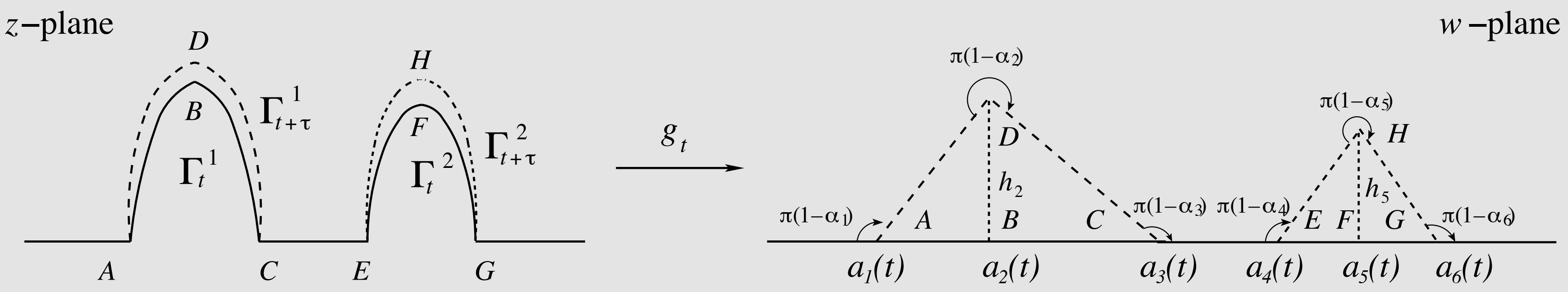
- Asymmetric interfaces:



$$d_2 = -1,\ d_3 = 0.8,\ d_4 = -0.5$$

- “Screening effect”: faster tip ‘screens’ slower tip

Multiple Interfaces



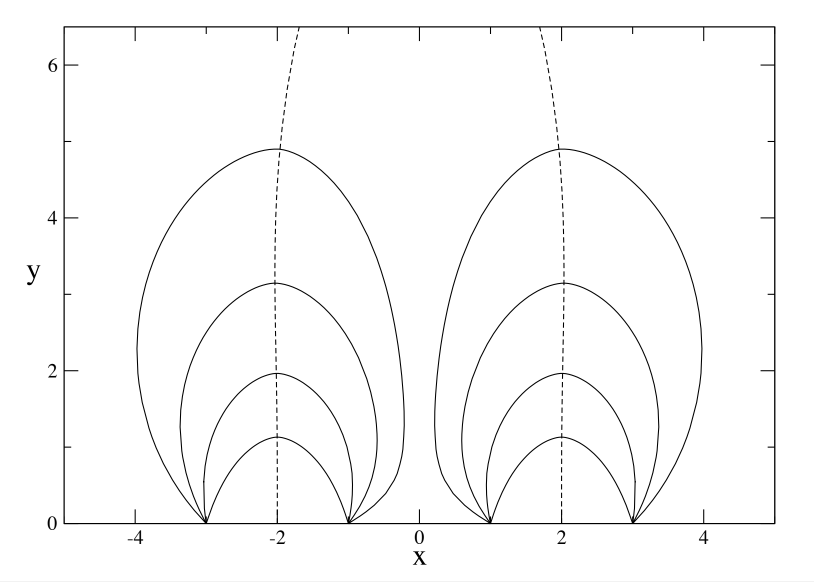
- Same Loewner equation:

$$\dot{g}_t(z) = \sum_{i=1}^{N} d_i(t)[g_t - a_i(t)]\ln[g_t - a_i(t)]$$

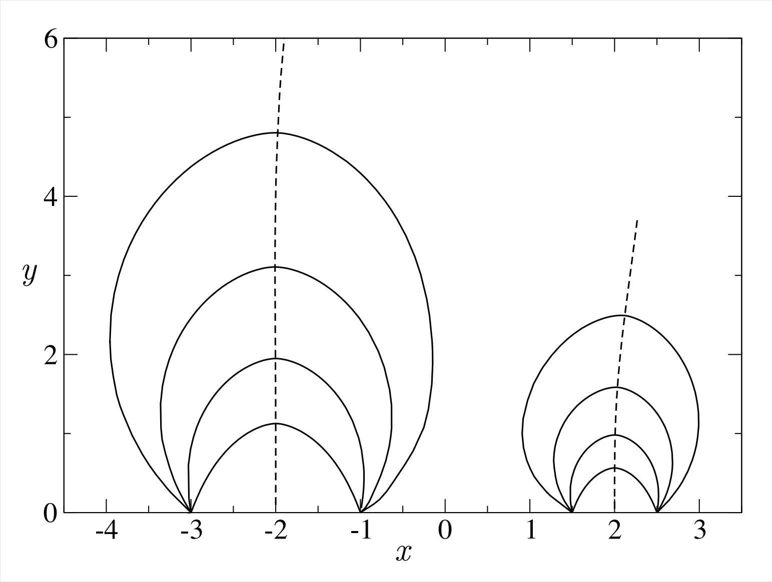
N : total number of vertices

Examples: Two Interfaces

symmetric interfaces



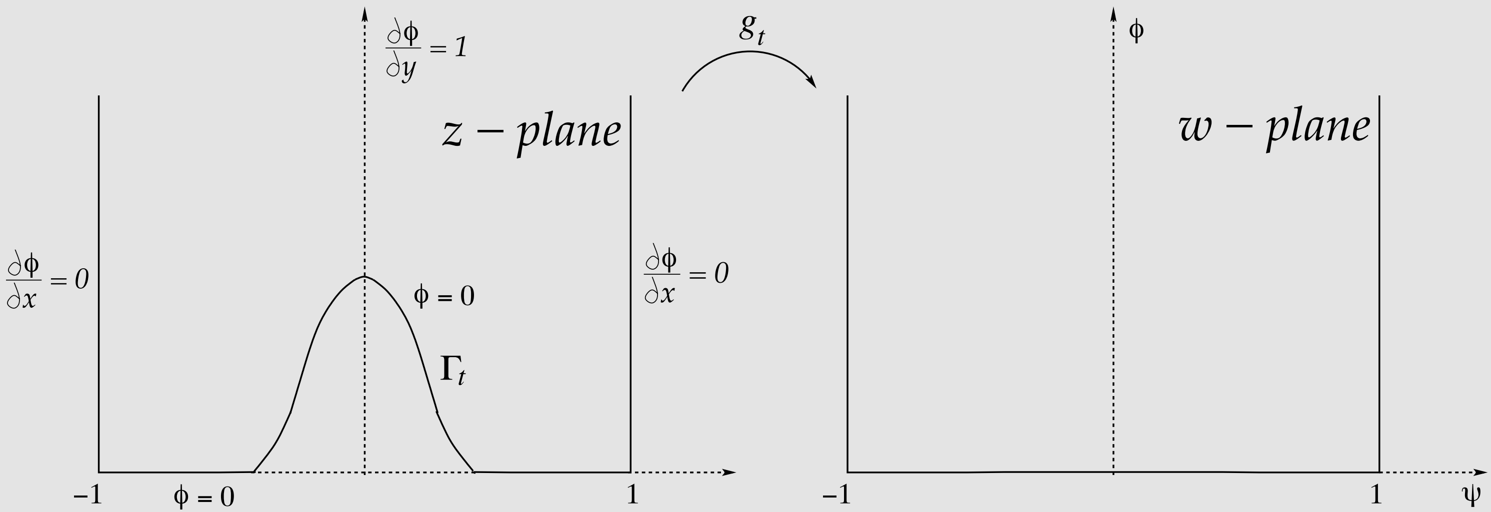
asymmetric interfaces



same 'growth factors' for both interfaces in both cases

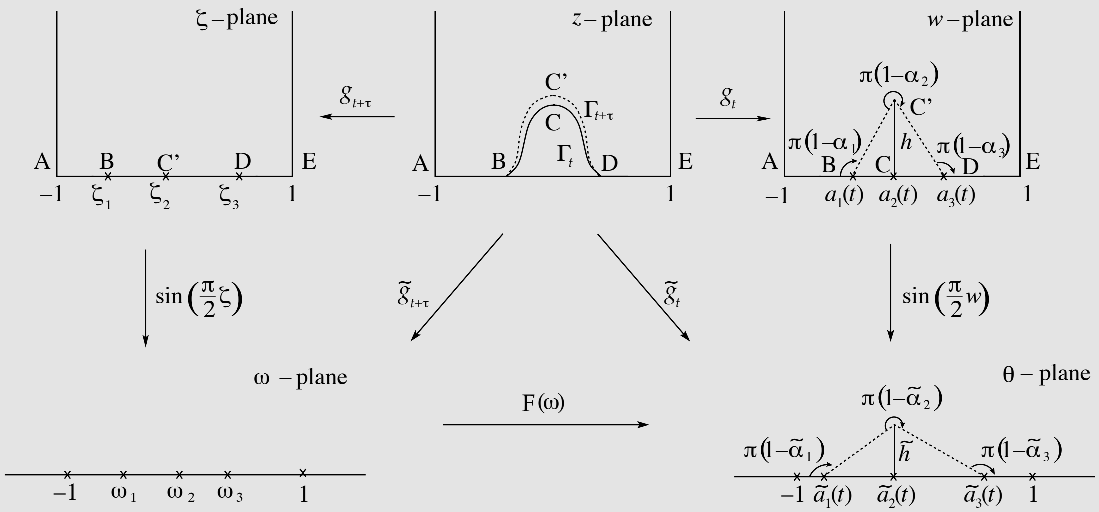
Broken symmetry $\Rightarrow$ "Screening effect"

Interface Growth in the Channel Geometry



