Control and Optimization
with Differential-Algebraic Constraints
Banff, October 24–29, 2010

Numerical Solution of Descriptor Riccati Equations for Linearized Navier-Stokes Control Problems

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supported by DFG SPP 1253

MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

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Linear Feedback Stabilization of Flow Problems

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Overview

Introduction

Optimal control-based stabilization for Navier-Stokes equations

- Stabilization to steady-state solution w of flows (with velocity field v and pressure χ), described by incompressible Navier-Stokes equations

$$\partial_t v + v \cdot \nabla v - \frac{1}{Re}\Delta v + \nabla \chi = f \tag{1a}$$

$$\operatorname{div} v = 0 \tag{1b}$$

on $Q_\infty := \Omega \times (0,\infty)$, $\Omega \subseteq \mathbb{R}^d$, $d = 2,3$, with smooth boundary $\Gamma := \partial\Omega$, and boundary and initial conditions

$$\begin{aligned} v &= g \quad \text{on } \Sigma_\infty := \Gamma \times (0,\infty), \\ v(0) &= w + z(0) \quad (w \text{ given velocity field}). \end{aligned}$$

- Existence of stabilizing linear state feedback control proved in 2D [Fernández-Cara et al 2004] and 3D [Fursikov 2004].
- Construction of stabilizing feedback control based on associated linear-quadratic optimal control problem:
  - for distributed control, see [Barbu 2003, Barbu/Sritharan 1998, Barbu/Triggiani 2004];
  - for boundary control, see [Barbu/Lasiecka/Triggiani 2006/07] (tangential) and [Raymond 2005–07, Bahdra 2009] (normal).

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Optimal control-based stabilization for NSEs

Analytical solution [Raymond''05–'07]

Assume w solves the stationary Navier-Stokes equations

$$w \cdot \nabla w - \frac{1}{Re}\Delta w + \nabla \chi_s = f, \qquad \operatorname{div} w = 0, \tag{2}$$

with Dirichlet boundary condition $w = g$ on Γ, w possibly unstable.

If we can determine a Dirichlet boundary control u so that the corresponding controlled system for $z := v - w$,

$$\begin{aligned}
\partial_t z + (z\cdot\nabla)w + (w\cdot\nabla)z + (z\cdot\nabla)z - \frac{1}{Re}\Delta z + \nabla p &= 0 \quad \text{in } Q_\infty,\\
\operatorname{div} z &= 0 \quad \text{in } Q_\infty,\\
z &= bu \quad \text{in } \Sigma_\infty,\\
z(0) &= z_0 \quad \text{in } \Omega,
\end{aligned}$$

is stable for "small" initial values $z_0 \in X(\Omega) \subset V_n^0(\Omega)$, where

$$V_n^0(\Omega) := L_2(\Omega) \cap \{\operatorname{div} z = 0\} \cap \{z \cdot n = 0 \text{ on } \Gamma\},$$

then $\exists$ constants $c, \omega > 0$ so that $\|z(t)\|_{X(\Omega)} \le ce^{-\omega t}$.

$\Longrightarrow \left\{ \begin{array}{l} \text{Solution to instationary Navier-Stokes equations with } v = w + z, \\ \chi = \chi_s + p, \text{ and } v(0) = w + z_0 \text{ in } \Omega \text{ is controlled to } w. \end{array}\right.$

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Linearized Navier-Stokes control system:

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ωz with $\omega > 0$ de-stabilizes the system further, needed to guarantee exponential stabilization, ω controls decay rate!

Cost functional (with P = Helmholtz projector)

$$J(z, u) = \frac{1}{2} \int_0^\infty \langle Pz, Pz \rangle_{L_2(\Omega)} + \rho u(t)^2 \, dt, \qquad (4)$$

the linear-quadratic optimal control problem associated to (3) becomes

$$\inf \{ J(z, u) \mid (z, u) \text{ satisfies (3)}, \; u \in L_2(0, \infty) \} . \qquad (5)$$

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Proposition [Raymond '05, Bahdra '09]

The solution to the instationary Navier-Stokes equations with perturbed initial data is exponentially controlled to the steady-state solution w by the feedback law

$$u = -\rho^{-1} B^* \Pi z_H,$$

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- $z_H := Pz$, with $P : L_2(\Omega) \mapsto V_n^0(\Omega)$ being the Helmholtz projector ($\rightsquigarrow \operatorname{div} z_H \equiv 0$);
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$$0 = I + (A + \omega I)^* \Pi + \Pi (A + \omega I) - \Pi (B_\tau B_\tau^* + \rho^{-1} B_n B_n^*) \Pi,$$

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Long-Term Plan

Apply optimal control-based feedback stabilization to (multi-)field problems with increasing complexity:

- **Proof of concept:** Navier-Stokes with **normal** boundary control for model problem (von Kármán vortex shedding).
- Navier-Stokes coupled with (passive) transport of (reactive) species.
- Phase transition liquid/solid with convection.
- Stabilization of a flow with a free capillary surface.
- Control for electrically conducting fluids in presence of outer magnetic fields (MHD).

All scenarios require

- formulation as abstract parabolic Cauchy problem,
- definition of quadratic cost functional,
- formulation of corresponding ARE,
- spatial discretization (FEM),
- numerical solution of large-scale ARE.

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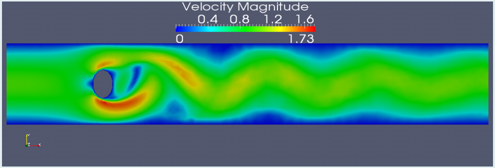
Introduction

Proof of concept: von Kármán vortex street

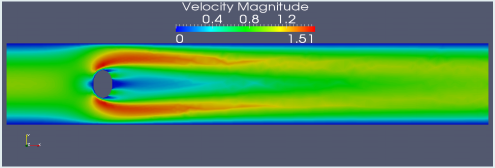
von Kármán vortex street ($Re = 300$)

Vortex suppression by blowing in at upper end of cylinder,

without ...



and with stabilizing feedback



Computations by Heiko Weichelt

Solving Large-Scale AREs

Low-Rank Newton-ADI for AREs

Consider

$$0 = \mathcal{R}(X) := C^T C + A^T X + XA - XBB^T X$$

Re-write Newton's method for AREs ($A_j := A - BB^T X_j$)

$$D\mathcal{R}(X_j)\,(N_j) = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j BB^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl '99/'08]

Solve Lyapunov equations for Z_{j+1} directly by factored ADI iteration and exploit 'sparse + low-rank' structure of A_j.

