Analyzing Controllability and Observability by Exploiting the Stratification of Matrix Pairs

Bo Kågström

Department of Computing Science and HPC2N
Umeå University, Sweden

Based on joint work with

Alan Edelman, Erik Elmroth, Peder Johansson, and **Stefan Johansson**

MATRIX commercial

- **The Matrix**
- **The Matrix Reloaded**
- **The Matrix Revolutions**
- **The Matrix Stratifications**

Coming soon to a PC near you!

Dense and Structured Matrix Computations @ Umeå

THEORY – ALGORITHMS – SOFTWARE TOOLS

- **Theme 1:** Matrix Pencil Computations in Computer-Aided Control System Design
  - Ill-posed eigenvalue problems
  - Canonical forms (Jordan, Kronecker, staircase)
  - Generalized Schur forms (GUPTRI, QZ)
  - Subspaces: eigenvalue reordering
  - Matrix equations (Sylvester, Lyapunov, Riccati)
  - Functions of matrices
  - Perturbation theory, condition estimation and error bounds
  - Periodic counterparts
    - Periodic Riccati differential equations
- **Theme 2:** High Performance and Parallel Computing

Dense and Structured Matrix Computations @ Umeå

THEORY – ALGORITHMS – SOFTWARE TOOLS

- **Theme 2:** High Performance and Parallel Computing
  - Blocking for memory hierarchies (DM, SM, hybrid, multicore, GPGPUs)
    - Explicit (multi level) blocking
    - Recursive blocking
    - Blocked hybrid data structures
  - Library software
    - Contributions to LAPACK, ScaLAPACK, SLICOT, ESSL
    - Matrix equations: RECSY and SCASY
  - Novel parallel QR algorithm
    - 30 times faster than current ScaLAPACK implementation!
    - Solved 100000×100000 dense nonsymmetric eigenvalue problems!

Outline

- Some motivation and background to stratification of orbits and bundles:
  - Canonical forms and structure information
  - Matrix and pencil spaces
  - Graph representation of a closure hierarchy
  - Nilpotent matrix orbit stratification (7×7)
  - Matrix bundle stratification (4×4)
- Controllability and observability matrix pairs
  - System pencils and equivalence orbits and bundles
  - Canonical forms of pairs (Kronecker and Brunovsky)
  - Closure and cover relations
- Applications in control system design and analysis:
  - Mechanical system - uniform platform with 2 springs
  - Linearized Boeing 747 model

Stratification [*Oxford advanced learner's dictionary*]

The division of something into different layers or groups

Some motivation

- Computation of canonical forms (e.g., Jordan, Kronecker, Brunovsky) are ill-posed problems
  - small perturbation of input data may drastically change the computed structure
- Compute canonical structure information using so called staircase algorithms (orthogonal transformations)
- Need to provide the user with more information:
  - What other structures are nearby?
  - Upper and lower bounds to other structures
- Applications in, e.g., control system design
  - Controllability
  - Observability

Leverage on the thery of matrix spaces

- *Objective:* Make use of the geometry of matrix and matrix pencil spaces to solve nearness problems related to Jordan and Kronecker canonical forms
- *Tools:* The theory of stratification of orbits and bundles (and versal deformations)

Our program:

To understand qualitative and quantitative properties of nearby Jordan and Kronecker structures

- *Deliverables:* Interactive tools and algorithms that make these complex theories easily available to end users

Matrix and matrix pencil spaces

- An $n \times n$ matrix can be viewed as a point in n^2–dim space
- Numerical computations – move from point to point or manifold to manifold

Orbit of a matrix

$$\mathcal{O}(A) = \left\{PAP^{-1} : \det P \neq 0\right\}$$

Manifold of all matrices with Jordan Normal Form (JNF) of A

Orbit of a pencil

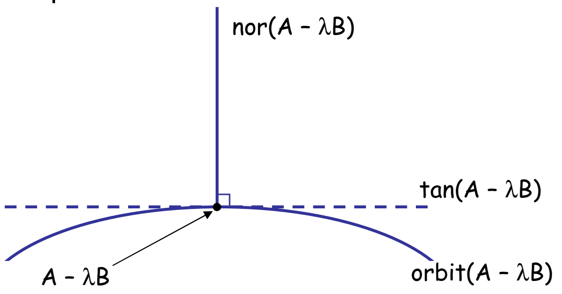
$$\mathcal{O}(A - \lambda B) = \{P(A - \lambda B)Q : \det P \det Q \neq 0\}$$

Manifold of all $m \times n$ pencils in $2mn$–dim space with the Kronecker Canonical Form (KCF) of $A - \lambda B$

- **Bundle:** $\mathcal{B}(\cdot)$ is the union of all orbits with the same canonical form but with eigenvalues unspecified

Dimensions and codimensions

$m \times n$ pencil case



- $\dim(\mathcal{O}(A-\lambda B)) \equiv \dim(\tan(\mathcal{O}(A-\lambda B))$
- $\operatorname{codim}(\mathcal{O}(A-\lambda B)) \equiv \dim(\operatorname{nor}(\mathcal{O}(A-\lambda B))$
- $\dim(\mathcal{O}(A-\lambda B)) + \operatorname{codim}(\mathcal{O}(A-\lambda B)) = 2mn$
- $\operatorname{codim}(\mathcal{B}(\cdot)) = \operatorname{codim}(\mathcal{O}(\cdot)) - k$,
  $k =$ number of unspecified eigenvalues

Stratification of orbits – matrix case

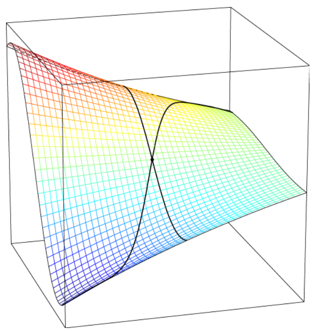
- Given a matrix and its orbit: What other structures are found within its closure?

