On a global minimum principle for DAE optimal control problems

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Outline

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1. Review of ODE results
2. Local minimum principle for a DAE optimal control problem
3. Global minimum principle for a DAE optimal control problem
4. Numerical exploitation

ODE Optimal Control Problem

Minimize

$$\varphi(x(0), x(1)) \qquad \text{(w.r.t } x \in W^{1,\infty}, u \in L^{\infty})$$

subject to

ODE

$$x'(t) = f(x(t), u(t))$$

control constraints

$$u(t) \in \mathcal{U}$$

boundary conditions

$$\psi(x(0), x(1)) = 0$$

Local vs global minimum principle: ODE case

Local minimum principle: variational inequality for a.a. $t \in [0,1]$

$$H'_u(x_*(t), u_*(t), \lambda(t))(u - u(t)) \geq 0 \qquad \forall u \in \mathcal{U}$$

Assumption: $\mathcal{U}$ convex, non-empty interior

Global minimum principle: for a.a. $t \in [0,1]$ it holds

$$H(x_*(t), u_*(t), \lambda(t)) \leq H(x_*(t), u, \lambda(t)) \qquad \forall u \in \mathcal{U}$$

$\mathcal{U}$ arbitrary, e.g. discrete set

Proof in 50ies: Pontryagin, Boltyanskii, Gamkrelidze, Mishchenko, and independently by Magnus R. Hestenes

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Local vs global minimum principle: ODE case

Relations: (if $\mathcal{U}$ convex, int($\mathcal{U}$) $\neq 0$)

- local minimum principle is a necessary optimality condition for the global minimum of H w.r.t. u, i.e.

$$\boxed{\text{global minimum principle} \quad \Rightarrow \quad \text{local minimum principle}}$$

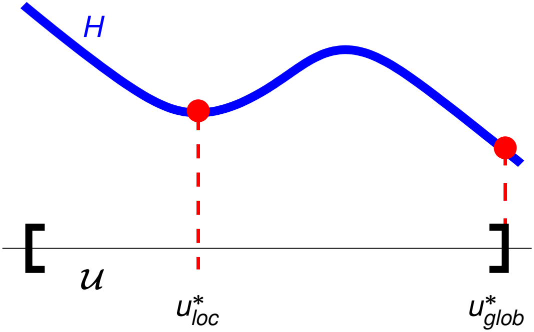
- if H is convex w.r.t. u,

$$\boxed{\text{global minimum principle} \quad \Leftrightarrow \quad \text{local minimum principle}}$$

Local vs global minimum principle: ODE case

Differences:

- global minimum principle requires a global minimum of H, whereas local minimum principle yields local minimum of H



- global minimum principle holds for arbitrary control sets $\mathcal{U}$
  $\rightarrow$ mixed-integer optimal control

Minimum Principle: Example

Example 1

$$\min \quad \frac{1}{2}\int_0^2 x(t)^2 dt \quad \text{s.t.} \quad \begin{array}{l} x'(t) = u(t),\ x(0) = 1,\ x(2) = 0, \\ u(t) \in \{-1, 0, 1\} \end{array}$$

Hamiltonian: $H(x,u,\lambda) := \frac{1}{2}x^2 + \lambda u$

Adjoint equation: $\lambda'(t) = -x(t)$

Minimality of Hamiltonian: a.e. in $[0,2]$:

$$u^*(t) = \begin{cases} -1, & \text{if } \lambda(t) > 0, \\ 1, & \text{if } \lambda(t) < 0, \\ \in \{-1,0,1\}, & \text{if } \lambda(t) = 0. \end{cases}$$

Solution:

$$u^*(t) = \begin{cases} -1, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } t > 1, \end{cases} \qquad x^*(t) = \begin{cases} 1-t, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } t > 1, \end{cases}$$

$$\lambda(t) = \begin{cases} \frac{1}{2}(1-t)^2, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } t > 1. \end{cases}$$

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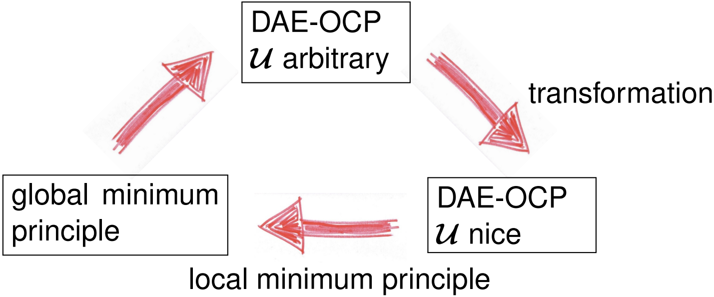
Extension to DAE Optimal Control Problems

Questions:

- local minimum principle for DAE-OCP?
- global minimum principle for DAE-OCP?
- relations and proof techniques?



- numerical exploitation?

Overview

Necessary conditions for DAE optimal control problems:

- de Pinho/Vinter [J. Math. Anal. Appl., 212, 1997]
  semi-explicit index one DAEs, no state constraints, global & local minimum principle
- Devdariani/Ledyaev [Appl. Math. Opt., 40, 1999]
  implicit systems, state and control constraints, global minimum principle
- Roubicek/Valasek [J. Math. Anal. Appl., 269, 2002]
  Hessenberg systems up to index 3, no state constraints, global minimum principle
- De Pinho/Ilchmann [Nonl. Anal., 48, 2002]
  non-smooth problems, local minimum principle
- Kurina/März [SICON, 42, 2004]
  linear-quadratic problems, no state or control constraints, local minimum principle
- Backes [PhD thesis, Humboldt-University, 2006]
  quasilinear DAEs, index one and higher, no state constraints, local and global minimum principles
- G. [JOTA, 130, 2006]
  index 2 Hessenberg systems with state constraints, local minimum principle
- Kunkel/Mehrmann [Math. Control. Signals Syst., 20, 2008]
  general DAEs, no state constraints, global minimum principle

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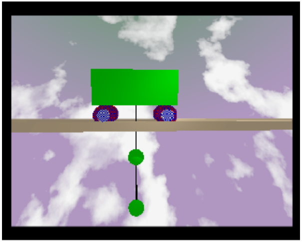
$$
\begin{aligned}
q' &= v - g'(q)^\top \cdot \mu \\
M(q)\cdot v' &= f(q,v,u) - g'(q)^\top \cdot \lambda \\
0 &= g'(q)\cdot v \\
0 &= g(q)
\end{aligned}
$$



Notation

diff./alg. state	:	$x = (q,v)^\top,\ \ y = (\lambda,\mu)^\top$
generalized positions	:	q
generalized velocities	:	v
generalized forces	:	$f(q,v,u)$
control	:	u
mass matrix	:	$M(q,u)$ non-singular

DAE Optimal Control Problem

Minimize

$$\varphi(x(0), x(1)) \qquad (\text{w.r.t } x \in W^{1,\infty}, y, u \in L^\infty)$$

subject to

semi-explicit index-2 DAE

$$\begin{aligned} x'(t) - f(x(t), y(t), u(t)) &= 0 \\ g(x(t)) &= 0 \end{aligned}$$

mixed/pure state constraints

$$\begin{aligned} c(x(t), y(t), u(t)) &\leq 0 \\ s(x(t)) &\leq 0 \end{aligned}$$

boundary conditions

$$\psi(x(0), x(1)) = 0$$

Derivation of Local Minimum Principle

Optimal control problem as infinite optimization problem

$$\begin{array}{lll} \min & F(x,y,u) & \\ \text{s.t.} & G(x,y,u) \in K & (K \text{ convex cone, } \operatorname{int}(K) \neq \emptyset) \\ & H(x,y,u) = \Theta & \\ & (x,y,u) \in S & (S \text{ convex, } \operatorname{int}(S) \neq \emptyset) \end{array}$$