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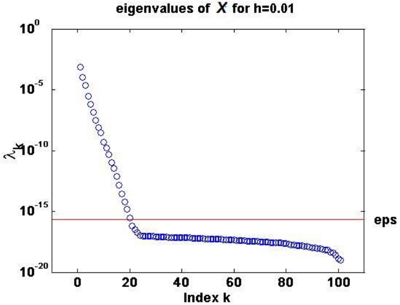
Low-Rank Approximation

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \Longrightarrow n = 101$.



Idea: $X = X^T \geq 0 \Longrightarrow$

$$X = ZZ^T = \sum_{k=1}^{n} \lambda_k z_k z_k^T \approx Z^{(r)}(Z^{(r)})^T = \sum_{k=1}^{r} \lambda_k z_k z_k^T.$$

$\Longrightarrow$ Goal: compute $Z^{(r)} \in \mathbb{R}^{n \times r}$ directly w/o ever forming X!

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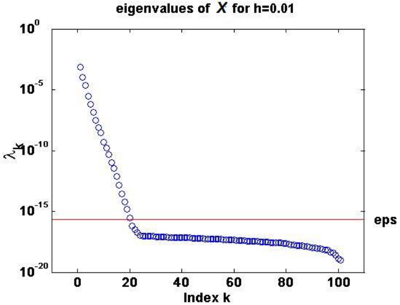

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Review: LRCF-ADI for Lyapunov Equations

Consider $$FX + XF^T = -GG^T$$

ADI iteration for the Lyapunov equation (LE) [Wachspress '95]

For $j = 1, \ldots, J$

$$\begin{aligned} X_0 &= 0 \\ (F + p_j I)X_{j-\frac{1}{2}} &= -GG^T - X_{j-1}(F^T - p_j I) \\ (F + p_j I)X_j^T &= -GG^T - X_{j-\frac{1}{2}}^T(F^T - p_j I) \end{aligned}$$

Rewrite as one step iteration and factorize $X_i = Z_i Z_i^T$, $i = 0, \ldots, J$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_j Z_j^T &= -2p_j (F + p_j I)^{-1} G G^T (F + p_j I)^{-T} \\ &\quad + (F + p_j I)^{-1}(F - p_j I) Z_{j-1} Z_{j-1}^T (F - p_j I)^T (F + p_j I)^{-T} \end{aligned}$$

$\ldots \rightsquigarrow$ low-rank Cholesky factor ADI

[Penzl '97/'00, Li/White '99/'02, B./Li/Penzl '99/'08, Gugercin/Sorensen/Antoulas '03]

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Review: LRCF-ADI for Lyapunov Equations

The Work Horse

Algorithm 1 Low-rank Cholesky factor ADI iteration (LRCF-ADI)

[Penzl '97/'00, Li/White '99/'02, B./Li/Penzl '99/'08]

Input: F, G defining $FX + XF^T = -GG^T$ and shifts $\{p_1, \ldots, p_{i_{max}}\}$
Output: $Z = Z_{i_{max}} \in \mathbb{C}^{n \times t_{i_{max}}}$, such that $ZZ^H \approx X$

1: Solve $(F + p_1 I)\, V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}G$ for V_1.
2: $Z_1 = V_1$
3: **for** $i = 2, 3, \ldots, i_{max}$ **do**
4: Solve $(F + p_i I)\tilde{V} = V_{i-1}$ for $\tilde{V}$.
5: $V_i = \sqrt{\mathrm{Re}\,(p_i)/\,\mathrm{Re}\,(p_{i-1})}\left(V_{i-1} - (p_i + \overline{p_{i-1}})\tilde{V}\right)$
6: $Z_i = [Z_{i-1} \;\; V_i]$
7: **end for**

Krylov Subspace Based Solvers for Lyapunov Equations

Consider Schur/singular value decomposition $X = U\Sigma U^T$,
$U \in \mathbb{R}^{n\times n}$, $U^T U = I$, $\Sigma = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_n)$ and $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq |\sigma_n|$.
The best rank-m Frobenius-norm approximation to X is thus given by

$$X_m := U \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} U^T = U_m \Sigma_m U_m^T.$$

Krylov projection idea [Saad '90, Jaimoukha/Kasenally '94]

Solve

$$(U_m^T F U_m) Y_m + Y_m (U_m^T F^T U_m) = -U_m^T G G^T U_m, \tag{6}$$

on $\mathrm{colspan}(U_m)$ and get X_m as

$$X_m = U_m Y_m U_m^T.$$

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The best rank-m Frobenius-norm approximation to X is thus given by

$$X_m := U \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix} U^T = U_m \Sigma_m U_m^T.$$

Note that a factorization

$$Z_m Z_m^T = X_m$$

can easily be computed from a Cholesky factorization of

$$Y_m = \tilde{Z}_m \tilde{Z}_m^T$$

as

$$Z_m = U_m \tilde{Z}_m.$$

Krylov Subspace Based Solvers for Lyapunov Equations

Basic Algorithm

Algorithm 2 Basic Krylov Subspace Method for the Lyapunov Equation

Input: F,G defining $FX + XF^T = -GG^T$, an initial Krylov subspace $\mathcal{V}$, e.g., $\mathcal{V} = \mathcal{K}_p(F, G)$ or $\mathcal{V} = \mathcal{K}_p(F, G) \cup \mathcal{K}_p(F^{-1}, G)$[1] with orthogonal basis $V \in \mathbb{C}^{n \times p}$.