Stratification:
The closure hierarchy of all possible Jordan structures

We make use of:

- Graphs to illustrate stratifications
- Dominance orderings for integer partitions in proofs and derivations

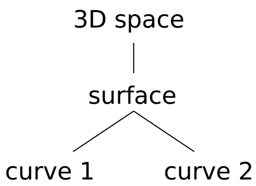
Closure hierarchy – graph representation



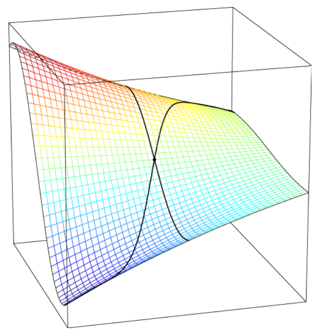
Surface: 2D manifold excluding the two 1D curves and the point

The 3D space covers the surface

The two curves are in the closure of the surface, which is in the closure of the 3D space



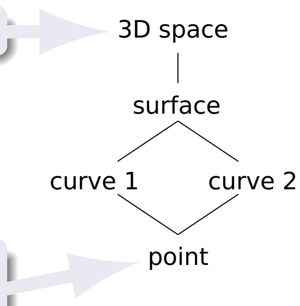
Closure hierarchy – graph representation



Surface: 2D manifold excluding the two 1D curves and the point

Most generic

Least generic (most degenerate)



Staircase form of nilpotent 7×7 matrix

$$Q^H(A+E)Q =$$

$$\begin{array}{ccc|cc|cc}
\multicolumn{3}{c|}{m_1} & \multicolumn{2}{c|}{m_2} & \multicolumn{2}{c}{m_3} \\
0 & 0 & 0 & x & x & x & x \\
 & 0 & 0 & x & x & x & x \\
 & & 0 & x & x & x & x \\
\hline
 & & & 0 & 0 & y & y \\
 & & & & 0 & y & y \\
\hline
 & & & & & 0 & 0 \\
 & & & & & & 0
\end{array}$$

Weyr: $(3, 2, 2)$

Q unitary

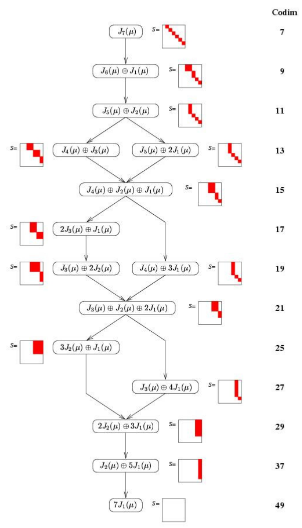
$$\begin{aligned}
m_1 &= 3 = \dim \mathcal{N}(A - \mu I), \\
m_1 + m_2 &= 5 = \dim \mathcal{N}((A - \mu I)^2) \\
m_1 + m_2 + m_3 &= 7 = \dim \mathcal{N}((A - \mu I)^3)
\end{aligned}$$

$$J_3(0) \oplus J_3(0) \oplus J_1(0)$$

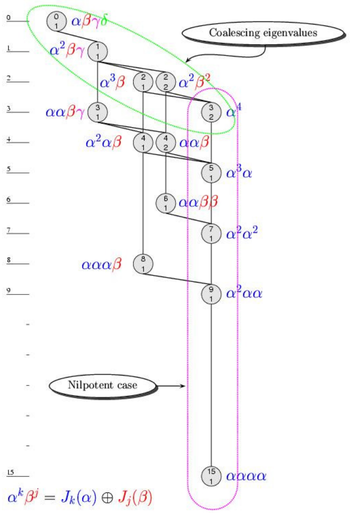
Segre: $(3, 3, 1)$

Nilpotent orbit stratification of 7×7 matrix

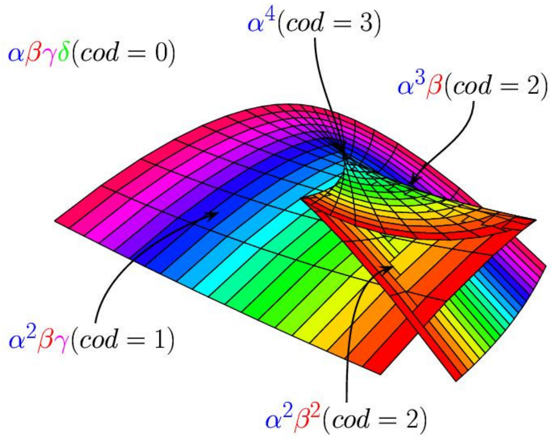
- Dominance ordering of the integer $n = 7$
- Deformations of stairs and ....
- ... versal deformations
- $A + Z(p)$
  $\text{span}(Z(p)) = \text{nor}(A)$



Stratification of 4×4 matrix bundle



Swallowtail – bundles of coalescing eigenvalues



Pencil representation

A (generalized) state-space system with the *state-space model*

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

can be represented in the form of a *system pencil*

$$\mathbf{S}(\lambda) = G - \lambda H = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},$$

with the corresponding general *matrix pencil* $G - \lambda H$

In short form, $\mathbf{S}(\lambda)$ is represented by a *matrix quadruple* (A, B, C, D) ($E = I$)

Controllable and observable systems

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

- **Controllable system:** There exists an input signal (vector) $u(t), t_0 \leq t \leq t_f$ that takes every state variable from an initial state $x(t_0)$ to a desired final state x_f in finite time.
- **Observable system:** If it possible to find the initial state $x(t_0)$ from the input signal $u(t)$ and the output signal $y(t)$ measured over a finite interval $t_0 \leq t \leq t_f$.

