Necessary Conditions for the Optimal Control of Differential-Algebraic Equations with Arbitrary Index

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1 Preliminaries

We consider optimal control problems of the form

$$\mathcal{J}(x,u) = \mathcal{M}(x(\bar{t})) + \int_{\underline{t}}^{\bar{t}} \mathcal{K}(t, x(t), u(t))\ dt = \mathrm{min!}$$

subject to the differential-algebraic equation (DAE)

$$F(t, x, u, \dot{x}) = 0, \quad x(\underline{t}) = \underline{x},$$

where

$$\mathcal{M} \in C(\mathbb{D}_x, \mathbb{R}), \quad \mathcal{K} \in C(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_u, \mathbb{R}), \quad F \in C(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \times \mathbb{D}_u, \mathbb{R}^n)$$

with

$$\mathbb{I} = [\underline{t}, \bar{t}] \text{ and } \mathbb{D}_x, \mathbb{D}_{\dot{x}} \subseteq \mathbb{R}^n, \ \mathbb{D}_u \subseteq \mathbb{R}^l \text{ open.}$$

We assume that all functions are sufficiently smooth.

The corresponding linearized problem reads (omitting the argument t)

$$\mathcal{J}(x,u) = \frac{1}{2}x(\bar{t})^T M x(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} (x^T W x + 2x^T S u + u^T R u)\ dt = \text{min!}$$

subject to

$$E\dot{x} = Ax + Bu + f, \quad x(\underline{t}) = \underline{x},$$

where

$$M \in \mathbb{R}^{n,n}, \quad W \in C(\mathbb{I},\mathbb{R}^{n,n}), \quad S \in C(\mathbb{I},\mathbb{R}^{n,l}), \quad R \in C(\mathbb{I},\mathbb{R}^{l,l}),$$

and

$$E, A \in C(\mathbb{I},\mathbb{R}^{n,n}), \quad B \in C(\mathbb{I},\mathbb{R}^{n,l}), \quad f \in C(\mathbb{I},\mathbb{R}^{n})$$

are sufficiently smooth.

We are allowed to assume that M is symmetric and W und R are pointwise symmetric.

For the abstract problem

$$\mathcal{J}(z) = \text{min!}$$

subject to

$$\mathcal{F}(z) = 0,$$

where

$$\mathcal{J} \in C(\mathbb{D}, \mathbb{R}), \quad \mathcal{F} \in C(\mathbb{D}, \mathbb{Y}), \quad \mathbb{D} \subseteq \mathbb{Z} \text{ open},$$

with real Banach spaces $\mathbb{Y}, \mathbb{Z}$, we have the following (classical) result due to Ljusternik (1934):

THEOREM
Let $z^* \in \mathbb{Z}$ be a local minimum of the above problem and assume that

- $\mathcal{J}$ is Fréchet differentiable in z^*,
- $\mathcal{F}$ is a submersion in z^*, i.e., $\mathcal{F}$ is Fréchet differentiable in a neighborhood of z^* with Fréchet derivative $D\mathcal{F}(z^*) : \mathbb{Z} \to \mathbb{Y}$ surjective and kernel $D\mathcal{F}(z^*)$ continuously projectable.

Then there exists a functional Λ in the dual space $\mathbb{Y}^*$ of $\mathbb{Y}$ with

$$D\mathcal{J}(z^*)\Delta z + \Lambda(D\mathcal{F}(z^*)\Delta z) = 0 \quad \text{for all } \Delta z \in \mathbb{Z}.$$

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Properties

- The Lagrange multiplier Λ is unique.
- The above necessary conditions transform covariantly with diffeomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}$.

Questions

- What is a suitable abstract formulation of a DAE?
- What is a suitable representation of Λ?
- How do the necessary conditions in the case of DAEs look like?

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2 DAE theory

The system

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\begin{aligned}
\dot{x}_1 &= x_4, & \dot{x}_4 &= 2x_1x_7,\\
\dot{x}_2 &= x_5, & \dot{x}_5 &= 2x_2x_7,\\
\dot{x}_3 &= x_6, & \dot{x}_6 &= -1 - x_7,\\
0 &= x_3 - x_1^2 - x_2^2,
\end{aligned}
$$

describes the movement of a mass point on a paraboloid under the influence of gravity.

Differentiating the constraint twice and eliminating the arising derivatives of the unknowns yields

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\begin{aligned}
0 &= x_6 - 2x_1x_4 - 2x_2x_5,\\
0 &= -1 - x_7 - 2x_4^2 - 4x_1^2x_7 - 2x_5^2 - 4x_2^2x_7.
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Hence, we may replace the original problem by

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Observations

- In order to solve a DAE we may be forced to differentiate parts of the given DAE.
- If differentiation is necessary, we get additional (so-called hidden) constraints for the states.
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For the following, it is convenient not to distinguish between states and controls, so-called behavior approach. Introducing

$$z = \begin{bmatrix} x \\ u \end{bmatrix},$$

we write the DAE as

$$F(t, z, \dot{z}) = 0$$

or

$$\mathcal{E}\dot{z} = \mathcal{A}z + f, \quad \mathcal{E} = [\, E \;\; 0 \,], \; \mathcal{A} = [\, A \;\; B \,]$$

in the linear case.

We first study linear DAEs.

