Pseudospectral Methods in Optimal Control

William W. Hager

University of Florida

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Collaborators:

- Anil Rao + students:
  - David Benson
  - Chris Darby
  - Divya Garg
  - Geoffrey Huntington
  - Michael Patterson

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Software:

- GPOPS: General Pseudospectral OPtimal Control Software
  - MATLAB software with interface to optimizer such as SNOPT
  - Free (GNU license)
  - Gauss, Radau, Lobatto

Model Control Problem

$$\min \Phi(\mathbf{x}(1))$$

subject to $$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad -1 \le t \le +1, \quad \mathbf{x}(-1) = \mathbf{x}_0,$$

$$\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \qquad \mathbf{x}_0 \text{ given.}$$

Gauss Pseudospectral Scheme

- $\tau_1, \tau_2, \ldots, \tau_N$ = Gauss quadrature points
- $\tau_0 = -1$ and $\tau_{N+1} = +1$.
- Lagrange interpolating polynomials:

$$L_i(\tau) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \qquad (i = 0, \ldots, N).$$

- State approximation:

$$x_j^N(\tau) = \sum_{i=0}^{N} x_{ij} L_i(\tau)$$

- Derivative approximation:

$$\dot{x}_j^N(\tau_k) = \sum_{i=0}^{N} x_{ij} \dot{L}_i(\tau_k) = \sum_{i=0}^{N} D_{ki} x_{ij}, \qquad D_{ki} = \dot{L}_i(\tau_k)$$

Gauss Pseudospectral Scheme (continued)

- Dynamics matrix:

$$F_{ij}(\mathbf{X},\mathbf{U}) = f_j(\mathbf{X}_i,\mathbf{U}_i), \quad 1 \le i \le N, \quad 1 \le j \le n.$$

- Collocated dynamics:

$$\mathbf{DX} = \mathbf{F}(\mathbf{X},\mathbf{U})$$

- State at end point:

$$\mathbf{X}_{N+1,j} = x_j^N(1) = x_j^N(-1) + \int_{-1}^{+1} \dot{x}_j^N(\tau) d\tau$$

- End state after quadrature ($\mathbf{w}$ = quadrature weights):

$$\mathbf{X}_{N+1} = \mathbf{X}_0 + \mathbf{w}^{\mathsf{T}}\mathbf{DX} = \mathbf{w}^{\mathsf{T}}\mathbf{F}(\mathbf{X},\mathbf{U})$$

The Gauss Pseudospectral Problem

$$\begin{aligned} \text{minimize} \quad & \Phi(\mathbf{X}_{N+1}) \\ \text{subject to} \quad \mathbf{D}\mathbf{X} &= \mathbf{F}(\mathbf{X}, \mathbf{U}) \\ \mathbf{X}_{N+1} &= \mathbf{X}_0 + \mathbf{w}^{\mathsf{T}}\mathbf{F}(\mathbf{X}, \mathbf{U}) \\ \mathbf{X}_0 &= \mathbf{x}_0 \end{aligned}$$

The Counterexample

$$\text{minimize} \quad \int_0^1 (u(t) - 1)^2 dt$$

$$\text{subject to} \quad \frac{d\mathbf{x}}{dt} = \lambda x + u, \quad 0 \le t \le +1, \quad \mathbf{x}(0) = 0.$$

Obvious solution: $u := 1, \quad x(t) = (e^{\lambda t} - 1)/\lambda.$

The Pseudospectral approximation

$$\text{minimize} \quad \sum_{i=1}^{N} w_i(u_i - 1)^2$$

$$\text{subject to} \quad \bar{\mathbf{D}}\mathbf{X} = \lambda\mathbf{X} + \mathbf{U},$$

where

$$\mathbf{X} = (x_1, x_2, \ldots, x_N)^\mathsf{T}, \quad x_0 = 0$$

$$\mathbf{U} = (u_1, u_2, \ldots, u_N)^\mathsf{T}$$

$\bar{\mathbf{D}}$ is N by N

If λ is an eigenvalue of $\mathbf{D}$, then the linear system for $\mathbf{X}$ is singular!