Matrix pairs

Consider the *controllability pair* (A, B) and the *observability pair* (A, C), associated with the particular systems:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

System pencil representations:

$$\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

Matrix pairs

An orbit of a matrix pair

A manifold of equivalent matrix pairs:

$$\mathcal{O}(A, B) = \left\{ P(A, B) \begin{bmatrix} P^{-1} & 0 \\ R & Q^{-1} \end{bmatrix} : \det P \cdot \det Q \neq 0 \right\}$$

$$(PAP^{-1} + PBR, PBQ^{-1})$$

A bundle of a matrix pair

The union of all orbits with the same canonical form but with unspecified eigenvalues

$$\mathcal{B}(A, B) = \bigcup_{\mu_i} \mathcal{O}(A, B)$$

Canonical forms

A canonical form is the simplest or most symmetrical form a matrix or matrix pencil can be reduced to

- Matrices – Jordan canonical form
- Matrix pencils – Kronecker canonical form
- System pencils – (generalized) Brunovsky canonical form

A canonical form reveals the canonical structure information from which the system characteristics are deduced

All matrix pairs in the same orbit has the same canonical form

Kronecker canonical form

Any matrix pencil $G - \lambda H$ or system pencil $\mathbf{S}(\lambda)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations (U and V non-singular):

$$U^{-1}(\mathbf{S}(\lambda))V =$$
$$\text{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t), N_{s_1}, \ldots, N_{s_k}, L^T_{\eta_1}, \ldots, L^T_{\eta_q})$$

Singular part:

- $L_{\epsilon_1}, \ldots, L_{\epsilon_p}$ – Right singular blocks
- $L^T_{\eta_1}, \ldots, L^T_{\eta_q}$ – Left singular blocks

$$L_k = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}$$

Kronecker canonical form

Any matrix pencil $G - \lambda H$ or system pencil $\mathbf{S}(\lambda)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations (U and V non-singular):

$$U^{-1}(\mathbf{S}(\lambda))V =$$
$$\text{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t), N_{s_1}, \ldots, N_{s_k}, L^T_{\eta_1}, \ldots, L^T_{\eta_q})$$

Singular part:

- $L_{\epsilon_1}, \ldots, L_{\epsilon_p}$ – Right singular blocks
- $L^T_{\eta_1}, \ldots, L^T_{\eta_q}$ – Left singular blocks

$$J_k(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_i - \lambda \end{bmatrix}$$

Regular part:

- $J(\mu_1), \ldots, J(\mu_t)$ – Each $J(\mu_i)$ is block-diagonal with *Jordan* blocks corresponding to the finite eigenvalue μ_i
- $N_{s_1}, \ldots, N_{s_k}$ – Jordan blocks corresponding to the infinite eigenvalue

Kronecker canonical form – Matrix pairs

- $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 \end{bmatrix}$ has full row rank $\Rightarrow$ KCF of $\mathbf{S}_C(\lambda)$ can only have finite eigenvalues (uncontrollable modes) and L_k blocks:

$$U^{-1}\mathbf{S}_C(\lambda)V = \text{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t))$$

- $\mathbf{S}_O(\lambda) = \begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ has full column rank $\Rightarrow$ KCF of $\mathbf{S}_O(\lambda)$ can only have finite eigenvalues (unobservable modes) and L^T_k blocks:

$$U^{-1}\mathbf{S}_O(\lambda)V = \text{diag}(J(\mu_1), \ldots, J(\mu_t), L^T_{\eta_1}, \ldots, L^T_{\eta_p})$$

Brunovsky canonical form

Given a matrix pair (A, B) or (A, C), there exists a *feedback equivalent* matrix pair in *Brunovsky canonical form* (BCF), such that

$$P\begin{bmatrix} A-\lambda I_n & B\end{bmatrix}\begin{bmatrix} P^{-1} & 0\\ R & Q^{-1}\end{bmatrix} = \left[\begin{array}{cc|c} A_\epsilon & 0 & B_\epsilon \\ 0 & A_\mu & 0\end{array}\right]$$

or

$$\begin{bmatrix}P & S\\0 & T\end{bmatrix}\begin{bmatrix}A-\lambda I_n\\ C\end{bmatrix}P^{-1} = \left[\begin{array}{cc} A_\eta & 0\\ 0 & A_\mu \\ \hline C_\eta & 0\end{array}\right]$$

respectively, where

- (A_ϵ, B_ϵ) – *controllable* and corresponds to the L blocks
- (A_η, C_η) – *observable* and corresponds to the L^T blocks
- A_μ – block diagonal with Jordan blocks and corresponds to the *uncontrollable* and *unobservable eigenvalues*, respectively

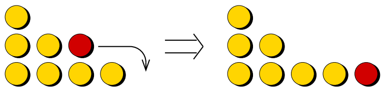
Canonical structure indices

- $\mathcal{R} = (r_0, r_1, \ldots)$ where $r_i = \#L_k$ blocks with $k \geq i$
- $\mathcal{L} = (l_0, l_1, \ldots)$ where $l_i = \#L_k^T$ blocks with $k \geq i$
- $\mathcal{J}_{\mu_i} = (j_1, j_2, \ldots)$ where $j_i = \#J_k(\mu_i)$ blocks with $k \geq i$. $\mathcal{J}_{\mu_i}$ is known as the *Weyr characteristics* of the finite eigenvalue μ_i
- $\mathcal{N} = (n_1, n_2, \ldots)$ where $n_i = \#N_k$ with $k \geq i$. $\mathcal{N}$ is known as the *Weyr characteristics* of the infinite eigenvalue

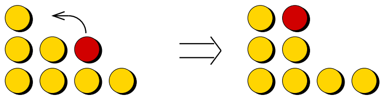
Integer partitions

A *partition* ν of an integer K is defined as $\nu = (\nu_1, \nu_2, \ldots)$ where $\nu_1 \geq \nu_2 \geq \cdots \geq 0$ and $K = \nu_1 + \nu_2 + \ldots$.