Loewner Equation in the Channel Geometry

Durán & GLV, 2010



- Loewner equation:

$$\dot{\tilde{g}}_t(z) = \sum_{i=1}^{N} \tilde{d}_i(t) \left\{ [\tilde{g}_t - \tilde{a}_i(t)] \ln[\tilde{g}_t - \tilde{a}_i(t)] - A_i^+(t)\tilde{g}_t + A_i^-(t) \right\}$$

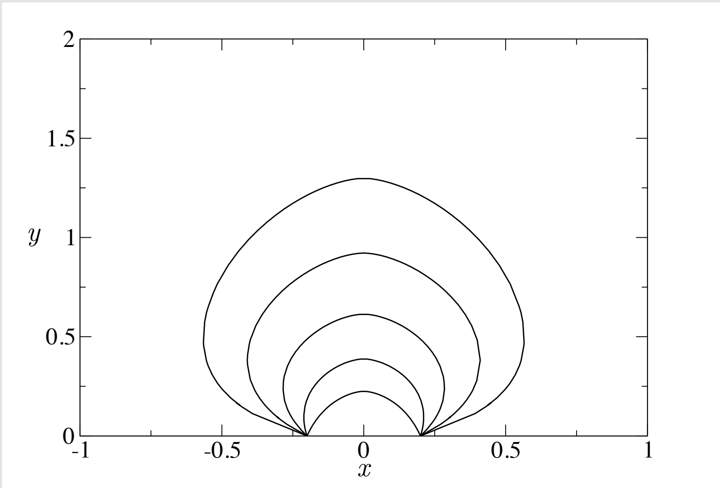
where

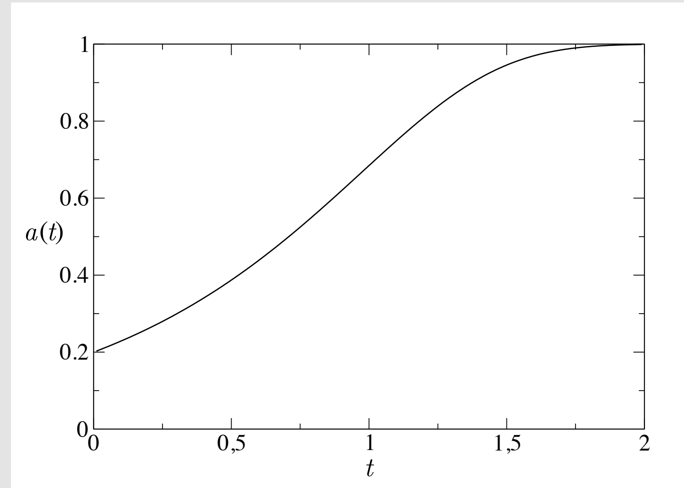
$$A_i^{\pm} = \frac{1}{2} \left\{ [1 + \tilde{a}_i(t)] \ln[1 + \tilde{a}_i(t)] \pm [1 - \tilde{a}_i(t)] \ln[1 - \tilde{a}_i(t)] \right\}$$