A Fix

- Dynamics: $\mathbf{X} = \lambda\bar{\mathbf{D}}^{-1}\mathbf{X} + \bar{\mathbf{D}}^{-1}\mathbf{U}$ where $\rho(\bar{\mathbf{D}}^{-1}) \le 2/3$ for $N \ge 2$
- Scaling: If the time interval is scaled by h, then $\bar{\mathbf{D}}^{-1}$ scales by h.
- hp: If we partition time interval into subintervals of width h, and use a pseudospectral scheme on each subinterval, then $\lambda\bar{\mathbf{D}}^{-1} = O(h) \to \mathbf{0}$ as $h \to 0$. Hence, the linear system for $\mathbf{X}$ is nonsingular when h is sufficiently small.
- Alternatively: Since $\rho(\mathbf{D}^{-1})$ tends to zero as N tends to infinity, it follows that by taking N sufficiently large, $\lambda\rho(\bar{\mathbf{D}}^{-1})$ tends to zero as $N \to \infty$ and the linear system for $\mathbf{X}$ becomes invertible.
- Note: Gaussian elimination with partial pivots should not work since the error could grow like 2^n in worst case; nonetheless, Gaussian elimination is routinely used to solve $\mathbf{Ax} = \mathbf{b}$.

Euler Discrete Control Problem

$$\min \Phi(\mathbf{x}_N)$$

$$\text{subject to} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + h\mathbf{f}(\mathbf{x}_k, \mathbf{u}_k), \quad 0 \le k \le N-1.$$

$h = 2/N$ = mesh spacing

Convergence Theory:

- $\mathbf{x}_k^* - \mathbf{x}^*(t_k) = O(h) = \mathbf{u}_k^* - \mathbf{u}^*(t_k)$
- theory developed in papers of Dontchev, Hager, Malanowski, Veliov
- Need $N \approx 1,000,000$ for error $\approx 10^{-6}$.

Pseudospectral Approach

- Approximate $\mathbf{x}$ by a polynomial
- Use collocation for system dynamics
- The hope: for $N = 10$ the error $\approx 10^{-6}$.
- Lobatto collocation: Fahroo, Kang, Ross, and Pietz
- Radau collocation: Larry Biegler and Shiva Kameswaran, Fahroo and Ross, Benson, Darby, Francilon, Garg, Hager, Huntington, Patterson, Rao
- Gauss collocation: Benson, Garg, Hager, Huntington, Patterson, Rao

Continuous Optimality Conditions (Pontryagin Minimum Principle)

$$
\begin{aligned}
\boldsymbol{\lambda}(-1) &= \boldsymbol{\mu} \\
\boldsymbol{\lambda}(1) &= \nabla\Phi(\mathbf{x}(1)) \\
\boldsymbol{\lambda}'(t) &= -\nabla_x\langle\boldsymbol{\lambda}(t), \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))\rangle \\
\mathbf{0} &= \nabla_u\langle\boldsymbol{\lambda}(t), \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))\rangle
\end{aligned}
$$

KKT Conditions

- Lagrangian: $\langle \mathbf{A}, \mathbf{B} \rangle = \sum_i \sum_j A_{ij} B_{ij}$

$$\mathcal{L}(\boldsymbol{\Lambda}, \boldsymbol{\Lambda}_{N+1}, \boldsymbol{\mu}, \mathbf{X}, \mathbf{X}_{N+1}, \mathbf{U}) = \Phi(\mathbf{X}_{N+1}) + \langle \boldsymbol{\Lambda}, \mathbf{F}(\mathbf{X}, \mathbf{U}) - \mathbf{D}\mathbf{X} \rangle$$
$$+ \langle \boldsymbol{\Lambda}_{N+1}, \mathbf{w}^{\mathsf{T}} \mathbf{F}(\mathbf{X}, \mathbf{U}) + \mathbf{X}_0 - \mathbf{X}_{N+1} \rangle + \langle \boldsymbol{\mu}, \mathbf{x}_0 - \mathbf{X}_0 \rangle.$$