Minimum rightward coin move: rightward *one* column or downward *one* row (keep partition monotonic)



Minimum leftward coin move: leftward *one* column or upward *one* row (keep partition monotonic)



Creates nearest neighbours in the dominance ordering of K [Edelman, Elmroth & Kågström; 1999]

Covering relations for (A, B) orbits

Find: canonical structures that are nearest neighbours

Theorem

Given the structure integer partitions $\mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of (A, B), one of the following if-and-only-if rules finds $(\tilde{A}, \tilde{B})$ such that:

$\mathcal{O}(A, B)$ **covers** $\mathcal{O}(\tilde{A}, \tilde{B})$

1. *Minimum rightward coin move in $\mathcal{R}$*
2. *If the rightmost column in $\mathcal{R}$ is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially)*
3. *Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$*

$\mathcal{O}(A, B)$ **is covered by** $\mathcal{O}(\tilde{A}, \tilde{B})$

1. *Minimum leftward coin move in $\mathcal{R}$*
2. *If the rightmost column in some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$*
3. *Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$*

Rules 1 and 2: Coin moves that affect $r_0 = \#L_k$-blocks are not allowed

Covering relations for (A, B) bundles

Theorem

Given the structure integer partitions $\mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of (A, B), one of the following if-and-only-if rules finds $(\tilde{A}, \tilde{B})$ such that:

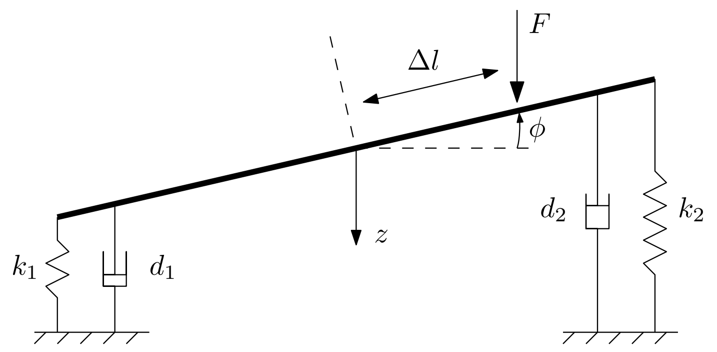
$\mathcal{B}(A, B)$ **covers** $\mathcal{B}(\tilde{A}, \tilde{B})$

1. *Minimum* rightward *coin move in $\mathcal{R}$*
2. *If the rightmost column in $\mathcal{R}$ is one single coin, move that coin to the first column of $\mathcal{J}_{\mu_i}$ for a new eigenvalue μ_i*
3. *Minimum* leftward *coin move in any $\mathcal{J}_{\mu_i}$*
4. *Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins*

$\mathcal{B}(A, B)$ **is covered by** $\mathcal{B}(\tilde{A}, \tilde{B})$

1. *Minimum* leftward *coin move in $\mathcal{R}$, without affecting r_0*
2. *If some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$*
3. *Minimum* rightward *coin move in any $\mathcal{J}_{\mu_i}$*
4. *For any $\mathcal{J}_{\mu_i}$, divide the set of coins into two new sets so that their union is $\mathcal{J}_{\mu_i}$*

Example 1 – Mechanical system



A uniform platform with mass m and length $2l$, supported in both ends by springs

The control parameter of the system is the force F applied at distance Δl from the center of the platform

Mechanical system – State-space model

By linearizing the equations of motion near the equilibrium the system can be expressed by the linear state-space model $\dot{x} = Ax(\tau) + Bu(\tau)$ [A. Mailybaev '03]:

$$\begin{bmatrix} \omega\dot{z}/l \\ \omega\dot{\phi} \\ \omega^2\ddot{z}/l \\ \omega^2\ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_1 & -c_2 & -f_1 & -f_2 \\ -3c_2 & -3c_1 & -3f_2 & -3f_1 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega\dot{z}/l \\ \omega\dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3\Delta \end{bmatrix} \frac{\omega^2}{ml} F$$

where Fixed elements!

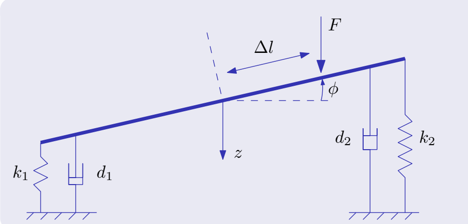
$$c_1 = \frac{(k_1 + k_2)\omega^2}{m}, \; c_2 = \frac{(k_1 - k_2)\omega^2}{m},$$
$$f_1 = \frac{(d_1 + d_2)\omega}{m}, \quad f_2 = \frac{(d_1 - d_2)\omega}{m},$$

and $\tau = t/\omega$ where ω is a time scale coefficient

Mechanical system – Canonical forms

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \end{bmatrix}$ is

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.0004 & 0 & -0.027 & 0 & 1 \\ 0 & -0.0012 & 0 & -0.08 & 0 \end{array}\right] - \lambda \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$



Mechanical system – Canonical forms

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \end{bmatrix}$ is

$$\mathbf{U}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.0004 & 0 & -0.027 & 0 & 1 \\ 0 & -0.0012 & 0 & -0.08 & 0 \end{array}\right] - \lambda \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]\mathbf{V}^{-1}$$