$$\tilde{g}_t = \sin\left(\frac{\pi}{2} g_t\right)$$

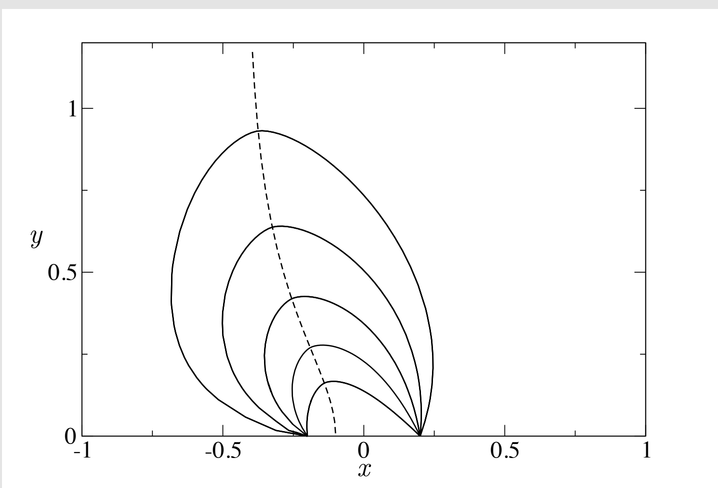
Examples: Single Tip

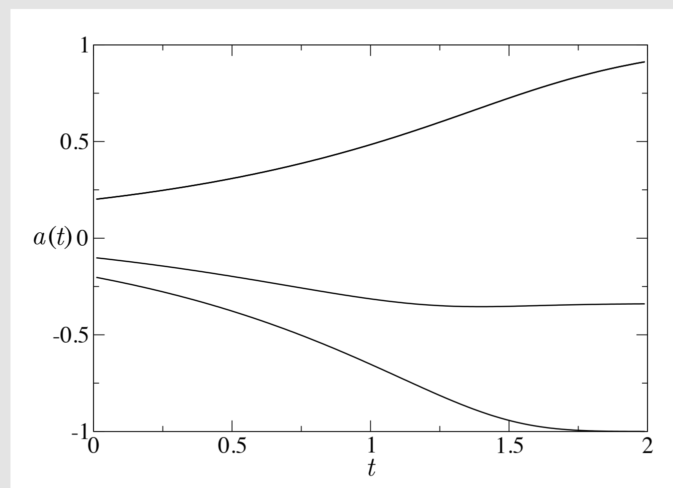
symmetrical





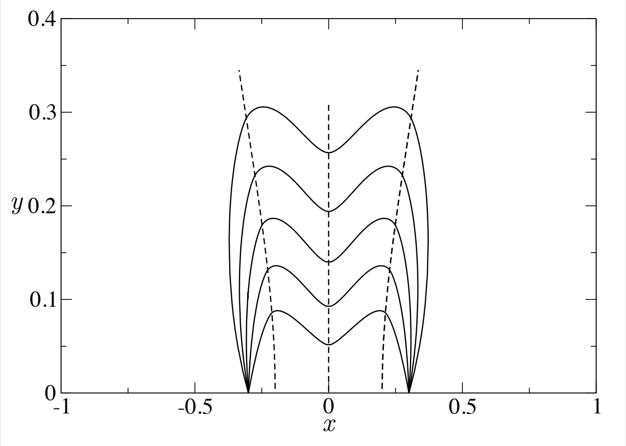
asymmetric



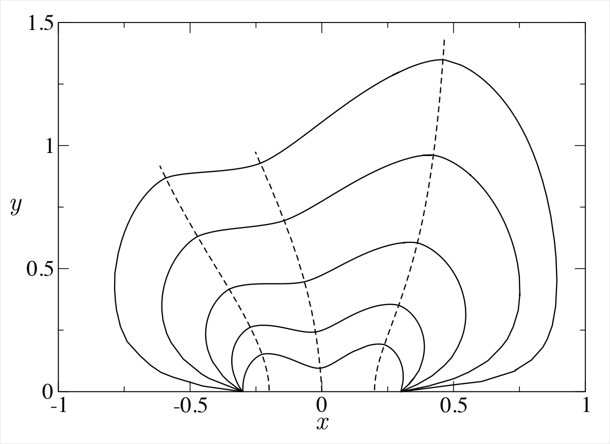


Examples: Two Tips

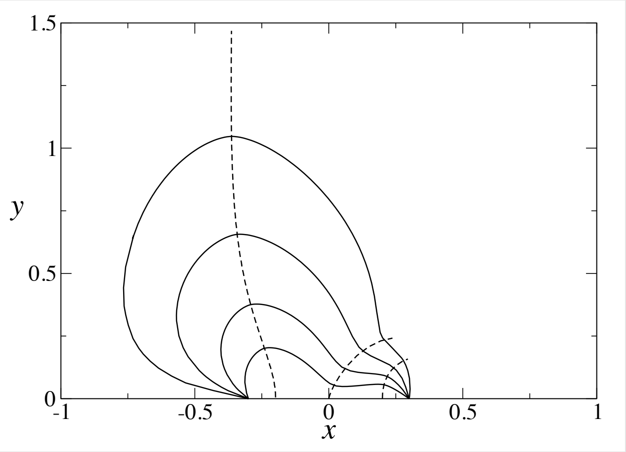
symmetrical interface



partial screening

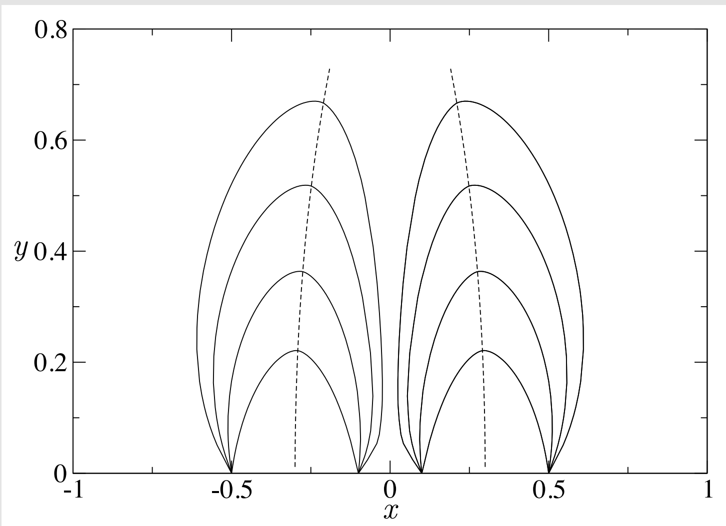


total screening

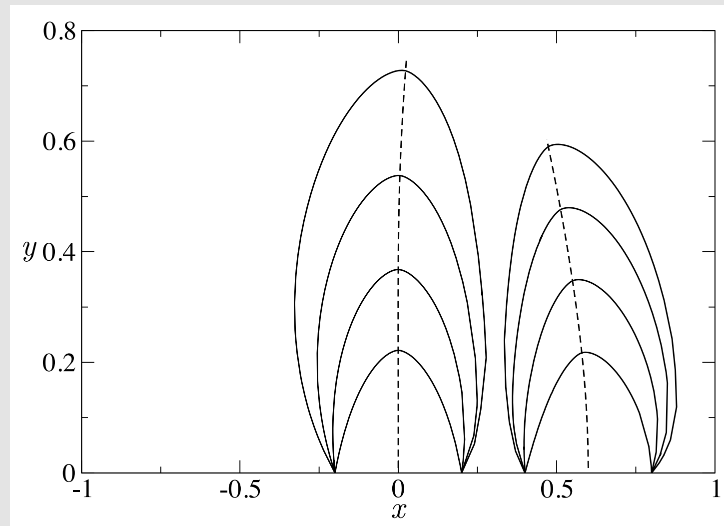


Examples: Multiple Interfaces

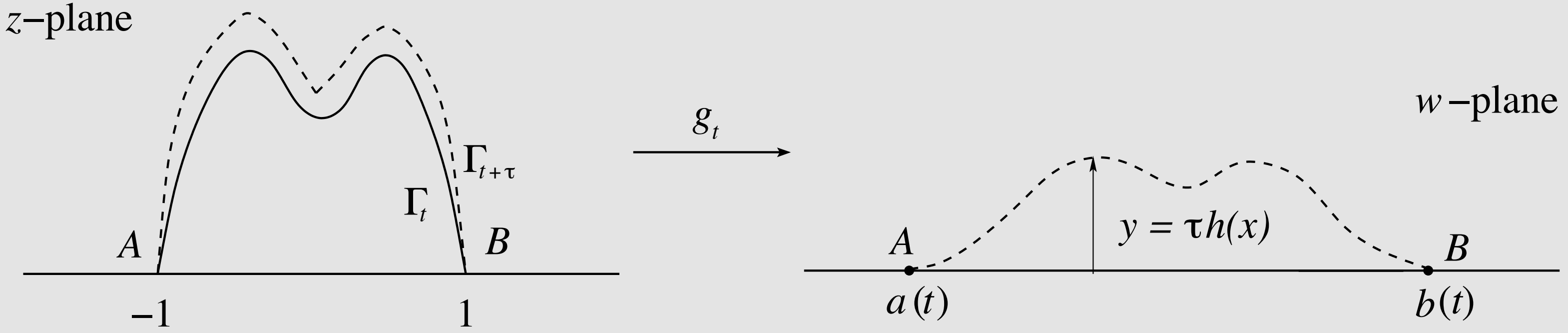