- Partial with respect to $\mathbf{X}_{N+1}$:

$$\boldsymbol{\Lambda}_{N+1} = \nabla_X \Phi(\mathbf{X}_{N+1})$$

- Partial with respect to $\mathbf{X}_j$:

$$\sum_{i=1}^{N} D_{ij} \boldsymbol{\Lambda}_i = \nabla_X \langle \boldsymbol{\Lambda}_j, \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j) \rangle + w_j \nabla_X \langle \boldsymbol{\Lambda}_{N+1}, \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j) \rangle, \quad 1 \le j \le N.$$

- Partial with respect to $\mathbf{X}_0$:

$$\boldsymbol{\mu} = \boldsymbol{\Lambda}_{N+1} - \mathbf{D}_0^{\mathsf{T}} \boldsymbol{\Lambda},$$

- Partial with respect to $\mathbf{U}_j$:

$$\nabla_U \langle \boldsymbol{\Lambda}_j, \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j) \rangle + w_j \nabla_U \langle \boldsymbol{\Lambda}_{N+1}, \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j) \rangle = 0, \quad 1 \le j \le N$$

Transformed Adjoint and Differentiation Matrix

$$
\begin{aligned}
\boldsymbol{\lambda}_i &= \boldsymbol{\Lambda}_i / w_i + \boldsymbol{\Lambda}_{N+1}, \quad 1 \le i \le N \\
\boldsymbol{\lambda}_{N+1} &= \boldsymbol{\Lambda}_{N+1} \\
\boldsymbol{\lambda}_0 &= \boldsymbol{\Lambda}_{N+1} - \mathbf{D}_0^\top \boldsymbol{\Lambda} \\
D_{ij}^\dagger &= -\frac{w_j}{w_i} D_{ji}, \quad (i,j) = 1, \ldots, N, \\
D_{i,N+1}^\dagger &= -\sum_{j=1}^{N} D_{ij}^\dagger, \quad i = 1, \ldots, N
\end{aligned}
$$

Transformed Adjoint Optimality Conditions

$$
\begin{aligned}
\boldsymbol{\lambda}_0 &= \boldsymbol{\mu} \\
\boldsymbol{\lambda}_{N+1} &= \nabla_X \Phi(\mathbf{X}_{N+1}) \\
\mathbf{D}^{\dagger}_{1:N}\boldsymbol{\lambda} + \mathbf{D}^{\dagger}_{N+1}\boldsymbol{\lambda}_{N+1} &= -\nabla_X \langle \boldsymbol{\lambda}, \mathbf{F}(\mathbf{X}, \mathbf{U}) \rangle \\
\boldsymbol{\lambda}_0 &= \boldsymbol{\lambda}_{N+1} + \sum_{j=1}^{N} w_j \nabla_X \langle \boldsymbol{\lambda}_j, \mathbf{f}(\mathbf{X}_j, \mathbf{U}_j) \rangle \\
\mathbf{0} &= \nabla_U \langle \boldsymbol{\lambda}, \mathbf{F}(\mathbf{X}, \mathbf{U}) \rangle
\end{aligned}
$$

Properties of $\mathbf{D}$ and $\mathbf{D}^\dagger$

1. $\mathbf{D}$ and $\mathbf{D}^\dagger$ are both differentiation matrices

2. $\mathbf{D}$ operates on polynomial values $p_i = p(\tau_i)$, $0 \le i \le N$:
$$(\mathbf{Dp})_i = p'(\tau_i), \quad 1 \le i \le N$$

3. $\mathbf{D}^\dagger$ operates on polynomial values $q_i = q(\tau_i)$, $1 \le i \le N+1$:
$$(\mathbf{D}^\dagger\mathbf{q})_i = q'(\tau_i), \quad 1 \le i \le N$$

4. $\mathbf{D}_{1:N}$ and $\mathbf{D}^\dagger_{1:N}$ are both invertible

5. $\mathbf{D}_{1:N}^{-1}\mathbf{D}_0 = -\mathbf{1} = (\mathbf{D}^\dagger_{1:N})^{-1}\mathbf{D}^\dagger_{N+1}$