Mechanical system – Canonical forms

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \end{bmatrix}$ is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.02 & 0 \\ 0 & 0 & 0 & 0 & -0.06 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = L_2 \oplus J_1(-0.02) \oplus J_1(-0.06)$$

Kronecker canonical form

Mechanical system – Canonical forms

With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \end{bmatrix}$ is

$$\mathbf{P}_{\mathbf{row}}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.02 & 0 \\ 0 & 0 & 0 & 0 & -0.06 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{P}_{\mathbf{col}}$$

Mechanical system – Canonical forms

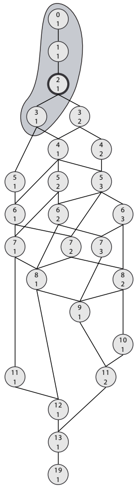
With the parameters $d_1 = 4$, $d_2 = 4$, $k_1 = 6$, $k_2 = 6$, $m = 3$, $l = 1$, $\omega = 0.01$, and $\Delta = 0$, the resulting controllability system pencil $\mathbf{S}_C(\lambda) = \begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \end{bmatrix}$ is

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.02 & 0 & 0 \\ 0 & 0 & 0 & -0.06 & 0 \end{array}\right] - \lambda \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Uncontrollable eigenvalues (modes)

Brunovsky canonical form

Mechanical system – Illustrating the bundle stratification

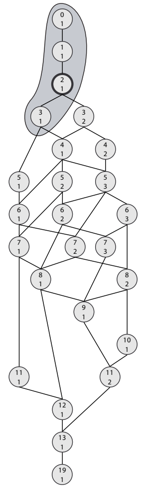


The software tool StratiGraph is used for computing and visualizing the stratification

[Elmroth, P. Johansson & Kågström; 2001]

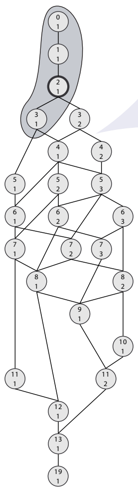
[P. Johansson; PhD Thesis 2006]

Mechanical system – Illustrating the bundle stratification



Each node represents a bundle (or orbit) of a canonical structure

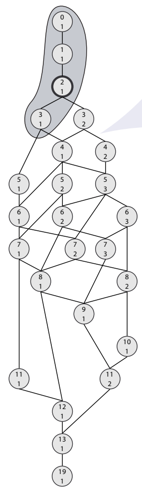
Mechanical system – Illustrating the bundle stratification



Each edge represents a cover relation

It is always possible to go from any canonical structure (node) to another higher up in the graph by a small perturbation

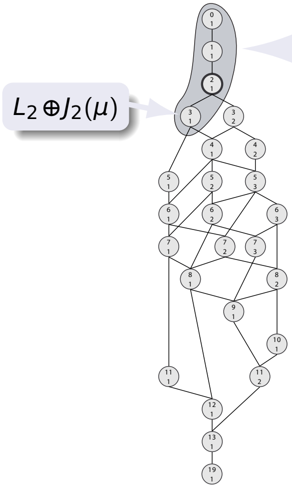
Mechanical system – Illustrating the bundle stratification



Each edge represents a cover relation

A cover relation is determined by the combinatorial rules acting on the integer sequences representing the canonical structure information

Mechanical system – Illustrating the bundle stratification



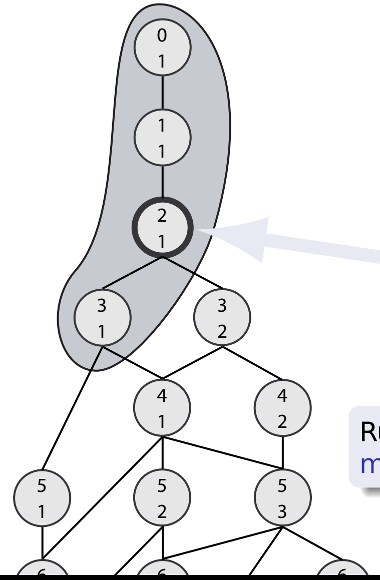
Bundles of most interest for the example

Least generic bundle

Let all free elements in the system matrices be zero ⇒ The least generic bundle of interest has the KCF $L_2 \oplus J_2(\mu)$

$$\begin{bmatrix} \omega\dot{z}/l \\ \omega\dot{\phi} \\ \omega^2\ddot{z}/l \\ \omega^2\ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z/l \\ \phi \\ \omega\dot{z}/l \\ \omega\dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{\omega^2}{ml} F$$

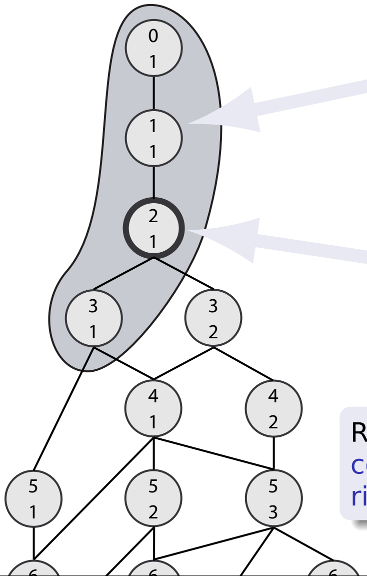
Mechanical system – Illustrating the bundle stratification



$L_2 \oplus$	$\mathcal{R}$: ○○○
$J_1(-0.02) \oplus$	$\mathcal{J}_{-0.02}$: ○
$J_1(-0.06)$	$\mathcal{J}_{-0.06}$: ○