Output: $Z \in \mathbb{C}^{n \times t}$, such that $ZZ^H \approx X$

repeat
 if not first step **then**
  increase dimension of $\mathcal{V}$ and update V.
 end if
 Solve the "small" LE for $\tilde{Z}$ with a classical solver:

$$(V^T FV)\tilde{Z}\tilde{Z}^T + \tilde{Z}\tilde{Z}^T(V^T F^T V) = -V^T GG^T V,$$

 Lift $\tilde{Z}$ to the full space: $Z = V\tilde{Z}$
until res(Z)$<$ TOL

[1](K-PIK, [Simoncini '07])

LRCF-ADI with Galerkin Projection

ADI and Rational Krylov

[Li '00; Theorem 2] interprets the column span of the ADI solution as a certain rational Krylov subspace

$$\mathcal{L}(F,G,\mathbf{p}) := span\left\{ \ldots, \prod_{i=-j}^{-1}(F+p_iI)^{-1}G,\ \ldots, (F+p_{-2}I)^{-1}(F+p_{-1}I)^{-1}G, \right.$$
$$(F+p_{-1}I)^{-1}G,\ G,\ (F+p_1I)G,$$
$$\left. (F+p_2I)(F+p_1I)G,\ \ldots,\ \prod_{i=1}^{j}(F+p_iI)G\ \ldots \right\}$$

Idea

Solve on current subspace of $\mathcal{L}(F,G,\mathbf{p})$ in the ADI step to increase the quality of the iterate.

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$$(F+p_{-1}I)^{-1}G,\ G,\ (F+p_1I)G,$$
$$\left. (F+p_2I)(F+p_1I)G, \ldots, \prod_{i=1}^{j}(F+p_iI)G \ \ldots \right\}$$

Idea

Solve on current subspace of $\mathcal{L}(F,G,\mathbf{p})$ in the ADI step to increase the quality of the iterate.

LRCF-ADI with Galerkin Projection

Projected ADI Step $\longrightarrow$ LRCF-ADI-GP

[B./Li/Truhar'09, Saak'09, B./Saak'10]

1. Compute the LRCF-ADI iterate Z_i
2. Compute orthogonal basis via RRQR factorization[a]: $Q_i R_i \Pi_i = Z_i$
3. Solve (for $\tilde{Z}$) the projected Lyapunov equation

$$(Q_i^T F Q_i)\tilde{Z}\tilde{Z}^T + \tilde{Z}\tilde{Z}^T(Q_i^T F^T Q_i) = -Q_i^T G G^T Q_i$$

4. Update Z_i according to $Z_i := Q_i \tilde{Z}$

[a] economy size QR with column pivoting; crucial to compute correct subspace if Z_i rank deficient.

- Need to ensure that projected systems remain stable, e.g., $F + F^T < 0$;
- may perform projected ADI step only every k-th step (e.g. $k = 5$) $\rightsquigarrow$ restarted ADI with shifts $\Lambda(Q_i^T F Q_i)$.

LRCF-ADI with Galerkin Projection

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LRCF-ADI with Galerkin Projection

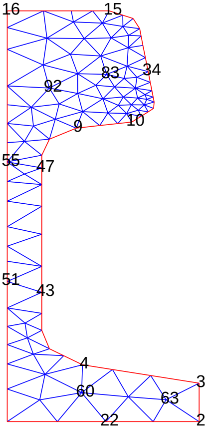
Test Example: Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa(u_k - x), && \xi \in \Gamma_k,\ 1 \leq k \leq 7, \\
\frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_0.
\end{aligned}
$$

$\Longrightarrow q = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$),
  1, 2, 3, 4 steps of mesh refinement $\Rightarrow$
  $n = 1\,357, 5\,177, 20\,209, 79\,841.$

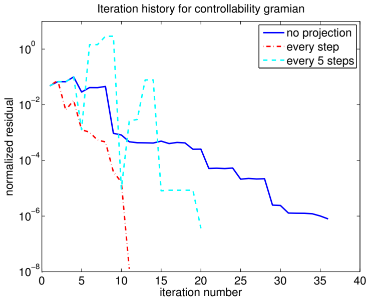


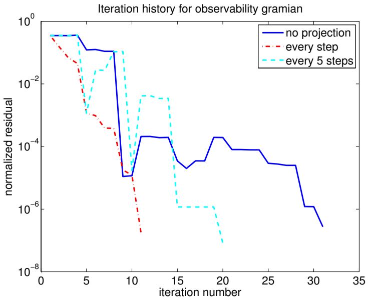
Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: Tröltzsch/Unger '99/'01, Penzl '99, S. '03.

LRCF-ADI with Galerkin Projection

Numerical Results

Steel profile n=20 209 good shifts





LRCF-ADI with Galerkin Projection

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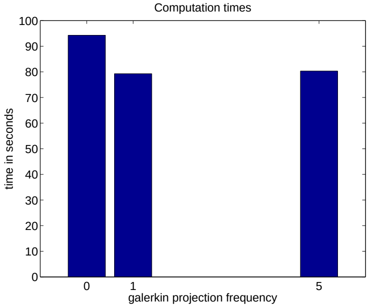

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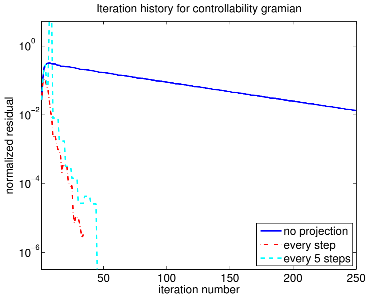

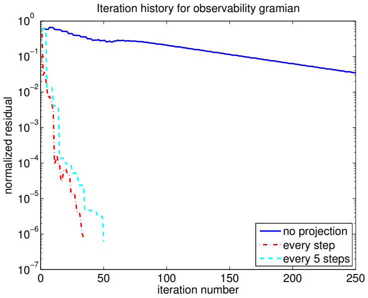

LRCF-ADI with Galerkin Projection

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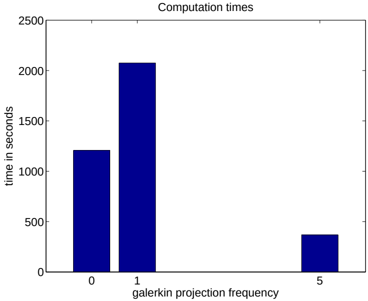

Solving Large-Scale AREs

LRCF-NM for the ARE

Consider $$\mathfrak{R}(X) := C^T C + A^T X + XA - XBB^T X = 0$$

Newton's Iteration for the ARE

$$\mathfrak{R}'|_X(N_\ell) = -\mathfrak{R}(X_\ell), \qquad X_{\ell+1} = X_\ell + N_\ell, \qquad \ell = 0, 1, \ldots$$

where the Frechét derivative of $\mathfrak{R}$ at X is the Lyapunov operator

$$\mathfrak{R}'|_X : \quad Q \mapsto (A - BB^T X)^T Q + Q(A - BB^T X),$$

i.e., in every Newton step solve a

Lyapunov Equation [Kleinman '68]

$$F_\ell^T X_{\ell+1} + X_{\ell+1} F_\ell = -G_\ell G_\ell^T,$$

where $F_\ell := A - BB^T X_\ell$, $G := [\,-C^T,\ -X_\ell B\,]$.