Inverses of $\mathrm{D}_{1:N}$ and $\mathrm{D}^{\dagger}_{1:N}$

$$
\begin{aligned}
[\mathbf{D}^{-1}_{1:N}]_{ij} &= \int_{-1}^{\tau_i} L^{\dagger}_{j}(\tau)d\tau \\
[(\mathbf{D}^{\dagger}_{1:N})^{-1}]_{ij} &= \int_{+1}^{\tau_i} L^{\dagger}_{j}(\tau)d\tau \\
L^{\dagger}_{j}(\tau) &= \prod_{\substack{i=1 \\ i\neq j}}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i}
\end{aligned}
$$

Compact Transformed Optimality Conditions

$$
\begin{aligned}
\mathbf{X}_i &= \mathbf{X}_0 + \mathbf{A}_i \mathbf{F}(\mathbf{X}, \mathbf{U}), && 1 \le i \le N+1 \\
\boldsymbol{\lambda}_i &= \boldsymbol{\lambda}_{N+1} - \mathbf{B}_i \nabla_X \langle \boldsymbol{\lambda}, \mathbf{F}(\mathbf{X}, \mathbf{U}) \rangle, && 0 \le i \le N \\
\boldsymbol{\lambda}_{N+1} &= \nabla_X \Phi(\mathbf{X}_{N+1}) \\
\mathbf{0} &= \nabla_U \langle \boldsymbol{\lambda}, \mathbf{F}(\mathbf{X}, \mathbf{U}) \rangle \\
\mathbf{A}_{1:N} &= \mathbf{D}_{1:N}^{-1}, \qquad \mathbf{A}_{N+1} = \mathbf{w}^{\mathsf{T}} \\
\mathbf{B}_{1:N} &= (\mathbf{D}_{1:N}^{\dagger})^{-1}, \quad \mathbf{B}_0 = -\mathbf{w}^{\mathsf{T}}
\end{aligned}
$$

Continuous Optimality Conditions (Pontryagin Minimum Principle)

$$
\begin{aligned}
\mathbf{x}'(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\
\boldsymbol{\lambda}'(t) &= -\nabla_x \langle \boldsymbol{\lambda}(t), \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \rangle \\
\boldsymbol{\lambda}(1) &= \nabla \Phi(\mathbf{x}(1)) \\
\mathbf{0} &= \nabla_u \langle \boldsymbol{\lambda}(t), \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \rangle
\end{aligned}
$$

Comment and Question

- If some given $\mathbf{U}$, $\boldsymbol{\lambda}$ and $\mathbf{X}$ satisfy the state and costate equations, then $\nabla_U \langle \boldsymbol{\lambda}, \mathbf{F}(\mathbf{X}, \mathbf{U}) \rangle$ is the gradient of objective function with respect to the control.

- What are the eigenvalues of $\mathbf{D}_{1:N}$?

- Suppose $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \gamma \mathbf{x} + \mathbf{g}(\mathbf{u})$:

$$\mathbf{D}_{1:N} \mathbf{X}_{1:N} = \gamma \mathbf{X}_{1:N} - \mathbf{D}_0 \mathbf{X}_0 + \mathbf{G}(\mathbf{U})$$