symmetrical



asymmetric



Loewner Domains



- Loewner equation: $$\dot{g}_t(z) = \int_{a(t)}^{b(t)} \kappa_t(x)[g_t(z) - x]\ln[g_t(z) - x]dx$$

  where $\kappa_t(x) = h_t''(x)$

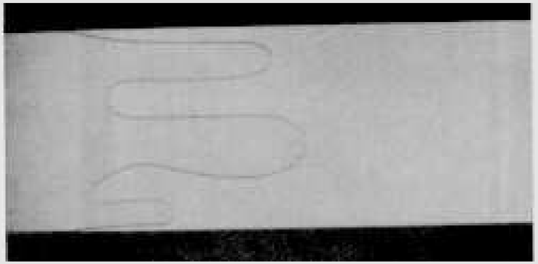
- More generally: $$\dot{g}_t(z) = \int_{\mathbb{R}} [g_t(z) - x]\ln[g_t(z) - x]d\mu_t(x)$$

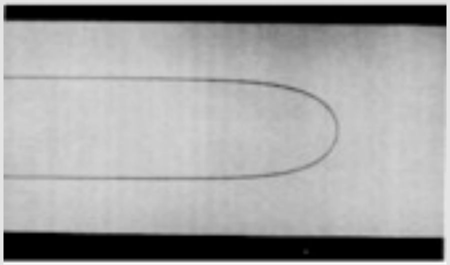
  where $\mu(x)$ is a signed measure with

$$\int_{\mathbb{R}} d\mu_t(x) = 0, \qquad \int_{\mathbb{R}} x d\mu_t(x) = 0$$

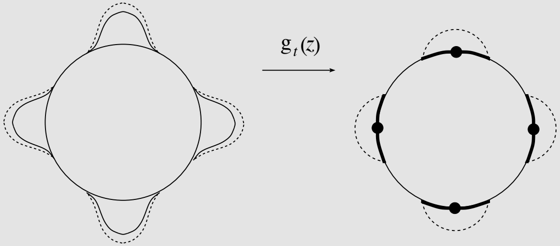