Solving Large-Scale AREs

LRCF-NM for the ARE

Factored Newton-Kleinman Iteration [B./Li/Penzl '99/'08]

$$F_\ell = A - BB^T X_\ell =: A - BK_\ell \qquad \text{is "sparse + low rank",}$$
$$G_\ell = [\, C^T,\ K_\ell^T \,] \qquad \text{is low rank factor.}$$

- Apply LRCF-ADI in every Newton step;
- exploit structure of F_ℓ using Sherman-Morrison-Woodbury formula:

$$(A - BK_\ell + p_k^{(\ell)} I_n)^{-1} =$$
$$(I_n + (A + p_k^{(\ell)} I_n)^{-1} B (I_m - K_\ell (A + p_k^{(\ell)} I_n)^{-1} B)^{-1} K_\ell)(A + p_k^{(\ell)} I_n)^{-1}$$

Solving Large-Scale AREs

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LRCF-NM for the ARE

Algorithms

Algorithm 3 Low-Rank Cholesky Factor Newton Method (LRCF-NM)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^T C + A^T X + XA - XBB^T X = 0.$$

1: **for** $k = 1, 2, \ldots, k_{max}$ **do**
2: Determine (sub)optimal ADI shift parameters $p_1^{(k)}, p_2^{(k)}, \ldots$ with respect to the matrix $F^{(k)} = A^T - K^{(k-1)} B^T$.
3: $G^{(k)} = \left[\begin{array}{cc} C^T & K^{(k-1)} \end{array} \right]$
4: Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) or (LRCF-ADI-GP) such that $F^{(k)} Z^{(k)} Z^{(k)^H} + Z^{(k)} Z^{(k)^H} F^{(k)^T} \approx -G^{(k)} G^{(k)^T}$.
5: $K^{(k)} = Z^{(k)} (Z^{(k)^H} B)$
6: **end for**

LRCF-NM for the ARE

Algorithms

Algorithm 4 Simplified Low-Rank Cholesky Factor Newton Method (LRCF-NM-S)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable

Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^T C + A^T X + XA - XBB^T X = 0.$$

1: Determine (sub)optimal ADI shift parameters $p_1, p_2, \dots$ with respect to $F^{(0)} = A^T - K^{(0)} B^T$ or $F^{(\infty)} = \lim_{k\to\infty} F^{(k)}$.

2: **for** $k = 1, 2, \dots, k_{max}$ **do**

3: $\quad G^{(k)} = \left[\begin{array}{cc} C^T & K^{(k-1)} \end{array}\right]$

4: $\quad$ Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) or (LRCF-ADI-GP) such that $\quad F^{(k)} Z^{(k)} Z^{(k)^H} + Z^{(k)} Z^{(k)^H} F^{(k)^T} \approx -G^{(k)} G^{(k)^T}$.

5: $\quad K^{(k)} = Z^{(k)} (Z^{(k)^H} B)$

6: **end for**

LRCF-NM for the ARE

Algorithms

Algorithm 5 Low-Rank Cholesky Factor Galerkin-Newton Method (LRCF-NM-GP)

Input: A, B, C, $K^{(0)}$ for which $A - BK^{(0)^T}$ is stable
Output: $Z = Z^{(k_{max})}$, such that ZZ^H approximates the solution X of

$$C^T C + A^T X + XA - XBB^T X = 0.$$

1: **for** $k = 1, 2, \ldots, k_{max}$ **do**
2: Determine (sub)optimal ADI shift parameters $p_1^{(k)}, p_2^{(k)}, \ldots$ with respect to the matrix $F^{(k)} = A^T - K^{(k-1)} B^T$.
3: $G^{(k)} = \begin{bmatrix} C^T & K^{(k-1)} \end{bmatrix}$
4: Compute $Z^{(k)}$ using Algorithm 1 (LRCF-ADI) or (LRCF-ADI-GP) such that $F^{(k)} Z^{(k)} Z^{(k)^H} + Z^{(k)} Z^{(k)^H} F^{(k)^T} \approx -G^{(k)} G^{(k)^T}$.
5: Project ARE, solve and prolongate solution.
6: $K^{(k)} = Z^{(k)} (Z^{(k)^H} B)$
7: **end for**

Solving for the Feedback Operator

Feedback Iteration

$Z_{j,k} = [Z_{j,k-1}, V_{j,k}]$,

j: Newton index,

k: ADI index.

Optimal feedback

$$K_* = B^T X_* = B^T Z_* Z_*^T$$

can be computed by direct feedback iteration:

- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \xrightarrow{j\rightarrow\infty} K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, $A_j = BK_j$
  $\Rightarrow$ no need to form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m\times n}$!

Related to earlier work by [Banks/Ito '91].

Feedback Iteration

Test Examples

Example 1: 3d Convection-Diffusion Equation

- FDM for 3D convection-diffusion equation on $[0, 1]^3$
- proposed in [SIMONCINI '07], $q = p = 1$
- non-symmetric $A \in \mathbb{R}^{n \times n}$, $n = 10\,648$

Example 2: 2d Convection-Diffusion Equation

- FDM for 2D convection-diffusion equations on $[0, 1]^2$
- LyaPack benchmark, $q = p = 1$, e.g., `demo_l1`
- non-symmetric $A \in \mathbb{R}^{n \times n}$, $n = 22\,500$.