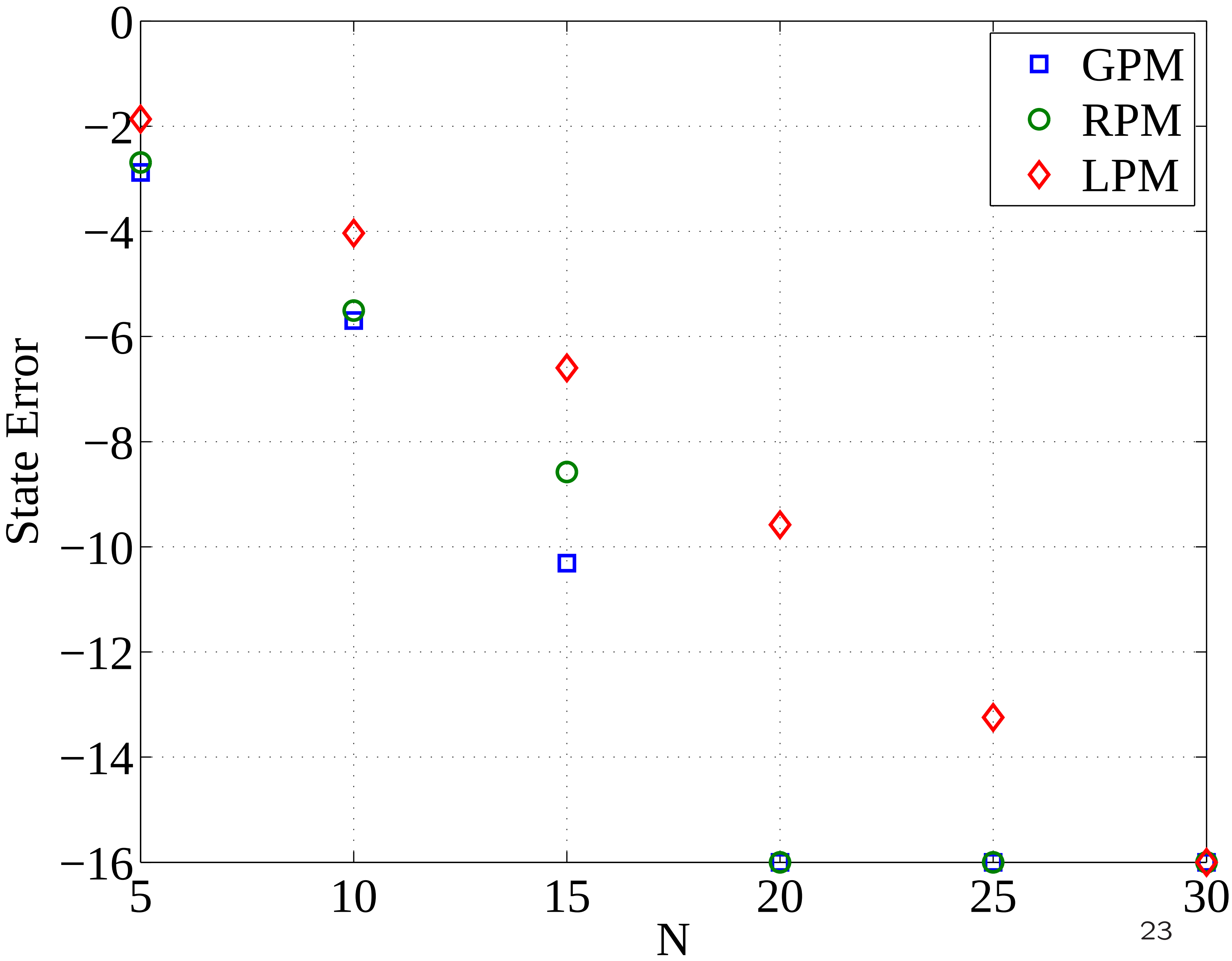
  When γ is an eigenvalue of $\mathbf{D}_{1:N}$, cannot solve for $\mathbf{X}_{1:N}$ as a function of $\mathbf{U}$.

