HOW OFTEN SHOULD ONE COOPERATE?

PARETO-INEFFICIENCY OF PURE NASH EQUILIBRIUM IN SOME FINITE RANDOM GAMES

Christine Taylor
Harvard University
Prisoner’s Dilemma

\[
\begin{array}{cc}
C & D \\
\hline
C & b-c & -c \\
D & b & 0 \\
\end{array}
\]
Penalty Kick Game

How to take penalties: Freakonomics explains, S. J. Dubner and S. D. Levitt, Times Online, June 12, 2010
Penalty Kick Game

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Penalty Kick Game

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<td>L</td>
<td>-0.5,0.3</td>
<td>0.3,0.1</td>
</tr>
<tr>
<td>R</td>
<td>0.5,-0.8</td>
<td>-0.2,-0.7</td>
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Finite random games


Thursday, June 17, 2010
Finite random games

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* Cooperation arises: when NE is Pareto dominated by another strategic profile in which every player fares at least as well, and some fares better.

Finding Nash equilibria
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* Finding NE is NP hard....

* But finding PNE is easy.
Finite random games

* Probability distribution of \( k \), the number of PNEs:

\[
P(k, m_1, m_2) = \sum_{j=0}^{k} (-1)^j \binom{k+j}{k} \binom{m_1}{k+j} (m_1m_2)^{(k+j)} \binom{m_2}{k+j} (k+j)!
\]

* Probability that a PNE is PPO is given by

\[
\int_{x\in[0,1]^n} (1 - \prod_{p=1}^{n} (1 - x_p)) \prod_{p=1}^{n} m_p - \sum_{p=1}^{n} m_p + n-1 \prod_{p=1}^{n} (m_p x_p^{m_p-1} dx_p)
\]

* If all \( m_p = m \), the probability that a PNE is PPO is not monotonic in \( n \), the number of players. However, the probability that a PNE is PPO decreases as \( m_p \) increases.

* For fixed \( n \), the probability that a PNE is PPO is bounded from below by \( 1/e \) when all \( m_p \) tends to infinity.

* If all players have the same number of strategies, as \( n \) tends to infinity, a PNE is always PPO.


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(A, B): a random two-person m-strategy game

A, B are $m \times m$ payoff matrices, one for each player. The $m^2$ payoff entries $a_{ij}$ and $b_{ij}$ are i.i.d. (real-valued, independent, identically distributed continuous random variables), we shall assume them to be U(0, 1) for this talk.

The pure strategy pair $(i^*, j^*)$ is a PNE if

$$a_{i^*, j^*} = \max_i a_{i, j^*}, \quad b_{i^*, j^*} = \max_j b_{i^*, j}$$

In symmetric random games, $a_{ij} = b_{ji}$

In zero-sum games $a_{ij} = -b_{ij}$

In common payoffs games, $a_{ij} = b_{ij}$
2-person 2-strategy two-role games

* Trust Game: PNE (3,3) is Pareto-Dominated by (1,1), (4,4), (1,2), (2,4), (1,4)

<table>
<thead>
<tr>
<th>e₁</th>
<th>f₁</th>
<th>f₂</th>
</tr>
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<tbody>
<tr>
<td>e₂</td>
<td>(β - c, rc - β)</td>
<td>(−c, rc)</td>
</tr>
<tr>
<td></td>
<td>(0, 0)</td>
<td>(0, 0)</td>
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\(c < β < rc\)

\(G₁ = e₁ \mathbf{f₁}, \quad G₂ = e₂ \mathbf{f₁}, \quad G₃ = e₂ \mathbf{f₂}, \quad G₄ = e₁ \mathbf{f₂}\)

* Ultimatum Game: PNEs (1,1) and (3,3) are PPO

<table>
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<tr>
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</tr>
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<tbody>
<tr>
<td>e₂</td>
<td>(1 - h, h)</td>
<td>(1 - h, h)</td>
</tr>
<tr>
<td></td>
<td>(0, 0)</td>
<td>(1 - l, l)</td>
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\(0 < l < h < 1\)


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2-person m-strategy
two-role games

Consider a game with two roles I and II and m strategies for each role. Let $a_{ij}$ and $b_{ij}$ be the respective payoffs to role I and II players when the role I player uses strategy $i$ and the role II player uses strategy $j$. $A$ and $B$ are $mxm$ payoff matrices whose entries are independent $U(0,1)$ distribution. A coin toss decides which role to assign to each player. The resulting game is a 2-person $m^2$-strategy symmetric game whose $m^2$-$xm^2$ payoff matrix $C$ has entries given by $c_{ij,kl} = a_{il} + b_{kj}$

The strategic profile $(i^*j^*, k^*l^*)$ is PNE if

$$a_{i^*l^*} + b_{k^*j^*} = \max_{(i,j)} a_{il^*} + b_{k^*j} , \quad a_{i^*l^*} + b_{k^*j^*} = \max_{(k,l)} a_{i^*l} + b_{kj^*}$$
Some main questions
Some main questions

- How often is there a PNE? What is the probability distribution of the number of PNEs?
Some main questions

* How often is there a PNE? What is the probability distribution of the number of PNEs?

