Who laughs last?
perturbation theory of games

Tibor Antal, Program for Evolutionary Dynamics, Harvard

- games in phenotype space
- perturbation method: two key aspects
- further examples, general results

with: C Tarnita, H Ohtsuki, J Wakeley, P Taylor, A Traulsen, F Fu, N Wage, M Nowak
What is the question?

Two strategies: A and B: Which one is better?

John Forbes Nash,
John Maynard Smith
fixation probabilities ...
C Taylor, Nowak

Or: Which outnumbers the other in the long run?
with two way mutation \( u \)
(who laughs last?)

\[
\langle x \rangle > 1/2 \quad u \to 0
\]

Kandori '93
fixation probabilities
\[
\rho_A > \rho_B
\]

general \( u \)
Evolution in phenotype space

\[ N = 7 \]

birth

\[ \beta \]

\[ 1 - 2\beta \]

random death
Evolution in phenotype space

\[ N = 7 \]

\[ \beta, 1 - 2\beta, \beta \]

random death

birth
Evolution in phenotype space

$N = 7$

disperse or condense?
Group of size \( \sqrt{N\beta} \) diffuses as \( D = N\beta/2 \).
Colors

\[ y = \Pr(S_k = S_q) \]
\[ z = \Pr(X_k = X_q) \]
\[ g = \Pr(S_k = S_q, X_k = X_q) \]
\[ h = \Pr(S_l = S_k, X_k = X_q) \]

\[ \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \]

\[ \left( \begin{array}{c} b \\ -c \end{array} \right)^* = \frac{z - h}{g - h} \]
\( \beta = 1/2, \quad \mu = 1/2 \)

\[ \frac{1 + 12 \sqrt{2}}{7} \]

\[ \mu = 2Nu \]

\[ r = 2N\beta \]
\[
\begin{pmatrix}
1 & \hat{S} \\
\hat{T} & 0
\end{pmatrix}
\]

\[
\hat{T} < \hat{S} + 1 + \sqrt{3}
\]
Perturbation method: 2 key points

Payoff = \( 1 + \delta \times \text{payoff of} \)

Wright-Fisher

\[
\langle x \rangle = \frac{1}{2} + \frac{1 - u}{u} \langle \Delta x^{\text{sel}} \rangle
\]

\[
\Delta x^{\text{tot}} = \Delta x^{\text{sel}} - \frac{u}{2}(x + \Delta x^{\text{sel}}) + \frac{u}{2}(1 - x - \Delta x^{\text{sel}})
\]

\[
\langle x \rangle > \frac{1}{2} \iff \langle \Delta x^{\text{sel}} \rangle > 0
\]
Perturbation method: 2 key points

Payoff = \( 1 + \delta \times \text{payoff of} \)

\[
\begin{array}{c|cc}
\text{A} & \text{a}_{11} & \text{a}_{12} \\
\text{B} & \text{a}_{21} & \text{a}_{22} \\
\end{array}
\]

\( x \) frequency of A

\( \delta \) selection strength

\( u \) mutation probability

Easy perturbation method for small \( \delta \)

\[ \langle \Delta x \rangle = \sum \Delta x_i \pi_i \]
\[ \Delta x_i = 0 + \delta \Delta x_i^{(1)} \]
\[ \pi_i = \pi_i^{(0)} + \delta \pi_i^{(1)} \]

\[ \langle \Delta x \rangle = \delta \sum \Delta x_i^{(1)} \pi_i^{(0)} + \mathcal{O}(\delta^2) \]

neutral probabilities only!
Higher dimensions

phenotype II

phenotype I

Graphs

Sets

Time

Phenotype Space

Tarnita '09

Allen '10
One parameter to rule them all

A wins iff \( \sigma a + b > c + \sigma d \)

single parameter for all structures

classical well mixed \( \sigma = 1 \)

\[ a + b > c + d \quad \text{(risk dominance)} \]

phenotype game \( \sigma = 1 + \sqrt{3} \)

more strategies on structure?

\[
\begin{array}{cc}
\text{# strategies} & \text{# parameters} \\
2 & 1 \\
\geq 3 & 2
\end{array}
\]
Relations to relatedness

A wins iff \( \frac{b}{c} > \frac{1}{R} \)  

(Hamilton's rule)

same size islands

\[
R = \frac{\Pr(S_k = S_q | X_k = X_q) - \Pr(S_k = S_q)}{1 - \Pr(S_k = S_q)}
\]

fluctuating size islands, phenotype walk

\[
\left(\frac{b}{c}\right)^* = \frac{z - h}{g - h}
\]

\[
R = \frac{\Pr(S_k = S_q | X_k = X_q) - \Pr(S_l = S_k | X_k = X_q)}{1 - \Pr(S_l = S_k | X_k = X_q)}
\]

more general structures

is there always a relatedness interpretation of the general formulas?

TA '09, Taylor '10
Final slide

general method to study weak selection

TA, Ohtsuki, Wakeley, Taylor, Nowak, PNAS '09

some papers can be found on my website

thanks