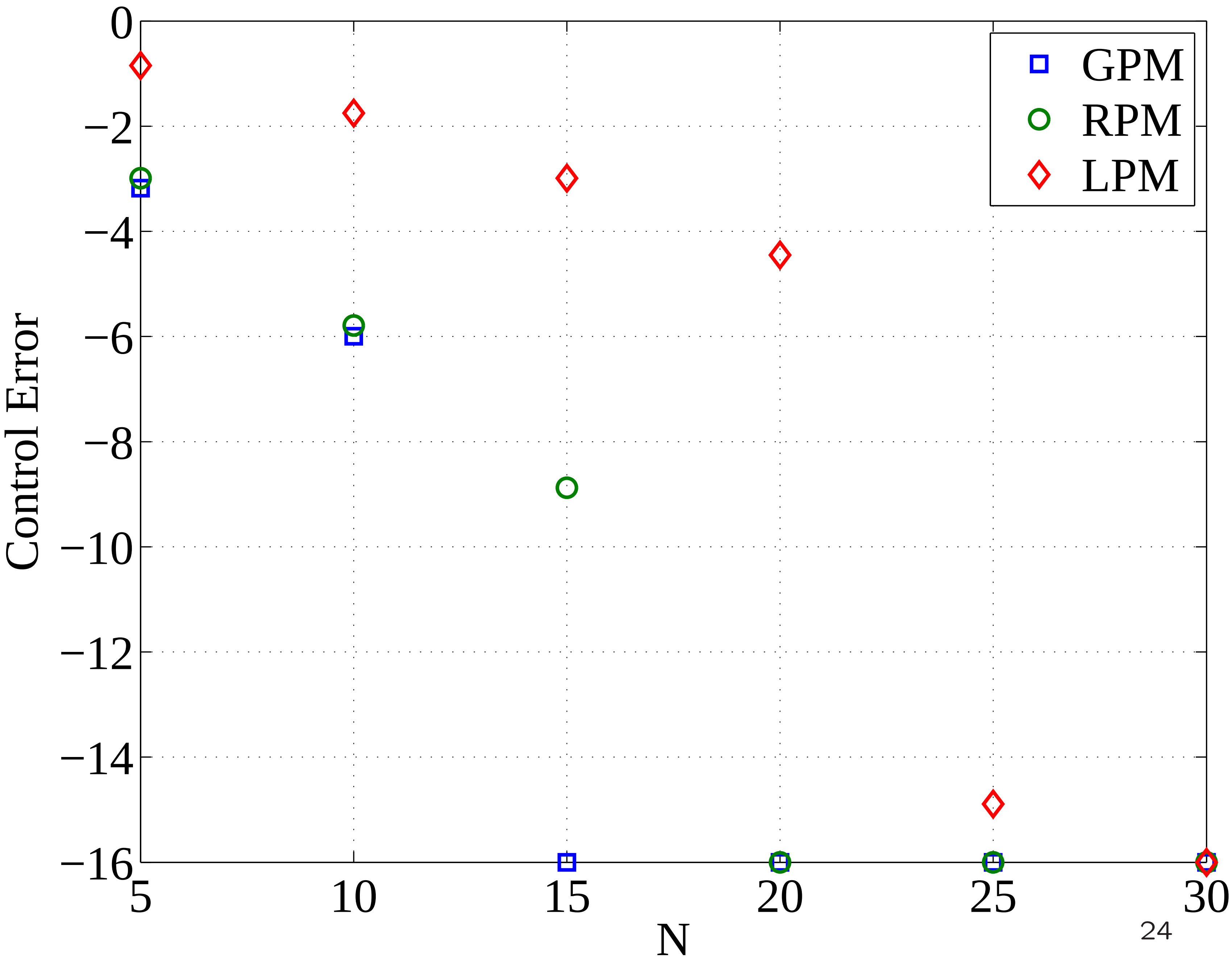
Example

$$\min -y(5) \qquad \text{subject to} \qquad y' = -y + yu - u^2, \quad y(0) = 1.$$