Notation:

$$G = \begin{pmatrix} \text{pure state constr.} \\ \text{mixed control-state constr.} \end{pmatrix} \in \begin{pmatrix} C([0,1],\mathbb{R}^{n_s}) \\ L^\infty([0,1],\mathbb{R}^{n_c}) \end{pmatrix}$$

$$H = \begin{pmatrix} \text{differential eq.} \\ \text{algebraic eq.} \\ \text{boundary cond.} \end{pmatrix} \in \begin{pmatrix} L^\infty([0,1],\mathbb{R}^{n_x}) \\ W^{1,\infty}([0,1],\mathbb{R}^{n_y}) \\ \mathbf{R}^{n_\psi} \end{pmatrix}$$

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Derivation of Local Minimum Principle

Non-density condition:

$$\mathrm{im}\left(H'(\hat{x},\hat{y},\hat{u})\right) \text{ closed}$$

Necessary conditions (Fritz-John type):

$$\begin{aligned} l_0 &\geq 0 \\ \eta^* &\in K^+ && \text{(non-negativity)} \\ \eta^*(G(\hat{x},\hat{y},\hat{u})) &= 0 && \text{(complementarity)} \\ l_0 F'(\hat{x},\hat{y},\hat{u})(x,y,u) & \\ -\eta^*(G'(\hat{x},\hat{y},\hat{u})(x,y,u)) & \\ -\lambda^*(H'(\hat{x},\hat{y},\hat{u})(x,y,u)) &\geq 0, \quad \forall (x,y,u) \in S - \{(\hat{x},\hat{y},\hat{u})\} \end{aligned}$$

(η: multiplier for G, λ: multiplier for H)

Derivation of Local Minimum Principle

Tool to show non-density condition and multiplier representation:

explicit respresentation of solution of linearized index-2 DAE

$$\begin{aligned} x'(t) &= A_1(t)x(t) + B_1(t)y(t) + h_1(t) \\ 0 &= A_2(t)x(t) + h_2(t) \end{aligned}$$

Algebraic equation in $W^{1,\infty}$ not in C!

Derivation of Local Minimum Principle

Necessary conditions imply

$$\lambda_f^*, \lambda_g^*, \eta_1^* \in (L^\infty)^*, \qquad \eta_2^* \in C^*$$

Derive explicit representations of multipliers:

$$\begin{aligned}
\eta_2^*(h) &= \sum_{i=1}^{n_s} \int_0^1 h_i(t) d\mu_i(t) \\
\eta_1^*(k(\cdot)) &= \int_0^1 \eta(t)^\top k(t) dt \\
\lambda_f^*(h_1(\cdot)) &= -\int_0^1 \left(p_f(t)^\top + p_g(t)^\top g_x'[t] \right) h_1(t) dt \\
\lambda_g^*(h_2(\cdot)) &= -\zeta^\top h_2(t_0) - \int_0^1 p_g(t)^\top h_2'(t) dt
\end{aligned}$$

Use explicit representations to obtain minimum principle via variational lemmas.

Minimum principle [G.'06]

Assumptions:

(i) Smoothness: $\varphi, f, c, s, \psi \in C^1$, $g \in C^2$.

(ii) $(\hat{x}, \hat{y}, \hat{u})$ (weak) local minimum.

(iii) $\operatorname{rank}\left(c_u'\right) = n_c$,
$g_x'\left(f_y' - f_u'(c_u')^+ c_y'\right)$ non-singular a.e. in $[0,1]$

(iv) Index-2 DAE:
$g_x' \cdot f_y'$ non-singular a.e. in $[0,1]$ with essentially bounded inverse.

Local Minimum Principle [G.'06]

(Augmented) Hamiltonian:

$$H := \lambda_f^\top f(x,y,u) + \lambda_g^\top g_x'(x)\cdot f(x,y,u) + \eta^\top c(x,y,u)$$

Adjoint equation: (index 1 DAE)

$$\begin{aligned}\lambda_f(t) &= \lambda_f(1) + \int_t^1 H_x'[\tau]^\top d\tau + \int_t^1 s_x'[\tau]^\top d\mu(\tau)\\ 0 &= H_y'[t]\end{aligned}$$

Transversality conditions:

$$\lambda_f(0)^\top = -\ell_0\varphi_{x_0}' - \psi_{x_0}'^\top\sigma - g_x'^\top\zeta, \qquad \lambda_f(1)^\top = \ell_0\varphi_{x_1}' + \psi_{x_1}'^\top\sigma$$

Stationarity of Hamiltonian:

$$H_u'[t] = 0$$

Complementarity conditions:

$$0 \le \eta(t) \quad \perp \quad -c[t] \ge 0$$

μ is monotonically increasing and constant on inactive parts of s.

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Transversality conditions:

$$\lambda_f(0)^\top = -\ell_0 \varphi'_{x_0} - {\psi'_{x_0}}^\top \sigma - {g'_x}^\top \zeta, \qquad \lambda_f(1)^\top = \ell_0 \varphi'_{x_1} + {\psi'_{x_1}}^\top \sigma$$

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Minimum principle [G.'06]

The integral equation for λ_f can be replaced:

Differential equation:

$$\lambda_f'(t) = -H_x'[t]^\top - s_x'[t]^\top \mu'(t)$$

Jump conditions:

At every point $t_j \in (0,1)$ of discontinuity of the multiplier μ:

$$\lambda_f(t_j) - \lambda_f(t_j-) = -s_x'(\hat{x}(t_j))^\top \left(\mu(t_j) - \mu(t_j-)\right)$$

Open (?) Question

Is there a minimum principle for the Hamiltonian

$$H := \boldsymbol{\lambda}_f^\top f(x,y,u) + \boldsymbol{\lambda}_g^\top g(x) + \boldsymbol{\eta}^\top c(x,y,u)$$

(reminds of Jacobsen/Lele/Speyer)

rather than for

$$H := \boldsymbol{\lambda}_f^\top f(x,y,u) + \boldsymbol{\lambda}_g^\top g_x'(x) \cdot f(x,y,u) + \boldsymbol{\eta}^\top c(x,y,u) \quad ?$$

(reminds of Bryson/Denham/Dreyfus)

Difficulty: well-posedness of index-2 adjoint DAE

2D-Stokes Equation

Minimize

$$\frac{1}{2}\|z - z_d\|_2^2 + \frac{\alpha}{2}\|u\|_2^2$$

subject to

$$\begin{aligned} z_t &= \Delta z - \nabla p + u, \\ 0 &= \operatorname{div}(z), \\ z(0,x,y) &= 0, \qquad (x,y) \in \Omega, \\ z(t,x,y) &= 0, \qquad (t,x,y) \in (0,T) \times \partial\Omega \end{aligned}$$

