Zero dynamics of linear DAEs

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Systems class: (E, A, B, C)

$$\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}$$

$E, A \in \mathbb{R}^{n\times n}$

$B, C^\top \in \mathbb{R}^{n\times m}$

Regularity: $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

Solultions: $(x, u, y) \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}^1(\mathbb{R};\mathbb{R}^m)$,
ν index of $sE - A$

Transfer function $G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m\times m}$

has proper inverse: $\lim_{s\to\infty} G(s)^{-1} \in \mathbb{R}^{m\times m}$

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Def: *non-positive strict relative degree*

$$\operatorname{sr\,deg} G(s) = \sup\left\{ k \in \mathbb{Z} \;\middle|\; \lim_{s\to\infty} s^k\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \right\} \le 0 \quad \text{exists}$$

Strict relative degree and proper inverse

Prop

$$\operatorname{sr\,deg} G(s) \le 0 \quad \overset{\Longrightarrow}{\underset{\text{i.a.}}{\not\Longleftarrow}} \quad G(s) \text{ has 'proper inverse'}$$

Proof

$\rho \le 0$ is largest integer : $\lim_{s\to\infty} s^\rho \, G(s) \in \mathbf{Gl}_m(\mathbb{R})$

$\Longrightarrow$

$$G(s) = P(s) + G_{\rm sp}(s), \qquad P(s) = \sum_{i=0}^N P_i s^i$$

$\Longrightarrow$

$$\operatorname{sr\,deg} G(s) = -\operatorname{sr\,deg} P(s), \quad P_N \in \mathbf{Gl}_m(\mathbb{R})$$

$\Longrightarrow$

$\exists \; P^{-1}(s)$

$\Longrightarrow$

Sherman-Morrison-Woodbury formula:

$$G^{-1}(s) = P^{-1}(s) - P^{-1}(s) G_{\rm sp}(s) \left[I + P^{-1}(s) G_{\rm sp}(s) \right]^{-1} P^{-1}(s)$$

$\Longrightarrow$

$\exists \; G^{-1}(s)$ and $G^{-1}(s)$ proper $\square$

Theorem: Zero Dynamics Form

$$\boxed{\begin{aligned} E\,\dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}}$$

& $C(sE - A)^{-1}B$ has proper inverse

$\Longrightarrow$

$$\exists\ T \in \mathbf{Gl}_n(\mathbb{R})\ :\ \begin{pmatrix} y(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = T\,x(t)$$

solves

$$\boxed{\begin{aligned} 0 &= A_{11}\,y(t) + A_{12}\,x_2(t) + u(t) \\ \dot{x}_2(t) &= Q\,x_2(t) + A_{21}\,y(t) \\ x_3(t) &= \sum_{i=0}^{\nu-1} N_{33}^i\,E_{31}\,y^{(i+1)}(t) \\ x_4(t) &= 0 \end{aligned}}$$

unique: $\dim x_i$, A_{11}

unique mod similarity: Q, N_{33}

Negative strict relative degree

$$
\begin{aligned}
0 &= A_{11}\ y(t) + A_{12}\ x_2(t) + u(t)\\
\dot{x}_2(t) &= Q\ x_2(t) + A_{21}\ y(t)\\
x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\ y^{(i+1)}(t)\\
x_4(t) &= 0
\end{aligned}
$$

$$C(sE-A)^{-1}B = -\left(A_{11} + A_{12}(sI-Q)^{-1}A_{21}\right)^{-1}$$

Prop

$$
\begin{aligned}
E\ \dot{x}(t) &= A\ x(t) + B\ u(t)\\
y(t) &= C\ x(t)
\end{aligned}
$$

has strict relative degree $\rho \leq -1$

$\iff$

$$A_{11} = 0 \quad \wedge \quad \mathrm{sr\,deg}\ \left(A_{12}(sI-Q)^{-1}A_{21}\right) = -\rho$$

$\Longrightarrow$

$$
\begin{aligned}
\dot{x}_2(t) &= Q\ x_2(t) + A_{21}\ y(t)\\
u(t) &= -A_{12}\ x_2(t)
\end{aligned}
$$

Zero dynamics

$$\boxed{\begin{aligned} E\,\dot x(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}}$$

$C(sE - A)^{-1}B$ has proper inverse

Def

$$\mathcal{ZD}_{(E,A,B,C)} := \left\{ (x,u,y) \;\middle|\; \begin{array}{l} (x,u,y) \text{ solves } \text{(E,A,B,C)} \\ \text{and} \quad y \equiv 0 \end{array} \right\}$$

Prop

$$(x,u,y) \in \mathcal{ZD}_{(E,A,B,C)} \iff x(t) \in \mathcal{V}^*(A,E,B;\ker C) \quad \forall t \in \mathbb{R}$$

where $\mathcal{V}^*$ denotes the maximal subspace such that

$$A\,\mathcal{V}^* \subset E\,\mathcal{V}^* + \operatorname{im} B \qquad \text{and} \qquad \mathcal{V}^* \in \ker C$$