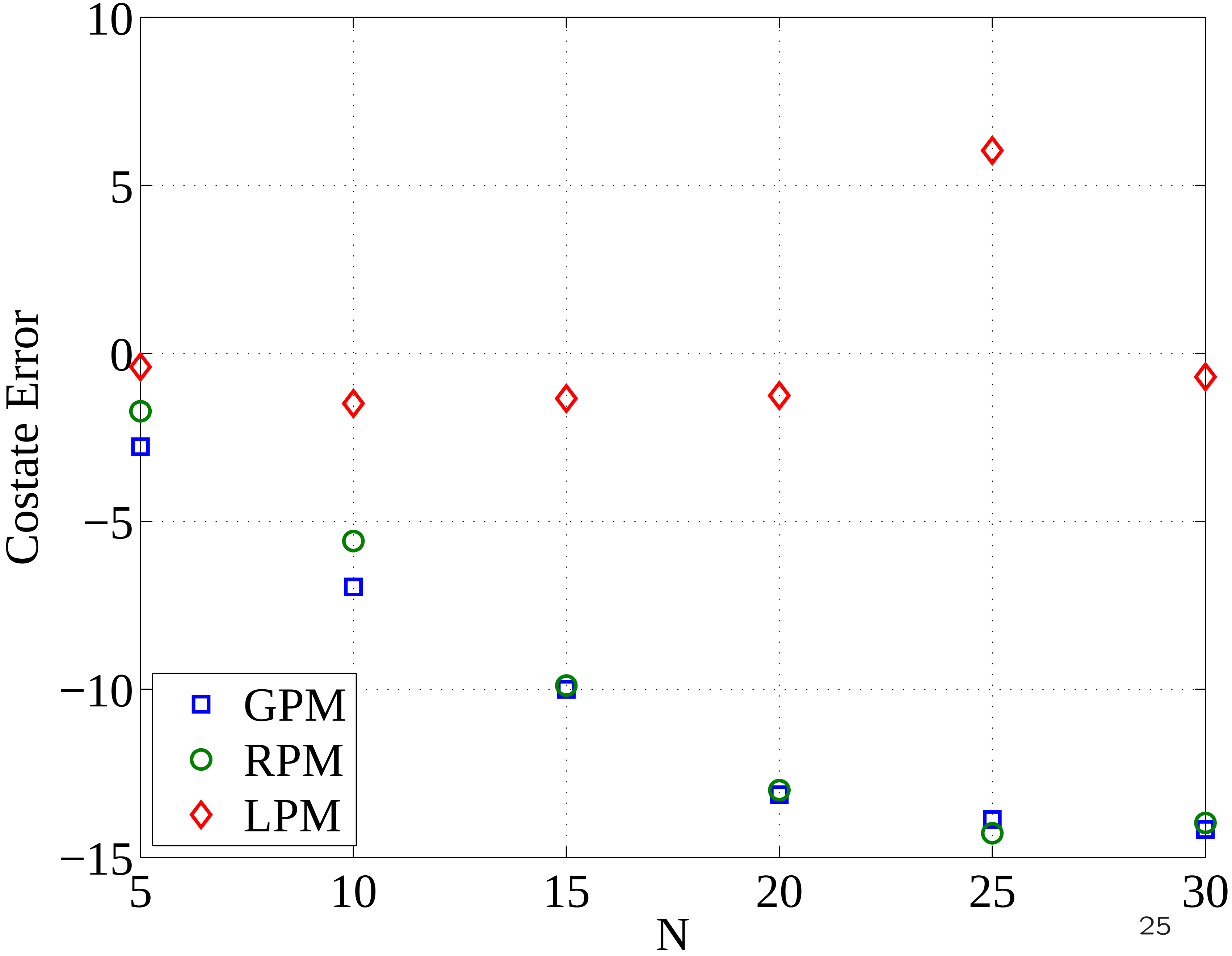
Solution:

$$\begin{aligned}
y^*(t) &= 4/(1 + 3\exp(t)), \\
u^*(t) &= y^*(t)/2, \\
\lambda_y^*(t) &= -\exp(2\ln(1 + 3\exp(t)) - t)/(\exp(-5) + 6 + 9\exp(5)).
\end{aligned}$$