Rule 1: Minimum leftward coin move in $\mathcal{R}$, without affecting r_0

Mechanical system – Illustrating the bundle stratification



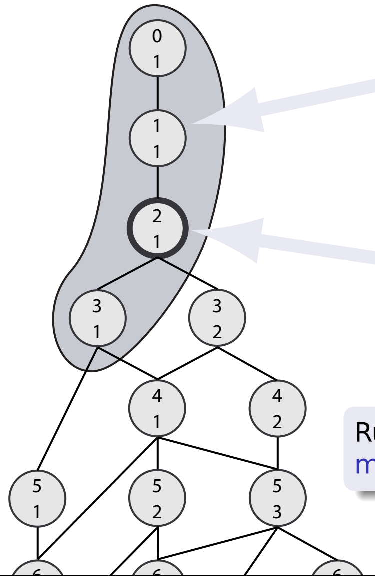
$L_3 \oplus$	$\mathcal{R}$: ○○○●
$J_1(\beta)$	$\mathcal{J}_\beta$: ○

⇑

$L_2 \oplus$	$\mathcal{R}$: ○○○
$J_1(-0.02) \oplus$	$\mathcal{J}_{-0.02}$: ●
$J_1(-0.06)$	$\mathcal{J}_{-0.06}$: ○

Rule 2: If some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$

Mechanical system – Illustrating the bundle stratification

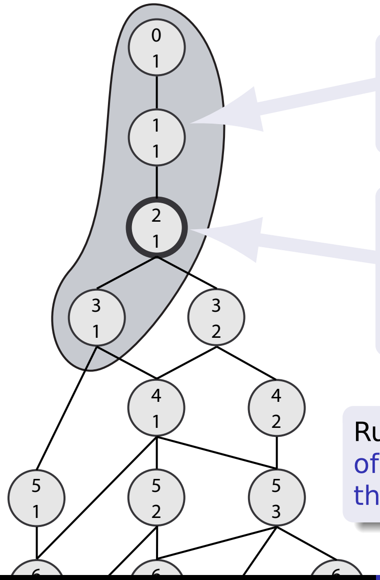


$L_3 \oplus$
$J_1(\beta)$

$L_2 \oplus$	$\mathcal{R}$: ○○○
$J_1(-0.02) \oplus$	$\mathcal{J}_{-0.02}$: ○
$J_1(-0.06)$	$\mathcal{J}_{-0.06}$: ○

Rule 3: Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$

Mechanical system – Illustrating the bundle stratification

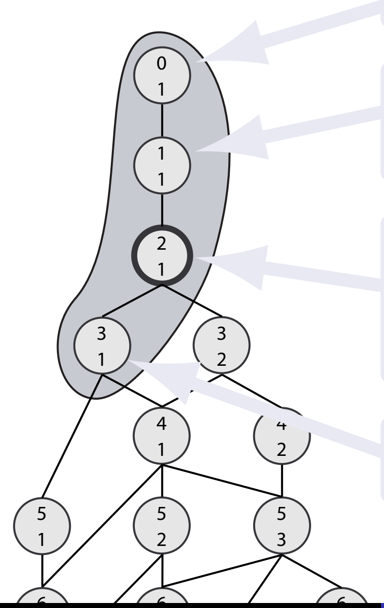


$L_3 \oplus$
$J_1(\beta)$

$L_2 \oplus$ $\mathcal{R}$: ○○○
$J_1(-0.02) \oplus$ $\mathcal{J}_{-0.02}$: ○
$J_1(-0.06)$ $\mathcal{J}_{-0.06}$: ○

Rule 4: For any $\mathcal{J}_{\mu_i}$, divide the set of coins into two new sets so that their union is $\mathcal{J}_{\mu_i}$

Mechanical system – Illustrating the bundle stratification



L_4 (most generic)

$L_3 \oplus$
$J_1(\beta)$

$L_2 \oplus$
$J_1(-0.02) \oplus$
$J_1(-0.06)$

$L_2 \oplus J_2(\alpha)$
(least generic)

Example 2 – Boeing 747

A Boeing 747 under straight-and-level flight at altitude 600 *m* with speed 92.6 *m/s*, flap setting at 20°, and landing gears up. The aircraft has mass = 317,000 *kg* and the center of gravity coordinates are $X_{cg} = 25\%$, $Y_{cg} = 0$, and $Z_{cg} = 0$

Boeing 747 – State-space model

A linearized nominal longitudinal model with 5 states and 5 inputs [A. Varga '07]:

$$\dot{x} = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04\cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98\cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30\cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix} x(t) +$$

$$\begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} u(t)$$

$$x = \begin{bmatrix} \delta q \\ \delta V_{TAS} \\ \delta\alpha \\ \delta\theta \\ \delta h_e \end{bmatrix} \begin{pmatrix} \text{pitch rate } (rad/s) \\ \text{true airspeed } (m/s) \\ \text{angle of attack } (rad) \\ \text{pitch angle } (rad) \\ \text{altitude } (m) \end{pmatrix}, \quad u = \begin{bmatrix} \delta_{ei} \\ \delta_{eo} \\ \delta_{ih} \\ \delta EPR_{1,4} \\ \delta EPR_{2,3} \end{bmatrix} \begin{pmatrix} \text{total inner elevator } (rad) \\ \text{total outer elevator } (rad) \\ \text{stabilizer trim angle } (rad) \\ \text{total thrust engine \#1 and \#4 } (rad) \\ \text{total thrust engine \#2 and \#3 } (rad) \end{pmatrix}$$

Boeing 747 – State-space model

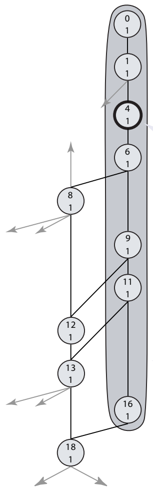
Goal

Find all *possible* closest uncontrollable systems which can be reached by a perturbation of the system matrices, and distance bounds to uncontrollability