Method of lines: Minimize

$$\frac{1}{2}\|z_h - z_{d,h}\|_2^2 + \frac{\alpha}{2}\|u_h\|_2^2$$

subject to Index-2 DAE

$$\begin{aligned} z_h'(t) &= A_h z_h(t) + B_h p_h(t) + u_h(t), \qquad z_h(0) = 0, \\ 0 &= B_h^\top z_h(t) \end{aligned}$$

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2D-Stokes Equation

Hamiltonian:

$$H = \frac{1}{2}\|z_h - z_{d,h}\|^2 + \frac{\alpha}{2}\|u_h\|^2 + (\lambda_f + B_h\lambda_g)^\top (A_h z_h + B_h p_h + u_h)$$

Stationarity:

$$0 = H'_u \qquad \Rightarrow \qquad u = -\frac{1}{\alpha}(\lambda_f + B_h\lambda_g)$$

Adjoint equation:

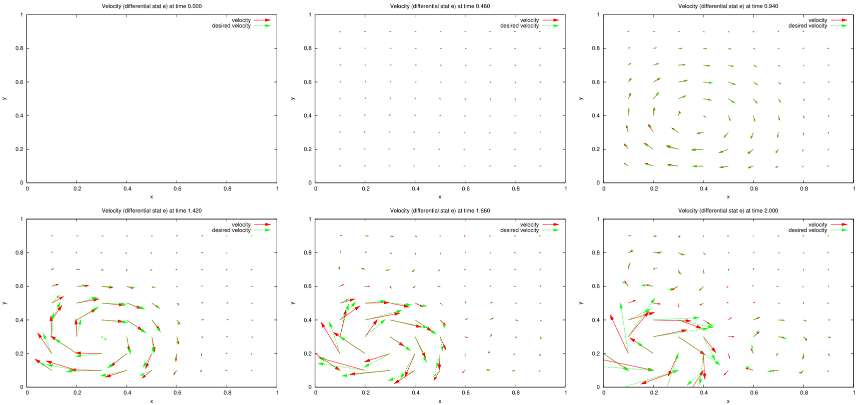
$$\begin{aligned} \lambda'_f &= -(z_h - z_{d,h}) - A_h^\top (\lambda_f + B_h\lambda_g) \\ 0 &= B_h^\top \lambda_f + B_h^\top B_h \lambda_g \end{aligned}$$

Transversality condition:

$$\lambda_f(T) = 0$$

2D-Stokes Equation, Results, $T = 2$, $\alpha = 10^{-6}$

Snapshots of differential state (velocity)



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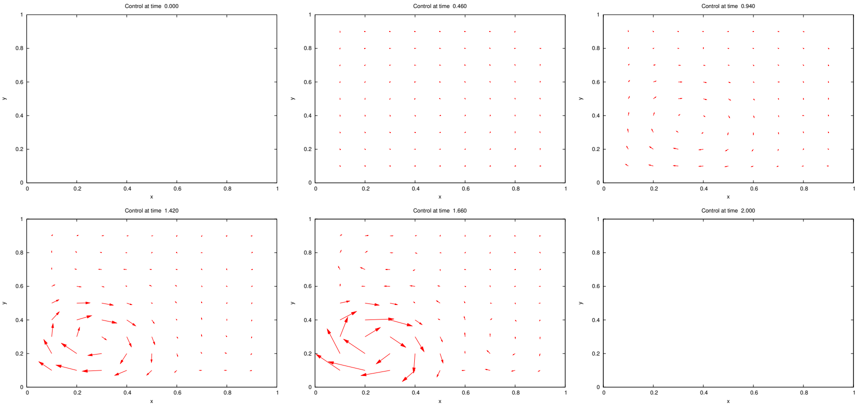
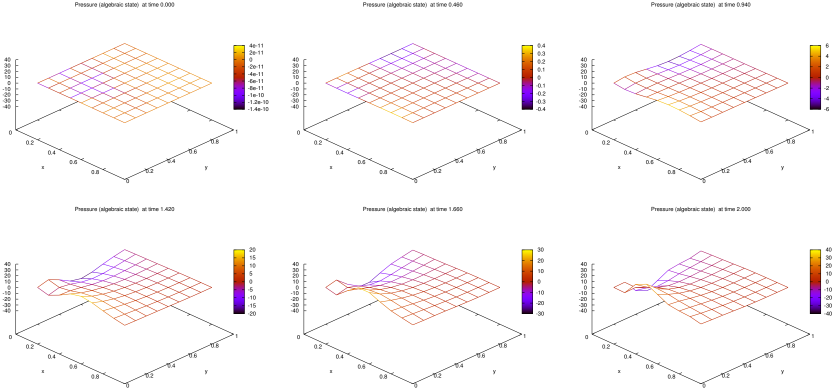
2D-Stokes Equation, Results, $T = 2$, $\alpha = 10^{-6}$

Snapshots control



2D-Stokes Equation, Results, $T = 2$, $\alpha = 10^{-6}$

Snapshots pressure



2D Navier Stokes Equations

Minimize

$$\frac{1}{2}\int_{(0,T)\times\Omega} \|z - z_d\|^2 dxdt + \frac{\alpha}{2}\int_{(0,T)\times\Omega} \|u\|^2 dxdt$$

subject to

$$\begin{aligned} z_t &= \frac{1}{Re}\Delta z - (z\cdot\nabla)z - \nabla p + u \\ 0 &= \operatorname{div}(z) \\ z(0,x) &= 0, \quad x \in \Omega \\ z(t,x) &= 0, \quad (t,x) \in (0,T)\times\partial\Omega \\ |u| &\leq u_{max} \end{aligned}$$

ITER	α_{k-1}	$\|F(z^k)\|_2$
0		3.3059005e+04
1	1.00000	1.5773644e+04
2	1.00000	2.1003213e+03
3	1.00000	2.6396508e+02
4	1.00000	5.5511370e+00
5	1.00000	5.2706671e+02
6	1.00000	1.6632141e+01
7	1.00000	1.1927847e+01
8	1.00000	4.7810060e-02
9	1.00000	8.5254261e-02
10	1.00000	5.3724008e-01
11	1.00000	2.7892178e-01
12	1.00000	1.7395834e-05
13	1.00000	5.7641406e-06
14	1.00000	5.5494592e-07
15	1.00000	1.5265781e-07
16	1.00000	1.2116406e-09

Numerics: Method of lines and finite differences (5-point-scheme for Δ, forward FD for ∇, backward FD for div)

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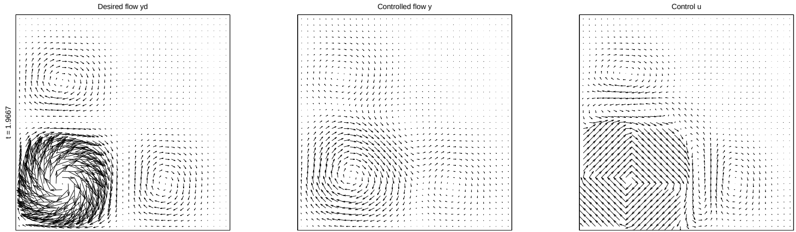
2D Navier Stokes Equations [M. Kunkel]



Details:

- linear Index-2 DAE BVP's: index reduction, symmetric collocation for index-1 DAE's [Stöver/Kunkel'02]
- linear equations: PARDISO [Schenk/Gärtner'02,'06]
- data: $T = 2$, $\alpha = 5 \cdot 10^{-6}$, $Re = 1$, $u_{max} = 200$, $N = 36$, $N_t = 60$, $n_x = n_u = 2450$, $n_y = 1225$, $leq = 2649675$

Outline

DAE Optimal Control Problem

Minimize

$$\varphi(x(0), x(1)) \qquad (\text{w.r.t } x \in W^{1,\infty}, y, u \in L^\infty)$$

subject to

semi-explicit index-2 DAE

$$\begin{aligned} x'(t) - f(x(t), y(t), u(t)) &= 0 \\ g(x(t)) &= 0 \end{aligned}$$

control constraints

$$u(t) \in \mathcal{U}$$

boundary conditions

$$\psi(x(0), x(1)) = 0$$

Global Minimum Principle

Approach 1: exploitation of ODE results

- $\rightarrow$ reduction to index-1 constraint
- $\rightarrow$ solve for algebraic variable (implicit function theorem)
- $\rightarrow$ apply ODE global minimum principle
- $\rightarrow$ backtransformation to DAE

[Roubicek/Valasek'02]

Approach 2: exploitation of local minimum principle

Dubovitskii–Milyutin trick [Dubovitskii/Milyutin'65, Girsanov'72, Ioffe/Tihomirov'79]

DAE Optimal Control Problem

DAE OCP

Minimize

$$\varphi(x(0), x(1))$$

subject to

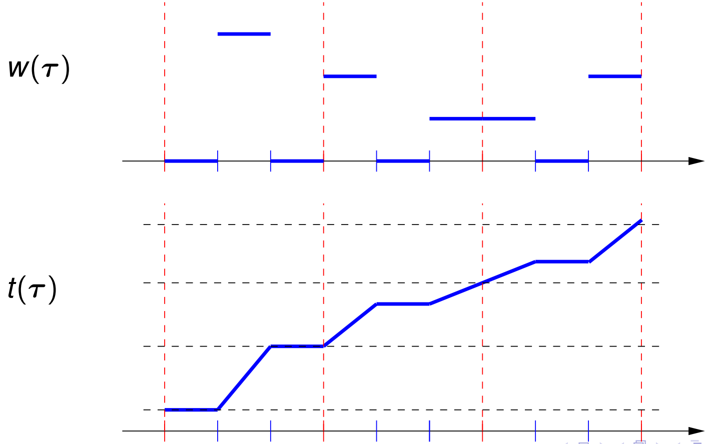
$$\begin{aligned} x'(t) - f(x(t), y(t), u(t)) &= 0 \\ g(x(t)) &= 0 \\ u(t) &\in \mathcal{U} \quad (\mathcal{U} \text{ arbitrary!}) \\ \psi(x(0), x(1)) &= 0 \end{aligned}$$

Optimal solution:

$$(x_*, y_*, u_*)$$

Time transformation

$$t(\tau) = \int_0^\tau w(s)ds, \quad t(0) = 0, \quad t(1) = 1, \quad w(\cdot) \geq 0$$



Application to OCP

Let $\Delta_w := \{\tau \in [0,1] \mid w(\tau) > 0\}$. Define

$$\begin{aligned}
\tilde{x}(\tau) &= x_*(t(\tau)), \\
\tilde{y}(\tau) &= \begin{cases} y_*(t(\tau)), & \text{if } \tau \in \Delta_w, \\ \text{suitable}, & \text{otherwise} \end{cases} \\
\tilde{u}(\tau) &= \begin{cases} u_*(t(\tau)), & \text{if } \tau \in \Delta_w, \\ \text{suitable}, & \text{otherwise} \end{cases}
\end{aligned}$$

Idea: [Dubovitskii, Milyutin'65]

- use $w(\cdot) \in \mathcal{W} = \{w \in L^\infty \mid w(\cdot) \geq 0\}$ as a control
- consider $\tilde{u}(\cdot)$ as fixed (on Δ_w)

Application to OCP

Transformed DAE OCP

Minimize

$$\varphi(\tilde{x}(0), \tilde{x}(1))$$

subject to

$$\begin{aligned}
\tilde{x}'(\tau) &= w(\tau) f(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{u}(\tau)) \\
t'(\tau) &= w(\tau),\ t(0) = 0,\ t(1) = 1 \\
g(\tilde{x}(\tau)) &= 0 \\
w(\tau) &\geq 0 \\
\psi(\tilde{x}(0), \tilde{x}(1)) &= 0
\end{aligned}$$

$\longrightarrow$ minimization w.r.t. $\tilde{x}$, $\tilde{y}$, w!

$\longrightarrow$ local minimum principle applicable!

Local minimum principle

(Augmented) Hamiltonian:

$$\tilde{H}(\tau,t,x,y,w,\lambda_f,\lambda_g,\lambda_t,\eta) := wH(x,y,\tilde{u}(\tau),\lambda_f,\lambda_g) + (\lambda_t - \eta)w$$

Adjoint equation: (index 1 DAE)

$$\begin{aligned} \tilde{\lambda}_f'(\tau) &= -wH_x'[\tau]^\top \\ \tilde{\lambda}_t'(\tau) &= 0, \\ 0 &= wH_y'[\tau] \end{aligned}$$

+ transversality conditions

Stationarity of Hamiltonian + Complementarity conditions:

$$H[\tau] + \tilde{\lambda}_t(\tau) \begin{cases} = 0, & \text{if } \tau \in \Delta_w, \\ \geq 0, & \text{otherwise} \end{cases}$$

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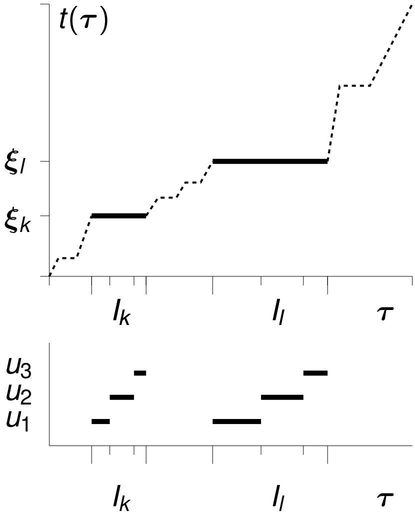
Note

If $w \equiv 0$ in some interval then

$$\begin{aligned} \tilde{x}' &\equiv 0 &\Rightarrow&\quad \tilde{x} \equiv \text{const} \\ t' &\equiv 0 &\Rightarrow&\quad t \equiv const \\ g(\tilde{x}(\tau)) &\equiv 0 &\Rightarrow&\quad \tilde{y} \text{ not defined} \end{aligned}$$

same holds for λ_f and λ_g!