- 16 shift parameters
- Penzl's heuristic from 50/25 Ritz/harmonic Ritz values of A

Feedback Iteration

Test Results (ADI-loop): Example 1

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	$9.97 \cdot 10^{-01}$	$9.27 \cdot 10^{-01}$	100
2	$3.67 \cdot 10^{-02}$	$9.58 \cdot 10^{-02}$	94
3	$1.36 \cdot 10^{-02}$	$1.09 \cdot 10^{-03}$	98
4	$3.48 \cdot 10^{-04}$	$1.01 \cdot 10^{-07}$	97
5	$6.41 \cdot 10^{-08}$	$1.34 \cdot 10^{-10}$	97
6	$\mathbf{7.47 \cdot 10^{-16}}$	$1.34 \cdot 10^{-10}$	97

CPU time: 4 805.8 sec.

Newton-Galerkin-ADI LRCF-ADI-GP(5)

NWT	rel. change	rel. residual	ADI
1	$9.97 \cdot 10^{-01}$	$9.29 \cdot 10^{-01}$	80
2	$3.67 \cdot 10^{-02}$	$9.60 \cdot 10^{-02}$	30
3	$1.36 \cdot 10^{-02}$	$1.09 \cdot 10^{-03}$	28
4	$3.47 \cdot 10^{-04}$	$1.01 \cdot 10^{-07}$	35
5	$6.41 \cdot 10^{-08}$	$1.03 \cdot 10^{-10}$	25
6	$1.23 \cdot 10^{-11}$	$\mathbf{1.98 \cdot 10^{-11}}$	27

CPU time: 1 460.1 sec.

test system: Intel® Xeon® 5160 3.00GHz ; 16 GB RAM;
64Bit-MATLAB ® (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

Feedback Iteration

Test Results (ADI-loop): Example 2

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	1	$1.70 \cdot 10^{+02}$	46
2	$2.88 \cdot 10^{-01}$	$4.25 \cdot 10^{+01}$	39
3	$2.13 \cdot 10^{-01}$	$1.06 \cdot 10^{+01}$	43
4	$1.77 \cdot 10^{-01}$	$2.58 \cdot 10^{+00}$	46
5	$2.47 \cdot 10^{-01}$	$5.15 \cdot 10^{-01}$	43
6	$3.04 \cdot 10^{-01}$	$3.26 \cdot 10^{-02}$	52
7	$1.78 \cdot 10^{-02}$	$6.90 \cdot 10^{-05}$	50
8	$2.60 \cdot 10^{-05}$	$1.08 \cdot 10^{-10}$	46
9	$\mathbf{2.75 \cdot 10^{-11}}$	$1.07 \cdot 10^{-10}$	50

CPU time: 493.81 sec.

Newton-Galerkin-ADI LRCF-ADI-GP(5)

NWT	rel. change	rel. residual	ADI
1	1	$1.70 \cdot 10^{+02}$	35
2	$2.88 \cdot 10^{-01}$	$4.25 \cdot 10^{+01}$	15
3	$2.13 \cdot 10^{-01}$	$1.06 \cdot 10^{+01}$	20
4	$1.77 \cdot 10^{-01}$	$2.58 \cdot 10^{+00}$	20
5	$2.47 \cdot 10^{-01}$	$5.15 \cdot 10^{-01}$	20
6	$3.04 \cdot 10^{-01}$	$3.26 \cdot 10^{-02}$	17
7	$1.78 \cdot 10^{-02}$	$6.90 \cdot 10^{-05}$	20
8	$2.60 \cdot 10^{-05}$	$1.10 \cdot 10^{-10}$	20
9	$2.75 \cdot 10^{-11}$	$\mathbf{1.92 \cdot 10^{-12}}$	20

CPU time: 280.55 sec.

test system: Intel® Core™2 Quad Q9400 2.66 GHz; 4 GB RAM;
64Bit-MATLAB (R2009a) using threaded BLAS (`reynolds`)
stopping criterion tolerances: 10^{-10}

Feedback Iteration

Test Results (both-loops): Example 1

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	$9.97 \cdot 10^{-01}$	$9.27 \cdot 10^{-01}$	100
2	$3.67 \cdot 10^{-02}$	$9.58 \cdot 10^{-02}$	94
3	$1.36 \cdot 10^{-02}$	$1.09 \cdot 10^{-03}$	98
4	$3.48 \cdot 10^{-04}$	$1.01 \cdot 10^{-07}$	97
5	$6.41 \cdot 10^{-08}$	$1.34 \cdot 10^{-10}$	97
6	$7.47 \cdot 10^{-16}$	$1.34 \cdot 10^{-10}$	97

CPU time: 4 805.8 sec.

NG-ADI inner= 5, outer= 1

NWT	rel. change	rel. residual	ADI
1	$9.98 \cdot 10^{-01}$	$5.04 \cdot 10^{-11}$	80

CPU time: 497.6 sec.

NG-ADI inner= 1, outer= 1

NWT	rel. change	rel. residual	ADI
1	$9.98 \cdot 10^{-01}$	$7.42 \cdot 10^{-11}$	71

CPU time: 856.6 sec.

NG-ADI inner= 0, outer= 1

NWT	rel. change	rel. residual	ADI
1	$9.98 \cdot 10^{-01}$	$6.46 \cdot 10^{-13}$	100

CPU time: 506.6 sec.

test system: Intel® Xeon® 5160 3.00GHz ; 16 GB RAM;
64Bit-MATLAB (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

Feedback Iteration

Test Results (both-loops): Example 2

Newton-ADI

NWT	rel. change	rel. residual	ADI
1	1	$1.70 \cdot 10^{+02}$	46
2	$2.88 \cdot 10^{-01}$	$4.25 \cdot 10^{+01}$	39
3	$2.13 \cdot 10^{-01}$	$1.06 \cdot 10^{+01}$	43
4	$1.77 \cdot 10^{-01}$	$2.58 \cdot 10^{+00}$	46
5	$2.47 \cdot 10^{-01}$	$5.15 \cdot 10^{-01}$	43
6	$3.04 \cdot 10^{-01}$	$3.26 \cdot 10^{-02}$	52
7	$1.78 \cdot 10^{-02}$	$6.90 \cdot 10^{-05}$	50
8	$2.60 \cdot 10^{-05}$	$1.08 \cdot 10^{-10}$	46
9	$\mathbf{2.75 \cdot 10^{-11}}$	$1.07 \cdot 10^{-10}$	50

CPU time: 493.81 sec.