* How often is a PNE not PPO? i.e. how often can cooperation lead to improvement for all players involved.
**Probability distribution of the number of PNES**

* random game

\[ P(k, m) = \nu(k, m) \sum_{i=0}^{m-k} (-1)^i \frac{1}{m^{2i}+2k} \nu(i, m-k), \quad \nu(k, m) = \binom{m}{k} \frac{m!}{(m-k)!} \]

* symmetric random game

\[ P(k, m) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{m!}{(k-2j)!2^j!m^k} \sum_{i=0}^{\lfloor \frac{m-k}{2} \rfloor} (-1)^i \frac{1}{i!2^i} \sum_{l=0}^{m-k-2i} (-1)^l \frac{1}{l!(m-k-2i-l)!m^l} \]

* zero-sum game

\[ P(0, m) = 1 - \frac{(m!)^2}{(2m-1)!}, \quad P(1, m) = \frac{(m!)^2}{(2m-1)!} \]

* common payoffs game

\[ P(k, m) = \frac{(m!)^2}{((m-k)!)^2k!} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \nu(j, m-k) \frac{(2m-1-k-j)!}{(2m-1)!} \]

* two-role game

\[ P(k, m^2) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(m!)^2}{(k-2j)!2^j!} \sum_{i=0}^{\lfloor \frac{m-k}{2} \rfloor} (-1)^i \frac{1}{i!2^i} \sum_{l=0}^{m-k-2i} \frac{(-1)^l}{m^{2k+4i+2l}l!(m-k-2i-l)!^2} \]
How often does PNE exist?

The diagram shows the probability of a PNE existing for different values of $m$. The x-axis represents $m$, and the y-axis represents the probability of a PNE existing. The legend indicates the types of games considered:

- random game
- symmetric random game
- zero-sum game
- common random game
- two-role game
Asymptotic behavior of number of PNEs for large $m$

* random game: $P(k, m) \to \frac{e^{-1}}{k!}$

* symmetric random game: $P(k, m) \to e^{-1.5} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{j!2^j (k - 2j)!}$

* zero-sum game: $P(0, m) \to 1, \quad P(1, m) \to 0$

* common payoffs game: $P(k, m) \to \frac{m^k}{k!2^k e^{m/2}}$

* two-role game $P(k, m^2) \to e^{-1.5} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{j!2^j (k - 2j)!}$
Expected the number of PNES

- random game: $1$
- symmetric random game: $1 \frac{m - 1}{m}$
- zero-sum game: $\frac{(m!)^2}{(2m - 1)!}$
- common random game: $\frac{m^2}{2m - 1}$
- two-role game: $1 \frac{(m - 1)^2}{m^2}$
Expected the number of PNES

Expected Number of PNEs in a 2-person m-strategy game

- random game
- symmetric random game
- zero-sum game
- common random game
- two-role random game

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How often is a PNE PPO?

- random game
  \[ \Pi = \int_0^1 \int_0^1 m^2 x^{m-1} y^{m-1} (1 - (1 - x)(1 - y))(m-1)^2 \, dy \, dx \]

- symmetric random game
  \[ \Pi = \frac{m}{2m-1} (J_m + \frac{m-1}{m} K_m) \]
  \[ J_m = \int_0^1 m x^{2(m-1)} (1 - (1 - x)^2)^{(m-2)(m-1)/2} \, dx \]
  \[ K_m = 2 \int_0^1 \int_0^x m^2 x^{2m-3} y^{m-1} (x^2 + 2(1 - x)y)^{(m-2)(m-3)/2} \, dy \, dx \]

- zero-sum game
  \[ \Pi = 1 \]

- common random game
  \[ \Pi = \frac{2m-1}{m^2} \]
How often is a PNE PPO?

Probability a PNE is PPO in a two-person m-strategy random game

- random game
- symmetric random game
- zero-sum game
- common random game
How often is a PNE PPO in a two-person game?
How often is a PNE PPO in a two-person game?

* Probability that a PNE is PPO is independent of distribution.
How often is a PNE PPO in a two-person game?

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- As $m$, the number of strategies increases, cooperation becomes more favorable.
How often is a PNE PPO in a two-person game?

- Probability that a PNE is PPO is independent of distribution.
- As $m$, the number of strategies increases, cooperation becomes more favorable.
- As the correlation between payoffs increases, cooperation becomes more desirable.
n-person 2-strategy symmetric random games

- 2 strategies A and B, with payoff values
  \[ \vec{\alpha} = (\alpha_1, \cdots, \alpha_n), \quad \vec{\beta} = (\beta_0, \beta_1, \cdots, \beta_{n-1}) \]
  - \( i^* \) is PNE if
    \[ \alpha_i > \beta_{i-1}, \quad \beta_i > \alpha_{i+1} \]
- Probability distribution of \( k \), the number of PNEs.
  \[ P(k, n) = \frac{1}{2^n} \binom{n+1}{2k-1}, \quad E(X) = \frac{n+3}{4} \]
n-person 2-strategy symmetric random games

* PNE 0* or n* are PPO with probability

$$\sum_{i=0}^{n-2} (-1)^i \binom{n-2}{i} 2^{n-2-i} \frac{2}{n+1+i}$$

* PNE 1* or (n-1)* are PPO with probability

$$\sum_{i=0}^{n-3} (-1)^i \binom{n-3}{i} \sum_{j=0}^{n-3-i} \binom{n-3-i}{j} \frac{4}{(n-1-j)(2+i+j)} \left(1 - \frac{1}{n+2+i}\right)$$

* PNE 2*, 3*,,..., (n-2)* are PPO with probability

$$\sum_{i=0}^{n-4} (-1)^i \binom{n-4}{i} \sum_{j=0}^{n-4-i} \binom{n-4-i}{j} \frac{4}{(n-2-j)(2+i+j)} \left(1 - \frac{2}{n+1+i} + \frac{2}{(n+2+i)(n+1+i)}\right)$$
n-person 2-strategy symmetric random games

asymmetric case: \( \Pi(2,2) > \Pi(2,2,2) < \Pi(2,2,2,2) < \Pi(2,2,2,2,2) < \Pi(2,2,2,2,2) \)
To be continued...

- Expected gain from cooperation.
- Evolution of cooperation in repeated finite random games.