Construction of w – exploitation of degrees of freedom



Construction: [Ioffe, Tihomirov'79]

- $\{\xi_1, \xi_2, \ldots\}$ countable dense subset of $[0,1]$, $\beta_k > 0$ with $\sum_k \beta_k = \frac{1}{2}$, $\tau_k = \frac{\xi_k}{2} + \sum_{j:\xi_j<\xi_k} \beta_j$
- $I_k := (\tau_k, \tau_k + \beta_k]$, $t(\bigcup_k I_k)$ dense in $[0,1]$
- define
$$w(\tau) := \begin{cases} 0, & \text{if } \tau \in \bigcup_k I_k, \\ 2, & \text{otherwise} \end{cases}$$
- $t(\tau) = \xi_k = t(\tau_k)$ for $\tau \in I_k$
- $I_k = \bigcup_j I_{kj}$, $I_{kj} \neq \emptyset$
- $\{u_1, u_2, \ldots\}$ countable dense subset of $\mathcal{U}$

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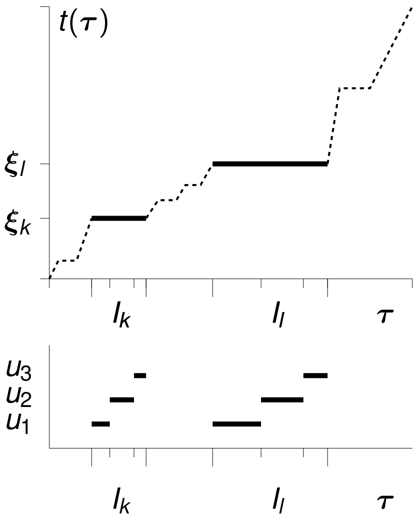

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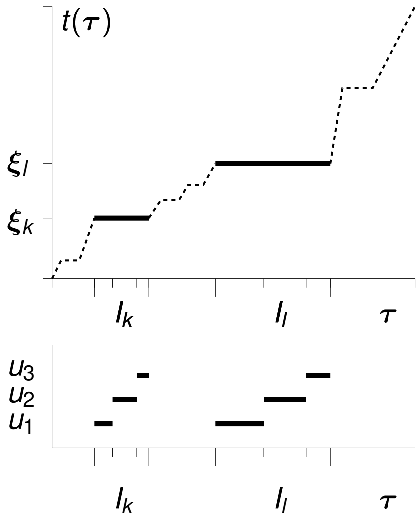

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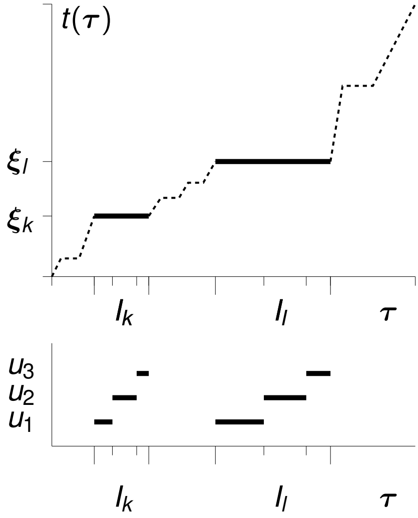

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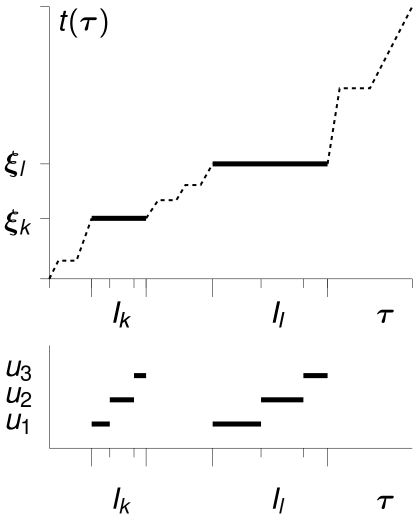

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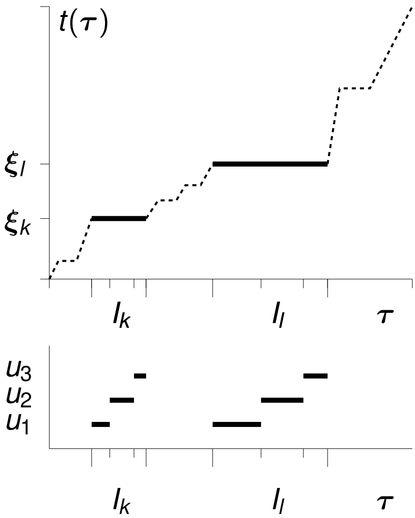

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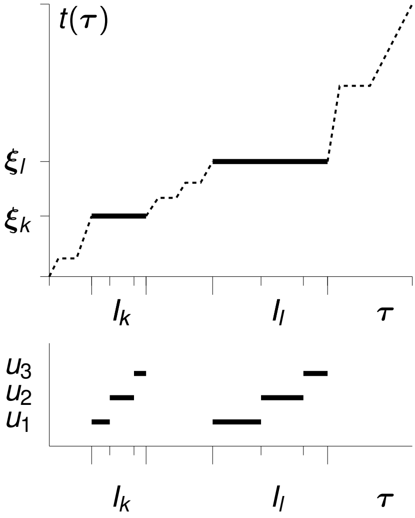

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- $\{u_1, u_2, \ldots\}$ countable dense subset of $\mathcal{U}$

Construction of u – exploitation of degrees of freedom

Let

$$\begin{aligned} \tilde{u}(\tau) &= u_j && \text{for } \tau \in I_{kj} \\ \tilde{y}(\tau) &= y_j && \text{for } \tau \in I_{kj} \\ \lambda_g(\tau) &= \lambda_{g,j} && \text{consistent} \end{aligned}$$

with

$$\begin{aligned} (u_j, y_j) &\in M(\tilde{x}(\tau)) \\ &:= \{(u,y) \in \mathcal{U} \times \mathbb{R}^{n_y} \mid g_x'(\tilde{x}(\tau)) f(\tilde{x}(\tau), y, u) = 0\} \end{aligned}$$

Then:

$$H[\tau] + \tilde{\lambda}_t(\tau) \geq 0 \qquad \text{a.e. in } \bigcup_k I_k$$

Construction of u – exploitation of degrees of freedom

Recall:

- $I_k = \bigcup_j I_{kj}$, $\text{int}(I_{kj}) \neq \emptyset$
- $t(\tau) = \xi_k$ for $\tau \in I_k$

Then:
For k and j there exists $\tau \in I_{kj}$ with

$$H[\tau] + \tilde{\lambda}_t(\tau) = H(x_*(\xi_k), y_j, u_j, \lambda_f(\xi_k), \lambda_{g,j}) + \lambda_t(\xi_k) \geq 0.$$

Continuity argument yields

$$H(x_*(t), y, u, \lambda_f(t), \lambda_g) + \lambda_t(t) \geq 0$$

for all $(u, y) \in M(x_*(t))$ and a.a. t.

Global Minimum Principle

On the other hand, a.e. it holds

$$H(x_*(t), y_*(t), u_*(t), \lambda_f(t), \lambda_g(t)) + \lambda_t(t) = 0$$

Together:

Global minimum principle

It holds

$$H(x_*(t), y_*(t), u_*(t), \lambda_f(t), \lambda_g(t)) \leq H(x_*(t), y, u, \lambda_f(t), \lambda_g(t))$$

for all $(u, y) \in M(x_*(t))$ a.e. in $[0, 1]$.

$$M(x) = \{(u, y) \in \mathcal{U} \times \mathbb{R}^{n_y} \mid g'_x(x) f(x, y, u) = 0\}$$

Example

Minimize

$$\int_0^1 x^2 + \alpha(y-u)^2 dt$$

subject to

$$\begin{aligned} x' &= y - u, \quad x(0) = 0 \\ 0 &= x \\ u &\in \mathcal{U} := [-1,1] \end{aligned}$$

Obviously, every feasible control is optimal!
Hamiltonian:

$$H = \ell_0 x^2 + \alpha(y-u)^2 + \lambda_f(y-u) + \lambda_g(y-u)$$

Minimization of H w.r.t. $(u,y) \in M = \{(u,y) \mid y-u=0\}$ yields that every u satisfies the global minimum principle.