Stable zero dynamics

Def

(E, A, B, C) *has* *stable zero dynamics* $\quad :\Longleftrightarrow$

$$(x, u, y) \in \mathcal{ZD}_{(E,A,B,C)} \quad \Longrightarrow \quad (x(t), u(t)) \to 0$$

Prop

$$(E, A, B, C) \text{ has stable zero dynamics} \quad \Longleftrightarrow \quad \sigma(Q) \subset \mathbb{C}_-$$

Proof

$$\begin{array}{rcl} 0 & = & A_{11}\ y(t) + A_{12}\ x_2(t) + u(t) \\ \dot{x}_2(t) & = & Q\ x_2(t) + A_{21}\ y(t) \\ x_3(t) & = & \sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\ y^{(i+1)}(t) \\ x_4(t) & = & 0 \end{array}$$

Theorem: Characterization of stable zero dynamics

Theorem

(E, A, B, C) has stable zero dynamics

$\iff$

$$\forall\, s \in \overline{\mathbb{C}}_+ \;:\; \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \neq 0$$

$\iff$

(i) (E, A, B, C) is stabilisable

(ii) (E, A, B, C) is detectable

(iii) $U(s)^{-1}\; C(sE - A)^{-1}B\; V(s)^{-1} = \operatorname{diag}\left\{ \frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_m(s)}{\psi_m(s)} \right\}$

$\forall s \in \overline{\mathbb{C}}_+ \;:\; \epsilon_i(s) \neq 0$

$\iff$

$\exists\, k^* \geq 0 \quad \forall\, k \in \mathbb{R}$ s.t. $|k| \geq k^* \quad : \; \lim_{t\to\infty} x(t) = 0$

where $x(\cdot)$ solves '$u(t) = ky(t)$ & (E, A, B, C)'

High-gain stabilization: sketch of the proof

$$
\begin{aligned}
0 &= A_{11}\ y(t) + A_{12}\ x_2(t) + u(t)\\
\dot{x}_2(t) &= Q\ x_2(t) + A_{21}\ y(t)\\
x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\ y^{(i+1)}(t)
\end{aligned}
$$

$$\overset{u(t)\,=\,k\,y(t)}{\Longrightarrow}$$

$$
\begin{aligned}
-(A_{11} + kI_m)\,y(t) &= A_{12}x_2(t)\\
\dot{x}_2(t) &= Qx_2(t) + A_{21}y(t)\\
x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\ y^{(i+1)}(t)
\end{aligned}
$$

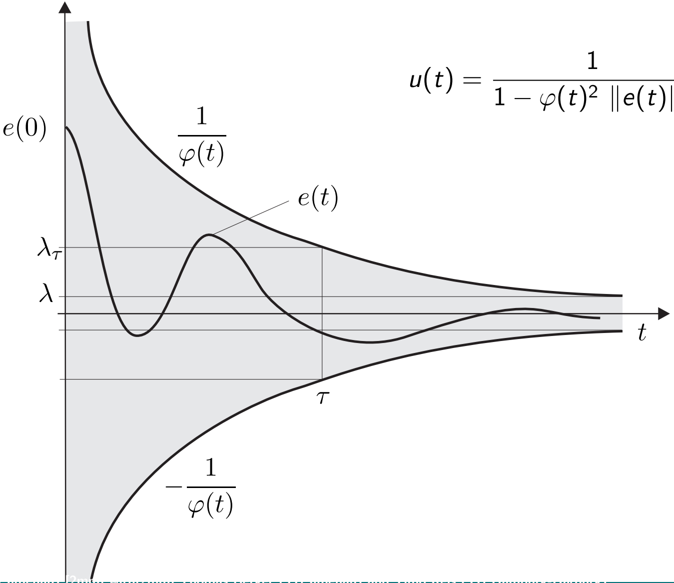
$$\overset{|k|\,>\,\|A_{11}\|}{\Longrightarrow}$$

$$
\begin{aligned}
\dot{x}_2(t) &= \left[Q - A_{21}(A_{11} + kI_m)^{-1}A_{12}\right]\ x_2(t) \overset{|k|\gg 1}{\approx} Q\,x_2(t)\\
x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\ y^{(i+1)}(t)
\end{aligned}
$$

□

Funnel

$$u(t) = \frac{1}{1 - \varphi(t)^2 \, \|e(t)\|^2} \, e(t)$$



Theorem: Funnel control

Suppose: (E, A, B, C) has stable zero dynamics

and $C(sE - A)^{-1}B$ has strict negative relative degree.

Then the funnel controller

$$
\boxed{
\begin{aligned}
u(t) &= k(t)\, e(t) \\
k(t) &= \frac{1}{1 - \varphi(t)^2 \, \|e(t)\|^2}
\end{aligned}
}
\qquad\qquad e(t) = y(t) - y_{\mathrm{ref}}(t)
$$

applied to (E, A, B, C) yields:

$$
x(\cdot) \in L^\infty,\ k(\cdot) \in L^\infty \qquad \wedge \qquad \exists\, \varepsilon > 0\ \forall\ t \geq 0:\ \|e(t)\| < \frac{1}{\varphi(t)} - \varepsilon\,.
$$