Means:

1. Identify all the controllable and the nearest uncontrollable systems in the orbit stratification
2. Determine the orbits of interest by considering the structural restrictions of the system matrices

Boeing 747 – Illustrating the orbit stratification



Complete orbit stratification:

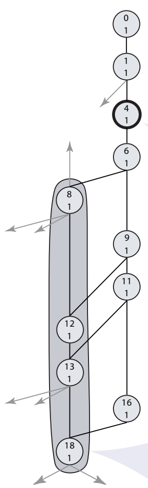
- 74 nodes and 133 edges
- Ranges from codimension 0 to 50

⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

Nodes corresponding to all controllable systems

Boeing 747 – Illustrating the orbit stratification



Complete orbit stratification:

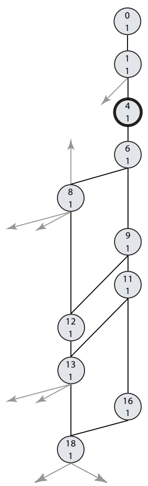
- 74 nodes and 133 edges
- Ranges from codimension 0 to 50

⇒ Identify only the nodes of interest!

Node corresponding to the orbit of the system under investigation with KCF $2L_2 \oplus L_1 \oplus 2L_0$

Nodes corresponding to the nearest uncontrollable systems (J_1-block)

Boeing 747 – Illustrating the orbit stratification



$5L_1$

$L_2 \oplus 3L_1 \oplus L_0$

NOT POSSIBLE!

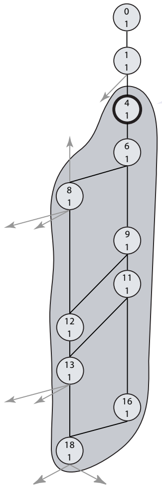
Most generic orbit

$\#L_0$ blocks $= \#$inputs $-\text{rank}(B)$, i.e., the most generic orbit must have at least 2 $(= 5 - 3)$ L_0 blocks

$$A = \begin{bmatrix} -0.4861 & 0.000317 & -0.5588 & 0 & -2.04 \cdot 10^{-6} \\ 0 & -0.0199 & 3.0796 & -9.8048 & 8.98 \cdot 10^{-5} \\ 1.0053 & -0.0021 & -0.5211 & 0 & 9.30 \cdot 10^{-6} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92.6 & 92.6 & 0 \end{bmatrix}$$

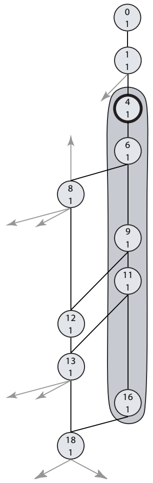
$$B = \begin{bmatrix} -0.291 & -0.2988 & -1.286 & 0.0026 & 0.007 \\ 0 & 0 & -0.3122 & 0.3998 & 0.3998 \\ -0.0142 & -0.0148 & -0.0676 & -0.0008 & -0.0008 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Boeing 747 – Illustrating the orbit stratification



Orbits of interest!

Boeing 747 – Controllable orbits



4: $2L_2 \oplus L_1 \oplus 2L_0$

$\tilde{u}_1$ controls $\tilde{x}_1, \tilde{x}_2$; $\tilde{u}_2$ controls $\tilde{x}_3, \tilde{x}_4$; $\tilde{u}_3$ controls $\tilde{x}_5$

6: $L_3 \oplus 2L_1 \oplus 2L_0$

$\tilde{u}_1$ controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; $\tilde{u}_2$ controls $\tilde{x}_4$; $\tilde{u}_3$ controls $\tilde{x}_5$

9:$L_3 \oplus L_2 \oplus 3L_0$

$\tilde{u}_1$ controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$; $\tilde{u}_2$ controls $\tilde{x}_4, \tilde{x}_5$;

11:$L_4 \oplus L_1 \oplus 3L_0$

$\tilde{u}_1$ controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$; $\tilde{u}_2$ controls $\tilde{x}_5$;

16:$L_5 \oplus 4L_0$

$\tilde{u}_1$ controls $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5$;

Distance to nearby structures – upper bounds

Given: $m \times n$ pencil $G - \lambda H$

Find: upper and lower bounds on the distance to the closest pencil (say $K - \lambda L$) with a specified KCF

Upper bound:

- Find perturbations $(\delta G, \delta H)$ such that $(G + \delta G) - \lambda(H + \delta H)$ has the KCF of $K - \lambda L$
- $(\delta G, \delta H)$ computed by a staircase algorithm that imposes the specified canonical structure (iGUPTRI)
- $\|(\delta G, \delta H)\|_F$ gives the upper bound

Distance to nearby structures – lower bounds

Lower bound:

- Use characterization of tangent space $\tan(G - \lambda H)$ of the orbit:
$$(XG - GY) - \lambda(XH - HY), \quad \forall X, Y$$
- Now, $\tan(G - \lambda H) = \text{range}(T)$, where
$$T \equiv \begin{bmatrix} G^T \otimes I_m & -I_n \otimes G \\ H^T \otimes I_m & -I_n \otimes H \end{bmatrix}$$
and $\text{nor}(G - \lambda H) = \text{kernel}(T^H)$
- Given $c = \text{cod}\,(G - \lambda H)$, a lower bound to a pencil $(G + \delta G) - \lambda(H + \delta H)$ with codimension $c + d$ is
$$\|(\delta G, \delta H)\|_F \geq \frac{1}{\sqrt{m+n}} \Big(\sum_{i=2mn-c-d+1}^{2mn} \sigma_i(T)^2 \Big)^{1/2}$$
where $\sigma_i(T) \geq \sigma_{i+1}(T)$