NG-ADI inner= 5, outer= 1

NWT	rel. change	rel. residual	ADI
1	1	$3.30 \cdot 10^{-11}$	35

CPU time: 24.1 sec.

NG-ADI inner= 1, outer= 1

NWT	rel. change	rel. residual	ADI
1	1	$1.31 \cdot 10^{-11}$	34

CPU time: 26.8 sec.

NG-ADI inner= 0, outer= 1

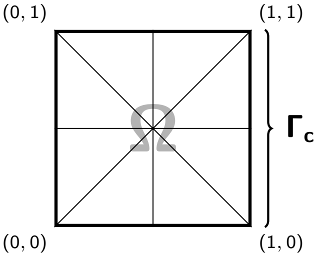
NWT	rel. change	rel. residual	ADI
1	1	$3.27 \cdot 10^{-15}$	46

CPU time: 24.0 sec.

test system: Intel® Core™2 Quad Q9400 2.66 GHz; 4 GB RAM;
64Bit-MATLAB (R2009a) using threaded BLAS (`reynolds`)
stopping criterion tolerances: 10^{-10}

Feedback Iteration

Computation Time Scales Linearly with Problem Size



$$\begin{aligned} \partial_t x(\xi,t) &= \Delta x(\xi,t) && \text{in } \Omega \\ \partial_\nu x &= b(\xi)\cdot u(t) - x && \text{on } \Gamma_c \\ \partial_\nu x &= -x && \text{on } \partial\Omega \setminus \Gamma_c \\ x(\xi,0) &= 1 \end{aligned}$$

Control operator: Here $b(\xi) = 4\,(1-\xi_2)\,\xi_2$ for $\xi \in \Gamma_c$ and 0 otherwise.
Output equation: $y = Cx$, where

$$\begin{aligned} C : \mathcal{L}^2(\Omega) &\to \mathbb{R} \\ x(\xi,t) &\mapsto y(t) = \int_\Omega x(\xi,t)\,d\xi, \end{aligned} \quad \Rightarrow C_h = \underline{1}\cdot M_h.$$

Cost functional:

$$\mathcal{J}(u) = \int_0^\infty y^2(t) + u^2(t)\,dt.$$

Feedback Iteration

Scaling results

simplified Low Rank Newton-Galerkin ADI

- generalized state space form implementation
- Penzl shifts (16/50/25) with respect to initial matrices
- projection acceleration in every outer iteration step
- projection acceleration in every 5-th inner iteration step

test system: Intel®Xeon® 5160 @ 3.00 GHz; 16 GB RAM;
64Bit-MATLAB (R2010a) using threaded BLAS (`romulus`)
stopping criterion tolerances: 10^{-10}

Feedback Iteration

Scaling results

Computation Times

discretization level	problem size	time in seconds
3	81	$5.53 \cdot 10^{-2}$
4	289	$1.33 \cdot 10^{-1}$
5	1 089	$2.84 \cdot 10^{-1}$
6	4 225	$1.51 \cdot 10^{+0}$
7	16 641	$9.52 \cdot 10^{+0}$
8	66 049	$5.97 \cdot 10^{+1}$
9	263 169	$4.72 \cdot 10^{+2}$
10	1 050 625	$6.89 \cdot 10^{+3}$
11	4 198 401	$8.08 \cdot 10^{+4}$

(Finest level: 8.813.287.577.601 unknowns, taking symmetry into account.)

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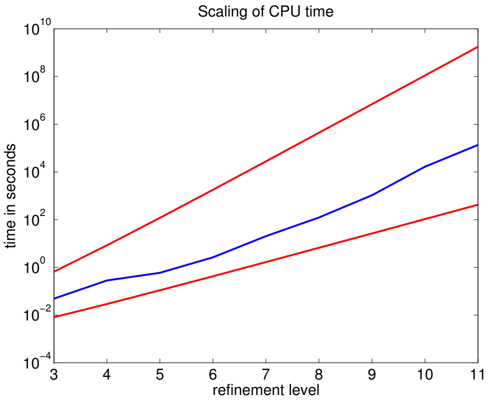
Feedback Iteration

Scaling results



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Solving AREs for Linearized Navier-Stokes Eqns.

$0 = M + (A + \omega M)^T X + X(A + \omega M) - MXBB^T XM$

Problems with Newton-Kleinman

1. Discretization of Helmholtz-projected linearized Navier-Stokes equations would need divergence-free finite elements.

   Here, we want to use standard discretization

   (Taylor-Hood elements available in flow solver NAVIER).

   Explicit projection of ansatz functions possible using application of Helmholtz projection, but too expensive in general.

2. Each step of Newton-Kleinman iteration: solve

   $$A_j^T Z_{j+1} Z_{j+1}^T M + M Z_{j+1} Z_{j+1}^T A_j = -M - K_j^T K_j$$

   $n_v := \mathrm{rank}\,(M) =$ dim of ansatz space for velocities.

   $\rightsquigarrow$ need to solve $n_v + m$ linear systems of equations in each step of Newton-ADI iteration!

3. Linearized system (i.e., $A + \omega M$) is unstable in general.

   But to start Newton iteration, a stabilizing initial guess is needed! √

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   $$\boxed{0 = \mathbf{I} + (\mathbf{A} + \omega \mathbf{I})^* \mathbf{X} + \mathbf{X}(\mathbf{A} + \omega \mathbf{I}) - \mathbf{X}(\mathbf{B}_\tau \mathbf{B}_\tau^* + \rho^{-1} \mathbf{B}_n \mathbf{B}_n^*)\mathbf{X}}$$

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   n_v

   $\rightsquigarrow$

   Newton-ADI iteration!

   [B. '08]*Partial Stabilization of Descriptor Systems Using Spectral Projectors*; to appear in V. Olshevsky et al (eds.), Numerical Linear Algebra in Signals, Systems and Control, Lecture Notes in Electrical Engineering, Springer-Verlag.