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Outline

Optimal Control Problem

Mixed-Integer OCP

$$\begin{aligned} \text{Minimize} \quad & \varphi(x(t_0), x(t_f)) \\ \text{s.t.} \quad & \dot{x}(t) - f(t, x(t), u(t), v(t)) = 0 \\ & s(t, x(t)) \leq 0 \\ & \psi(x(t_0), x(t_f)) = 0 \\ & u(t) \in \mathcal{U} \\ & v(t) \in \mathcal{V} \end{aligned}$$

Notation

state: $x \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_x})$

controls: $u \in L^{\infty}([t_0, t_f], \mathbb{R}^{n_u})$, $v \in L^{\infty}([t_0, t_f], \mathbb{R}^{n_v})$

$\mathcal{U} \subseteq \mathbb{R}^{n_u}$ convex, closed, $\mathcal{V} = \{v_1, \ldots, v_M\}$ discrete

Solution Approaches

Indirect approach:

exploit necessary optimality conditions (global minimum principle)

Direct discretization approaches:

(a) variable time transformation [Lee, Teo, Jennings, Rehbock, G., ...]
(b) Branch & Bound [von Stryk, G., ...]
(c) sum-up-rounding strategy [Sager]
(d) stochastic/heuristic optimization [Schlüter/G.,...]

Discretization

Main grid

$$\mathbb{G}_h \;:\; t_i = t_0 + ih, \qquad i = 0, \dots, N, \; h = \frac{t_f - t_0}{N}$$

Minor grid

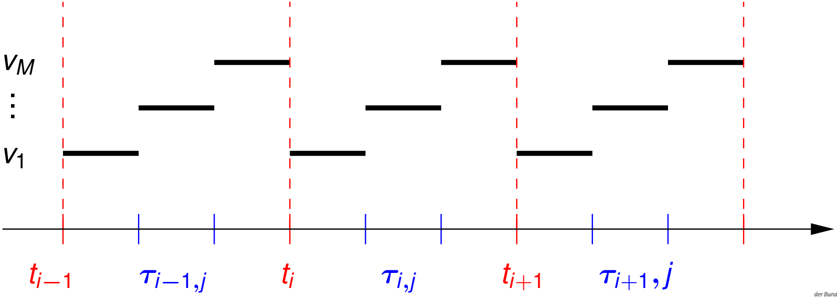
$$\mathbb{G}_{h,M} \;:\; \tau_{i,j} = t_i + j\frac{h}{M}, \qquad j = 0, \dots, M, \; i = 0, \dots, N-1$$

M = number of discrete values in $\mathcal{V} = \{v_1, \dots, v_M\}$

Idea

Replace the discrete control v by a fixed and piecewise constant function on the minor grid according to

$$v_{\mathbb{G}_{h,M}}(\tau) = v_j \quad \text{for } \tau \in [\tau_{i,j-1}, \tau_{i,j}],\ i = 0,\ldots,N-1,\ j = 1,\ldots,M$$



Idea: Variable Time Transformation

Variable time transformation:

$$t = t(\tau), \qquad t(\tau) := t_0 + \int_{t_0}^{\tau} w(s)ds, \qquad \tau \in [t_0, t_f]$$

and

$$t_f - t_0 = \int_{t_0}^{t_f} w(s)ds$$

Remark:

- w is the speed of running through $[t_0, t_f]$:

$$\frac{dt}{d\tau} = w(\tau), \qquad \tau \in [t_0, t_f]$$

- $w(\tau) = 0$ in $[\tau_{i,j}, \tau_{i,j+1}) \Rightarrow$ $[t(\tau_{i,j}), t(\tau_{i,j+1})]$ shrinks to $\{t(\tau_{i,j})\}$

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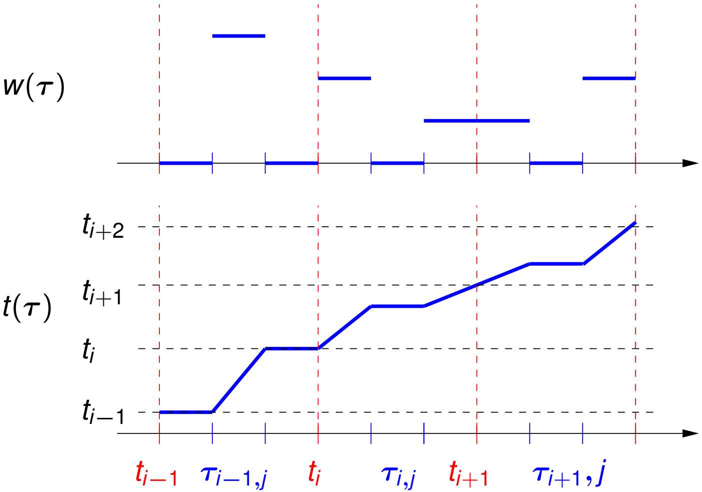
Time Transformation



New Control

Consider w as new control subject to the restrictions:

- $w(\tau) \geq 0$ for all τ ($\rightarrow$ no running back in time)
- $w(\tau)$ piecewise constant on the minor grid $\mathbb{G}_{h,M}$
- Major grid points are invariant under time transformation:

$$\int_{t_i}^{t_{i+1}} w(\tau) d\tau = t_{i+1} - t_i = h, \qquad i = 0, \ldots, N-1$$

Control set:

$$\mathcal{W} := \left\{ w \in L^\infty([t_0, t_f], \mathbb{R}) \;\middle|\; \begin{array}{l} w(\tau) \geq 0, \\ w \text{ piecewise constant on } \mathbb{G}_{h,M}, \\ \int_{t_i}^{t_{i+1}} w(\tau) d\tau = t_{i+1} - t_i \quad \forall i \end{array} \right\}$$

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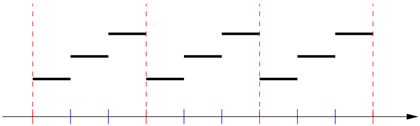
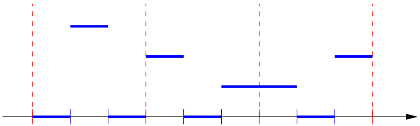
Backtransformation

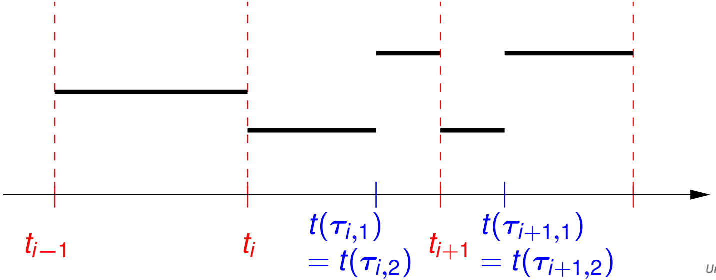
$v_{\mathbb{G}_{h,M}}(\tau)$:



$w(\tau)$:



Corresponding control $v(s) = v_{\mathbb{G}_{h,M}}(t^{-1}(s))$:



Transformed Optimal Control Problem

TOCP

$$
\begin{aligned}
\text{Minimize} \quad & \varphi(x(t_0), x(t_f)) \\
\text{s.t.} \quad & \dot{x}(\tau) - w(\tau) f(\tau, x(\tau), u(\tau), v_{\mathbb{G}_{h,M}}(\tau)) = 0 \\
& s(\tau, x(\tau)) \leq 0 \\
& \psi(x(t_0), x(t_f)) = 0 \\
& u(\tau) \in \mathcal{U} \\
& w \in \mathcal{W}
\end{aligned}
$$

Remarks:

- If $w(\tau) \equiv 0$ in $[\tau_{i,j}, \tau_{i,j+1}]$ then x remains constant therein!
- $v_{\mathbb{G}_{h,M}}$ is the fixed function defined before!
- TOCP has only 'continuous' controls, no discrete controls anymore!

Extension to Index-2 DAEs

Difficulty:

If $w \equiv 0$ in some interval then

$$\begin{aligned} x' &\equiv 0 \quad &\Rightarrow \quad & x \equiv \text{const} \\ g(x(\tau)) &\equiv 0 \quad &\Rightarrow \quad & y \text{ not defined} \end{aligned}$$

Remedy: Use stabilized DAE

$$\begin{aligned} x' &= w f(x, u, v_{\mathbb{G}_h, M}) - g_x'(x)^\top \mu \\ 0 &= g(x) \\ 0 &= g_x'(x) f(x, y, u) \end{aligned}$$

Theorem 2

If g_x' has full rank, then stabilized DAE has index 1 and $\mu \equiv 0$.

Virtual Test-driving

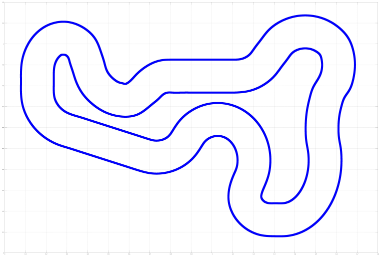
Simulation of test-drives:

- car model
  - → single track model (10 ODE)
  - → full car model (45 index-1 DAE)
  - → multibody model (index-3 DAE)
- model of track
  - → state constraints
  - → boundary conditions
  - → periodic splines
- model of driver
  - → optimal control problem
- objective
  - → fast, efficient, comfortable,...

Automatic Drive along a Test-course

Task:

Minimize Time + Steering effort!



Why?

- provide simulation tools useable in development process
- automatic/autonomous driving (fix influence of driver, standardized environment for set-up of cars)
- future: driving assistance system

Automatic Driving

Goals:

- fix influence of driver
- same testing environment (tuning, calibration)
- future: driver assistance system

Realization:

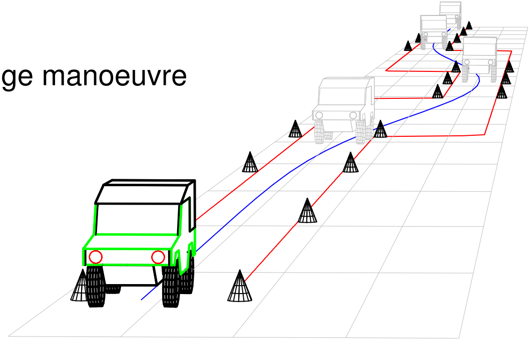
- explore unknown track
- compute optimal testdrive
- track optimal course using controllers and differential GPS

Virtual Test-Driving

Task:

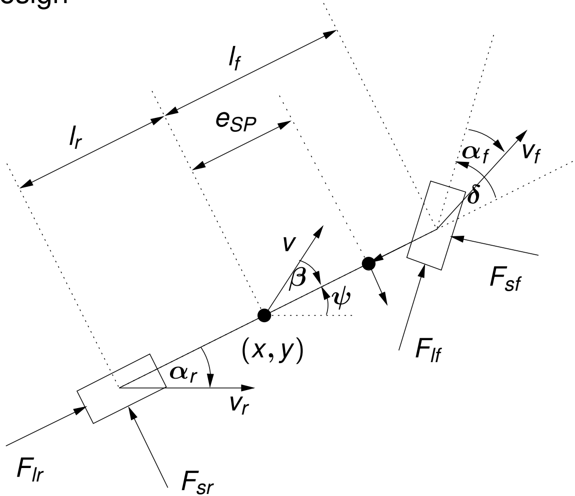
Simulation of test-drives of a car in the computer

Example:
double lane change manoeuvre
(Elktest)



Single-Track Model

Usage: basic investigation of lateral dynamics (up to $0.4g$), controller design



Equations of Motion

$$
\begin{aligned}
\dot{x} &= v\cos(\psi-\beta)\\
\dot{y} &= v\sin(\psi-\beta)\\
\dot{v} &= \Big[(F_{lr}-F_{Ax})\cos\beta + F_{lf}\cos(\delta+\beta) - (F_{sr}-F_{Ay})\sin\beta \\
&\qquad - F_{sf}\sin(\delta+\beta)\Big]/m\\
\dot{\beta} &= w_z - \Big[(F_{lr}-F_{Ax})\sin\beta + F_{lf}\sin(\delta+\beta) \\
&\qquad + (F_{sr}-F_{Ay})\cos\beta + F_{sf}\cos(\delta+\beta)\Big]/(m\cdot v)\\
\dot{\psi} &= w_z\\
\dot{w}_z &= \Big[F_{sf}\cdot l_f\cdot\cos\delta - F_{sr}\cdot l_r - F_{Ay}\cdot e_{SP} + F_{lf}\cdot l_f\cdot\sin\delta\Big]/I_{zz}\\
\dot{\delta} &= w_\delta
\end{aligned}
$$

Lateral tire forces F_{sf}, F_{sr}: 'magic formula' [Pacejka'93]

Equations of Motion: Details

Longitudinal tire forces (rear wheel drive):

$$F_{lf} = -F_{Bf} - F_{Rf}, \quad F_{lr} = \frac{M_{wheel}(\phi, \mu)}{R} - F_{Br} - F_{Rr}$$

(F_{Rf}, F_{Rr} rolling resistances front/rear)

Power train torque: [Neculau'92]

$$M_{wheel}(\phi, \mu) = i_g(\mu) \cdot i_t \cdot M_{mot}(\phi, \mu)$$

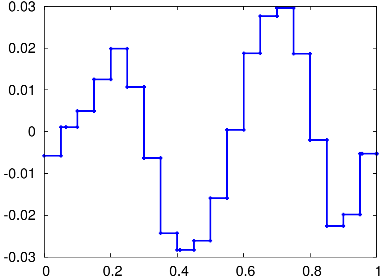
(i_g: gear transmission, i_t: motor torque transmission, M_{mot}: motor torque)

Controls:

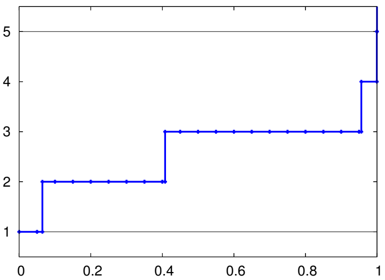
w_δ	$\in$	$[w_{min}, w_{max}]$	:	steering angle velocity
F_{Bf}, F_{Br}	$\in$	$[F_{min}, F_{max}]$	:	braking forces front/rear
ϕ	$\in$	$[0, 1]$	:	gas pedal position
μ	$\in$	$\{1, 2, 3, 4, 5\}$	:	gear

Result: 20 grid points

steering angle velocity w_δ:



gear shift μ:



Braking force: $F_B(t) \equiv 0$, **Gas pedal position:** $\phi(t) \equiv 1$

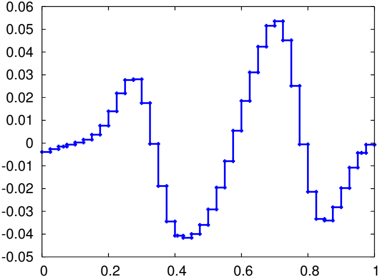
Complete enumeration (1 [s] to solve DOCP): $\approx 3 \cdot 10^6$ years

Branch & Bound: 23 m 52 s, objective value: 6.781922

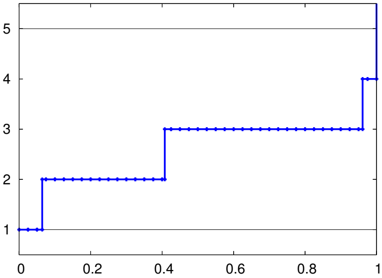
Transformation: 2 m 1.154 s, objective value: 6.774669

Result: 40 grid points

Steering angle velocity w_δ:



Gear shift μ:



Braking force: $F_B(t) \equiv 0$, **Gas pedal position:** $\phi(t) \equiv 1$

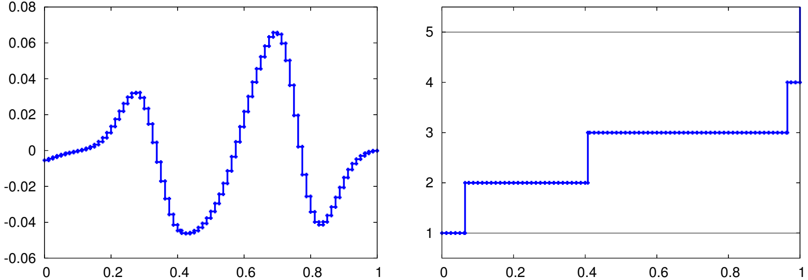
Complete enumeration (1 [s] to solve DOCP): $\approx 3 \cdot 10^{20}$ years

Branch & Bound: 232 h 25 m 31 s, objective value: 6.791374

Transformation: 9 m 39.664 s, objective value: 6.787982

Result: 80 grid points

Steering angle velocity w_δ: **Gear shift μ:**



Braking force: $F_B(t) \equiv 0$, **Gas pedal position:** $\phi(t) \equiv 1$

65 m 3.496 s, objective value: 6.795366

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F8 Aircraft

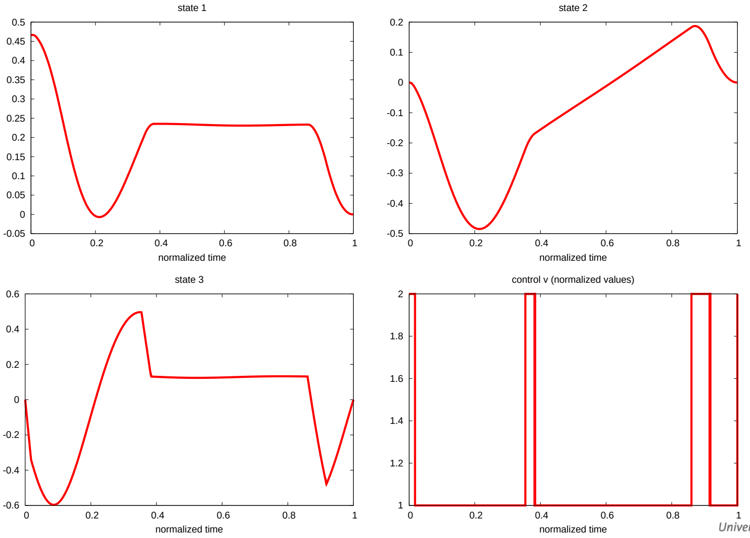
Minimize T subject to

$$\begin{aligned}
x_1' &= -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 \\
&\quad -x_1^2x_3 + 3.846x_1^3 - 0.215v + 0.28x_1^2v + 0.47x_1v^2 + 0.63v^3 \\
x_2' &= x_3 \\
x_3' &= -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967v \\
&\quad +6.265x_1^2v + 46x_1v^2 + 61.4v^3 \\
v &\in \{-0.05236, 0.05236\}
\end{aligned}$$

$$x(0) = (0.4655, 0, 0)^\top,\; x(T) = (0, 0, 0)^\top$$

Source: http://mintoc.de by Sebastian Sager

F8 Aircraft ($N = 500$, $T = 5.728674$)



Lotka-Volterra Fishing Problem

Minimize

$$\int_0^{12} (x_1(t)-1)^2 + (x_2(t)-1)^2 dt$$

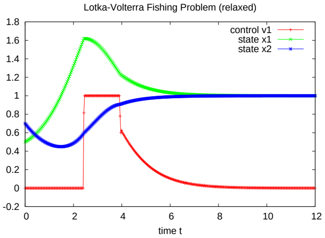
subject to

$$\begin{aligned} x_1'(t) &= x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)v(t) \\ x_2'(t) &= -x_2(t) + x_1(t)x_2(t) - 0.2x_2(t)v(t) \\ v(t) &\in \{0,1\} \\ x(0) &= (0.5, 0.7)^\top \end{aligned}$$

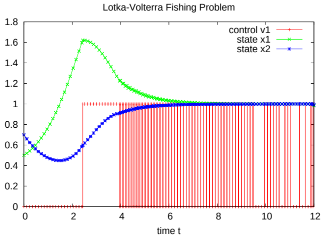
Source: http://mintoc.de by Sebastian Sager
Background: optimal fishing strategy on a fixed time horizon to bring the biomasses of both predator as prey fish to a prescribed steady state

Lotka-Volterra Fishing Problem – Solutions

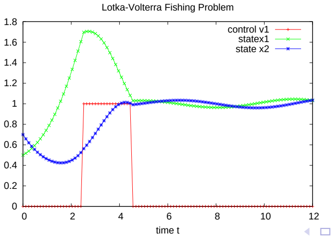
Relaxed Solution:



No switching costs:



With Switching costs:



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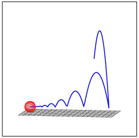
Outlook

- **Optimal control with complementarity constraints**
  - $\rightarrow$ mechanical systems with contacts
  - $\rightarrow$ Stackelberg games
  - $\rightarrow$ robust control



- **Realtime Optimal Control**
  - $\rightarrow$ parametric sensitivity analysis
  - $\rightarrow$ model predictive control
- **more general DAEs**
- ...

Thanks for your attention!

Questions?

Further Information:

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