Future Directions

- Can describe HS-like viscous fingering?

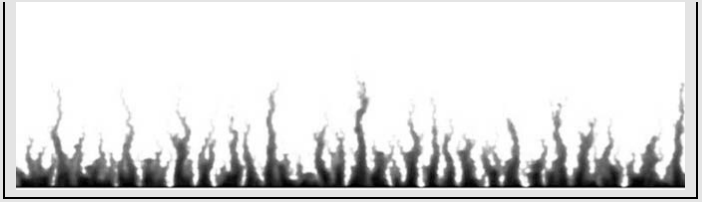




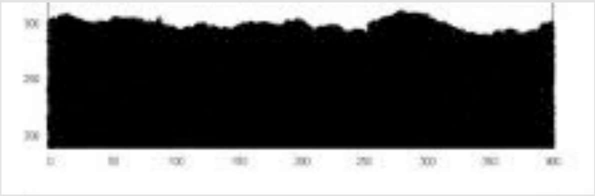
- Extension to radial geometry?



- Can generate random interfaces?



dissolving rock fractures



rough surface (KPZ, etc)

DLA-like pattern

Conclusions

- Interface growth model as a Loewner evolution.
- Loewner equation obtained for both upper half-plane and channel geometry.
- Interesting dynamical features: finger competition, screening, etc.
- Generalized model: Loewner domains

Thank you.