   [HEIN '10]*MPC/LQG-Based Optimal Control of Nonlinear Parabolic PDEs*; PhD thesis Chemnitz UT.

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Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

- incompressible Navier-Stokes-Equations

$$\frac{\partial \mathbf{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \mathbf{v} + \mathbf{v}.\nabla \mathbf{v} + \nabla \mathbf{p} = 0 \quad + \text{B.C.}$$
$$\nabla.\mathbf{v} = 0 \qquad \text{(NSE)}$$

- Spatial FE discretization

$$M\dot{v}(t) = K(v)v(t) - Gp(t) + B_1\mathbf{u}(t)$$
$$0 = G^T v(t) \qquad \text{(dNSE)}$$

- Linearization and change of notation

$$E_{11}\dot{v}(t) = A_{11}v(t) + A_{12}p(t) + B_1\mathbf{u}(t)$$
$$0 = A_{12}^T v(t) \qquad \text{(NSDAE)}$$

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Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

$$E_{11}\dot{v}(t) = A_{11}v(t) + A_{12}p(t) + B_1\mathbf{u}(t)$$
$$0 = A_{12}^T v(t)$$

Multiplication of line one from the left by $A_{12}^T E_{11}^{-1}$ together with

$0 = A_{12}^T v(t) \Rightarrow 0 = A_{12}^T \dot{v}(t)$ reveals the

hidden manifold

$$0 = A_{12}^T E_{11}^{-1} A_{11} v(t) + A_{12}^T E_{11}^{-1} A_{12} p(t) + A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t),$$

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which implies

$$p(t) = -\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} A_{11} v(t) - \left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t).$$

Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 1. Problem/no need for divergence free FE

Inserting p we find

$$\begin{aligned} E_{11}\dot{v}(t) &= \left(I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}\right) A_{11} v(t) \\ &\quad + \left(I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}\right) B_1 \mathbf{u}(t) \end{aligned}$$

Definition [Heinkenschloss/Sorensen/Sun '08]

$$\Pi := I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}$$

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Solving AREs for Linearized Navier-Stokes Eqns.

Derivation of the Projected State Space System and Matrix Equations

Definition [Heinkenschloss/Sorensen/Sun '08]

$$\Pi := I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}$$

Properties

- $\Pi^2 = \Pi$
- $\Pi E_{11} = E_{11}\Pi^T$
- $null(\Pi) = range(A_{12})$
- $range(\Pi) = null(A_{12}^T E_{11}^{-1})$

This implies

Lemma 1 [Heinkenschloss/Sorensen/Sun '08]

- Π is an oblique projector.
- $A_{12}^T z = 0 \Leftrightarrow \Pi^T z = z$
- $\Rightarrow \Pi^T v(t) = v(t)$

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Solving AREs for Linearized Navier-Stokes Eqns.

Derivation of the Projected State Space System and Matrix Equations

Thus (NSDAE) is equivalent to

Projected state space system

$$\Pi E_{11} \Pi^T \frac{d}{dt} v(t) = \Pi A_{11} \Pi^T v(t) + \Pi B_1 \mathbf{u}(t).$$

Leads to

projected Riccati equation

$$\begin{aligned} \Pi\Pi^T + \Pi A_{11}^T \Pi^T X \Pi E_{11} \Pi^T + \Pi E_{11}^T \Pi^T X \Pi A_{11} \Pi^T & \\ -\Pi E_{11}^T \Pi^T X \Pi B_1 B_1^T \Pi^T X \Pi E_{11} \Pi^T &= 0 \\ \Pi^T X \Pi &= X. \end{aligned}$$

If necessary p can be determined from

$$p(t) = -\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} A_{11} v(t) - \left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1} B_1 \mathbf{u}(t).$$

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Solving AREs for Linearized Navier-Stokes Eqns.

Solving the Projected Matrix Equations

Apply factored-Newton-ADI

Central question

How do we solve systems of equations $(A_\ell := A_{11} - BK_\ell)$

$$Z = \Pi^T Z, \qquad \Pi \left(E_{11} + p_\ell A_\ell\right) \Pi^T Z = \Pi \tilde{G}$$

in the (inner) ADI steps avoiding the computation of Π?

For $A_\ell = A_{11}$, i.e., $K_\ell = 0$:

Lemma [Heinkenschloss/Sorensen/Sun '08]

$$\begin{aligned} Z &= \Pi^T Z \\ \Pi \left(E_{11} + p_\ell A_{11}\right) \Pi^T Z &= \Pi \tilde{G} \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} E_{11} + p_\ell & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix}$$

Solving AREs for Linearized Navier-Stokes Eqns.

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For $A_\ell = A_{11}$, i.e., $K_\ell = 0$

tasks

- exploit "sparse + low rank" structure of A_ℓ,
- precondition our saddle point problem.
(joint work with A. Wathen/M. Stoll)

Lemma [Heinkenschloss/Sorensen/Sun '08]

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Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 2. Problem: remove W from r.h.s. of Lyapunov eqns. in Newton-ADI

One step of Newton-Kleinman iteration for ARE:

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1}A_j = -W - \underbrace{(X_j B)}_{=K_j^T} \underbrace{B^T X_j}_{=K_j} \qquad \text{for } j = 1, 2, \ldots$$

Subtract two consecutive equations $\Longrightarrow$

$$A_j^T N_j + N_j A_j = N_{j-1}^T B B^T N_{j-1} \qquad \text{for } j = 1, 2, \ldots$$

See [BANKS/ITO '91, B./HERNÁNDEZ/PASTOR '03, MORRIS/NAVASCA '05] for details and applications of this variant.

But: need $B^T N_0 = K_1 - K_0$!

Assuming K_0 is known, need to compute K_1.

Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 2. Problem: remove W from r.h.s. of Lyapunov eqns. in Newton-ADI

Solution idea:

$$\begin{aligned} K_1 &= B^T X_1 \\ &= B^T \int_0^\infty e^{(A-BK_0)^T t}(W + K_0^T K_0)e^{(A-BK_0)t}\,dt \\ &= \int_0^\infty g(t)\,dt \approx \sum_{\ell=0}^{N} \gamma_\ell g(t_\ell), \end{aligned}$$

where $g(t) = \left(\left(e^{(A-BK_0)t}B\right)^T (W + K_0^T K_0)\right) e^{(A-BK_0)t}$.