GPM
RPM
LPM
State Error
N
0
−2
−4
−6
−8
−10
−12
−14
−16
5
10
15
20
25
30



0
−2
−4
−6
−8
−10
−12
−14
−16
Control Error
5
10
15
20
25
30
N
GPM
RPM
LPM



Costate Error
N
GPM
RPM
LPM
10
5
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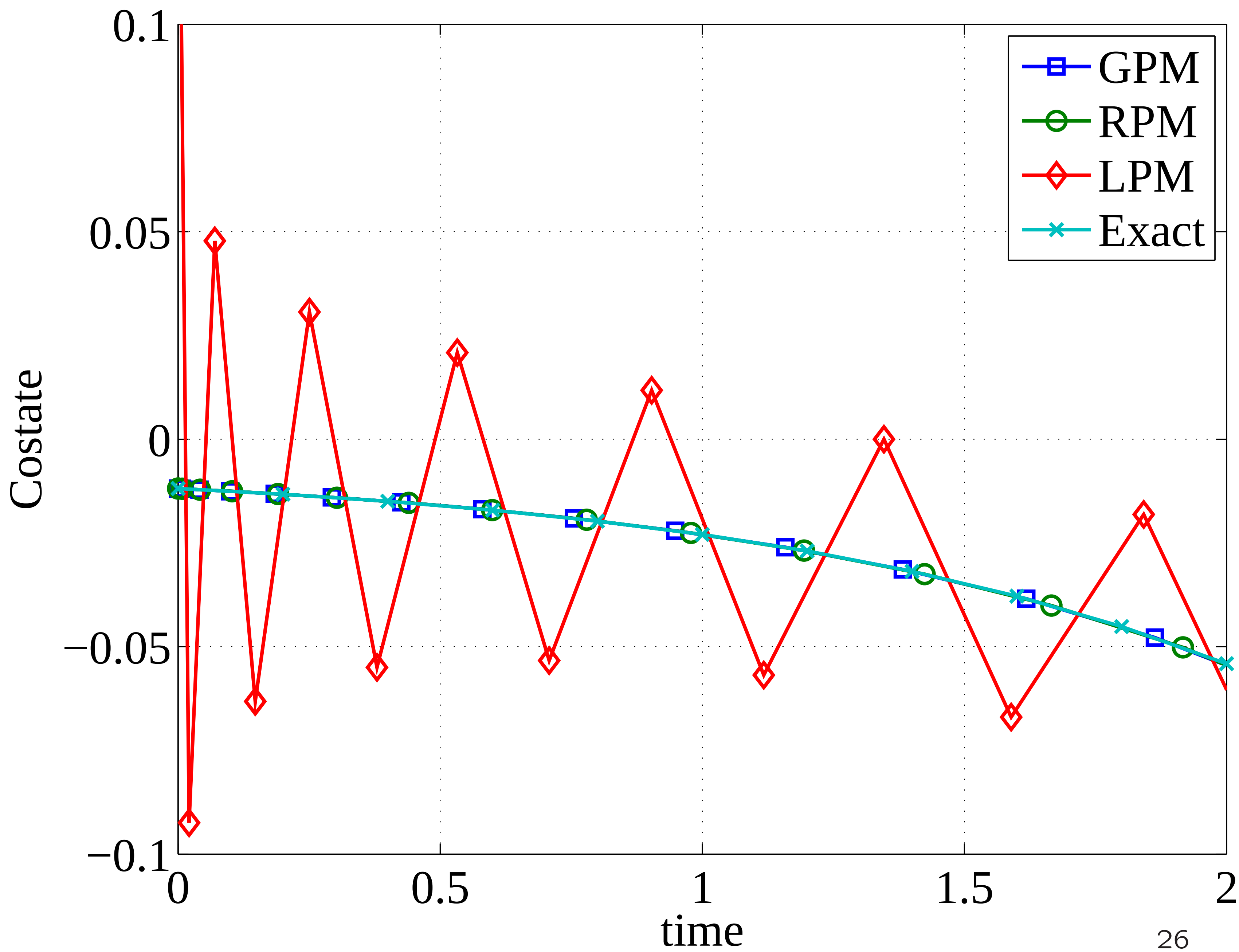
20
25
30



GPM
RPM
LPM
Exact
Costate
time
0.1
0.05
0
−0.05
−0.1
0
0.5
1
1.5
2

Journal articles at

http://www.math.ufl.edu/~hager/papers/Control