Lower bounds for matrix pairs

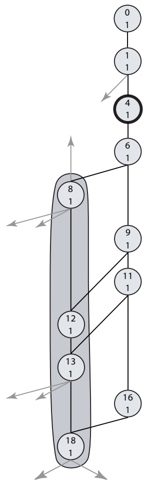
Similar characterizations give lower bounds for matrix pairs with tangent space represented as

$$T_{(A,B)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & I_n \otimes B & 0 \\ B^T \otimes I_n & 0 & I_m \otimes B \end{bmatrix} \quad \text{and}$$

$$T_{(A,C)} = \begin{bmatrix} A^T \otimes I_n - I_n \otimes A & C^T \otimes I_n & 0 \\ -I_n \otimes C & 0 & C^T \otimes I_p \end{bmatrix}$$

Matrix case: $T_A = I_n \oplus A - A^T \oplus I_n$

Boeing 747 – Illustrating the orbit stratification

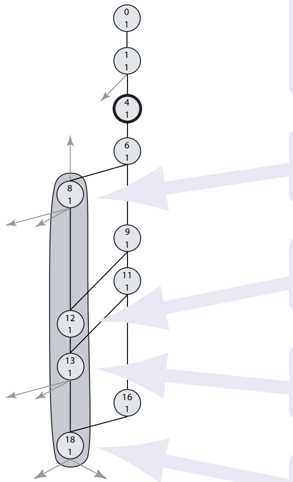


Distance to uncontrollability

$$\tau(A,B) = \min\left\{\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \| : (A+\Delta A, B+\Delta B) \text{ is uncontrollable}\right\}$$

where $\|\cdot\|$ denotes the 2-norm or Frobenius norm

Boeing 747 – Illustrating the orbit stratification



Computed distance to uncontrollability [Gu et al., 2006]: $3.03e-2$

Lower bound: $4.33e-4$
Upper bound: 1.0

Lower bound: $1.09e-3$
Upper bound: $2.48e-1$

Lower bound: $1.33e-3$
Upper bound: $1.79e-1$

Lower bound: $7.57e-2$
Upper bound: $5.56e-1$

Polynomial matrices – work in progress

Consider dynamical systems described by sets of differential equations:

$$P_d x^{(d)}(t) + \cdots + P_1 x^{(1)}(t) + P_0 x(t) = f(t), \quad P_i \text{ is } m \times n$$

Taking the Laplace transform yields the algebraic equation

$$P(s)\hat{x}(s) = \hat{f}(s) \quad \text{with} \quad P(s) := P_d s^d + \cdots + P_1 s + P_0$$

We study linearizations of

- $P(s)$ with full normal rank ($r = m$ or $r = n$)
- $P(s)\hat{x}(s) = \hat{f}(s)$ when $P(s)$ is monic, i.e., $P(s)$ is square with $P_d \equiv I_{n\times n}$

Polynomial matrices – work in progress

Consider dynamical systems described by sets of differential equations:

$$P_d x^{(d)}(t) + \cdots + P_1 x^{(1)}(t) + P_0 x(t) = f(t), \quad P_i \text{ is } m \times n$$

Taking the Laplace transform yields the algebraic equation

$$P(s)\hat{x}(s) = \hat{f}(s) \quad \text{with} \quad P(s) := P_d s^d + \cdots + P_1 s + P_0$$

We study linearizations of

- $P(s)$ with full normal rank ($r = m$ or $r = n$)
- $P(s)\hat{x}(s) = \hat{f}(s)$ when $P(s)$ is monic, i.e., $P(s)$ is square with $P_d \equiv I_{n \times n}$

Goal:

Derive stratification rules for *full rank polynomial matrices* $P(s)$ and $P(s)\hat{x}(s) = \hat{f}(s)$ where $P(s)$ is monic

References – only some of our work

Papers

- E. Elmroth, S. Johansson, and B. Kågström
Stratification of Controllability and Observability Pairs – Theory and Use in Applications. *SIAM J. Matrix Analysis and Applications*, Vol. 31, No. 2, 2009
- S. Johansson
Reviewing the Closure Hierarchy of Orbits and Bundles of System Pencils and Their Canonical Forms. Report UMINF-09.02, Umeå University, 2009
- E. Elmroth, P. Johansson, and B. Kågström
Bounds for the Distance Between Nearby Jordan and Kronecker Structures in a Closure Hierarchy. *Journal of Mathematical Sciences*, Vol. 114, No. 6, 2003
- A. Edelman, E. Elmroth, and B. Kågström
A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils. Part II: A Stratification-Enhanced Staircase Algorithm. *SIAM J. Matrix Analysis and Applications*, Vol. 20, No. 3, 1999
- A. Edelman, E. Elmroth, and B. Kågström
A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils. Part I: Versal Deformations. *SIAM J. Matrix Analysis and Applications*, Vol. 18, No. 3, 1997
(awarded ***SIAM/SIAG Linear Agebra Prize 2000***)

PhD Theses

- S. Johansson
Tools for Control System Design – Stratification of Matrix Pairs and Periodic Riccati Differential Equation Solvers. Department of Computing Science, Umeå University, 2009
- P. Johansson
Software Tools for Matrix Canonical Computations and Web-Based Software Library Environments. Department of Computing Science, Umeå University, 2006

Matrix Stratification Epilogue

- While **stratigraphy** is the key to understanding the geological evolution of the world, **StratiGraph** is the entry to understanding the "geometrical evolution" of orbits and bundles in the "world" of matrices and matrix pencils.
- But remember these worlds grow exponentially with matrix size!
- Thanks!