[Borgggaard/Stoyanov '08]:
evaluate $g(t_\ell)$ using ODE solver applied to $\dot{x} = (A - BK_0)x$ + adjoint eqn.

Solving AREs for Linearized Navier-Stokes Eqns.

Solution to 2. Problem: remove W from r.h.s. of Lyapunov eqns. in Newton-ADI

Better solution idea:

(related to frequency domain POD [Willcox/Peraire '02])

$$\begin{aligned} K_1 &= B^T X_1 \qquad\qquad (\text{Notation: } A_0 := A - BK_0) \\ &= B^T \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} (\jmath\omega I_n - A_0)^{-H} (W + K_0^T K_0)(\jmath\omega I_n - A_0)^{-1}\, d\omega \\ &= \int_{-\infty}^{\infty} f(\omega)\, d\omega \approx \sum_{\ell=0}^{N} \gamma_\ell f(\omega_\ell), \end{aligned}$$

where $f(\omega) = \left(-\left((\jmath\omega I_n + A_0)^{-1} B\right)^T (W + K_0^T K_0)\right) (\jmath\omega I_n - A_0)^{-1}$.

Evaluation of $f(\omega_\ell)$ requires

- 1 sparse LU decmposition (complex!),
- $2m$ forward/backward solves,
- m sparse and $2m$ low-rank matrix-vector products.

Use adaptive quadrature with high accuracy, e.g. Gauß-Kronrod (`quadgk` in MATLAB).

Further Applications

Navier-Stokes Coupled with (Passive) Transport of (Reactive) Species

Goal: stabilize concentration at certain level

Model equations:

$$\begin{aligned} \partial_t \mathbf{v} - \frac{1}{Re}\Delta \mathbf{v} + \mathbf{v}.\nabla \mathbf{v} + \nabla \mathbf{p} &= \mathbf{f} \\ \operatorname{div} \mathbf{v} &= 0 \\ \partial_t \mathbf{c} + \mathbf{v}.\nabla \mathbf{c} - \frac{1}{Re \cdot Sc}\Delta \mathbf{c} &= 0 \end{aligned}$$

with boundary conditions:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 & \mathbf{c} = \mathbf{c}_0 &= const & &\text{on } \Gamma_{in} \\ \mathbf{v} &= 0 & \partial_\nu \mathbf{c} &= 0 & &\text{on } \Gamma_{wall} \\ \mathbf{v} &= 0 & \mathbf{c} &= 0 & &\text{on } \Gamma_r, \end{aligned}$$

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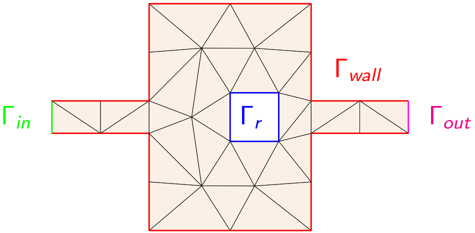
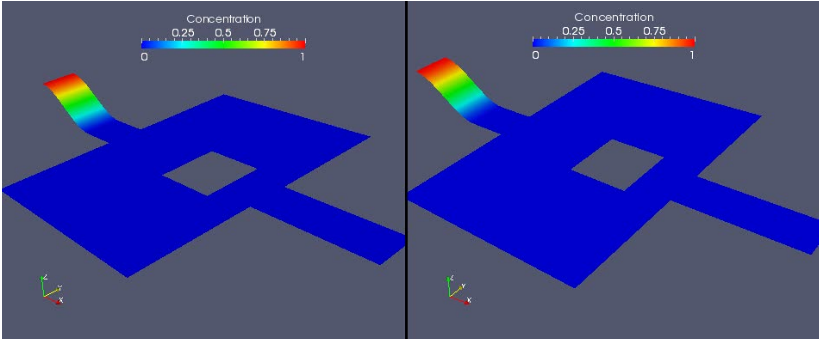
Domain:



Further Applications

Results for $Re = 10$, $Sc = 10$



movie

no control

piecewise constant feedback

Computations by Heiko Weichelt

Further Applications

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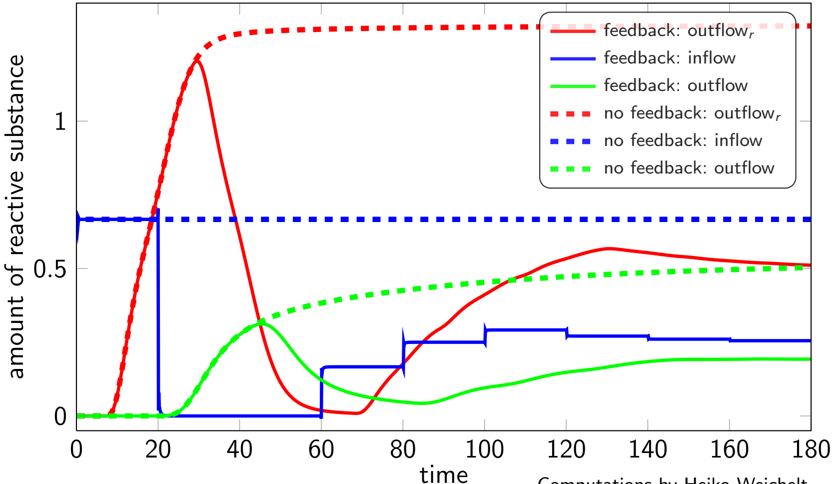

Computations by Heiko Weichelt

Conclusions and Future Work

- Progress in solving AREs in the last decade now allows application of Riccati feedback to realistic PDE control problems.
- Implementation for Navier-Stokes and multi-field flow problems in progress, requires many details not encountered for linear convection-diffusion or beam equations.
- For 3D problems, need dedicated preconditioned iterative "saddle point" solver.

  "(1,1)"-term is nonsymmetric sparse matrix + low-rank perturbation $\rightsquigarrow$ joint work with A. Wathen, M. Stoll.
- Model reduction based on LQG balanced truncation for flow problems in $L_2(0, \infty; V_n(\Omega))$ can be based on derived Riccati solver.

References

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