Since we may have to differentiate the given DAE in order to determine its solutions, we consider so-called derivative array equations

$$M_\ell \dot{z}_\ell = N_\ell z_\ell + g_\ell,$$

where

$$(M_\ell)_{i,j} = \binom{i}{j}\mathcal{E}^{(i-j)} - \binom{i}{j+1}\mathcal{A}^{(i-j-1)}, \ \ i,j = 0,\ldots,\ell,$$

$$(N_\ell)_{i,j} = \begin{cases} \mathcal{A}^{(i)} & \text{for } i = 0,\ldots,\ell, \ \ j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(z_\ell)_j = z^{(j)}, \ \ j = 0,\ldots,\ell,$$

$$(g_\ell)_i = f^{(i)}, \ \ i = 0,\ldots,\ell,$$

consisting of

$$\begin{aligned} \mathcal{E}\,\dot{z} \qquad\qquad\qquad\qquad\qquad\qquad &= \mathcal{A}z + f, \\ (\dot{\mathcal{E}} - \mathcal{A})\dot{z} + \qquad \mathcal{E}\,\ddot{z} \qquad\qquad &= \dot{\mathcal{A}}z + \dot{f}, \\ (\ddot{\mathcal{E}} - 2\dot{\mathcal{A}})\dot{z} + (2\dot{\mathcal{E}} - \mathcal{A})\ddot{z} + \mathcal{E}\,\dddot{z} &= \ddot{\mathcal{A}}z + \ddot{f}, \end{aligned}$$

etc.

Hypothesis

There are integers $\mu, a, d \in \mathbb{N}_0$ with $a + d = n$ such that

1. $\operatorname{rank} M_\mu = (\mu + 1)n - a$ on $\mathbb{I}$ and thus the existence of

   Z_2 smooth matrix function, max. rank a, orth. columns,
   $Z_2^T M_\mu = 0$ on $\mathbb{I}$,

2. $\operatorname{rank} Z_2^T N_\mu = a$ on $\mathbb{I}$ and thus the existence of

   T_2 smooth matrix function, max. rank $d + l$, orth. columns,
   $Z_2^T N_\mu [\, I_{n+l} \;\; 0 \;\; \cdots \;\; 0 \,]^T T_2 = 0$ on $\mathbb{I}$,

3. $\operatorname{rank} \mathcal{E} T_2 = d$ on $\mathbb{I}$ and thus the existence of

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Remarks

- In the case $l = 0$, the hypothesis is equivalent with the assumption of a well-defined differentiation index.

- Under the hypothesis, we can extract a so-called reduced DAE

  $$\hat{E}\dot{x} = \hat{A}x + \hat{B}u + \hat{f}, \quad \hat{E} = \begin{bmatrix} \hat{E}_1 \\ 0 \end{bmatrix}, \ \hat{A} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \ \hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix},$$

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  out of the derivative array equations satifying the hypothesis with $\mu = 0$.

- Original and reduced DAE have the same (smooth) solutions.

- For the reduced DAE there exists a linear feedback $u = Kx + w$ such that the closed loop problem

  $$\hat{E}\dot{x} = (\hat{A} + \hat{B}K)x + \hat{B}w + \hat{f}$$

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Consequences

- In the optimal control problem, one should replace the original DAE by the reduced DAE.
- Since a feedback defines a diffeomorphism with respect to z, we may assume that the reduced DAE satisfies the above hypothesis with $\mu = 0$ for every continuous u.
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Using

$$E\dot{x} = EE^+E\dot{x} = E\tfrac{d}{dt}(E^+Ex) - E\tfrac{d}{dt}(E^+E)x,$$

we interpret

$$E\dot{x} = Ax + Bu + f, \quad x(\underline{t}) = \underline{x}, \quad \underline{x} \in \text{range}\, E^+E(\underline{t})$$

as

$$E\tfrac{d}{dt}(E^+Ex) = (A + E\tfrac{d}{dt}(E^+E))x + Bu + f, \quad (E^+Ex)(\underline{t}) = \underline{x}.$$

Consider now

$$\mathcal{F} : \mathbb{Z} = \mathbb{X} \times \mathbb{U} \to \mathbb{Y}$$

with the Banach spaces

$$\mathbb{X} = C^1_{E^+E}(\mathbb{I}, \mathbb{R}^n) = \left\{ x \in C^0(\mathbb{I}, \mathbb{R}^n) \mid E^+Ex \in C^1(\mathbb{I}, \mathbb{R}^n) \right\},$$

$$\mathbb{U} = C^0(\mathbb{I}, \mathbb{R}^l),$$

$$\mathbb{Y} = C^0(\mathbb{I}, \mathbb{R}^n) \times \text{range}\, E^+E(\underline{t})$$

defined by

$$\mathcal{F}(z) = \left(E\tfrac{d}{dt}(E^+Ex) - (A + E\tfrac{d}{dt}(E^+E))x - Bu - f, (E^+Ex)(\underline{t}) - \underline{x} \right).$$

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Consider now

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with the Banach spaces

$$\mathbb{X} = C^{1}_{E^{+}E}(\mathbb{I}, \mathbb{R}^{n}) = \left\{ x \in C^{0}(\mathbb{I}, \mathbb{R}^{n}) \mid E^{+}Ex \in C^{1}(\mathbb{I}, \mathbb{R}^{n}) \right\},$$

$$\mathbb{U} = C^{0}(\mathbb{I}, \mathbb{R}^{l}),$$

$$\mathbb{Y} = C^{0}(\mathbb{I}, \mathbb{R}^{n}) \times \operatorname{range} E^{+}E(\underline{t})$$

defined by

$$\mathcal{F}(z) = \left(E\tfrac{d}{dt}(E^{+}Ex) - (A + E\tfrac{d}{dt}(E^{+}E))x - Bu - f, (E^{+}Ex)(\underline{t}) - \underline{x} \right).$$

THEOREM

The operator $\mathcal{F}$ is Fréchet differentiable and the restriction $\mathcal{F}(\,\cdot\,, u) : \mathbb{X} \to \mathbb{Y}$ is invertible for every $u \in \mathbb{U}$.

COROLLARY

The operator $\mathcal{F}$ is a submersion in every $z \in \mathbb{Z}$ with $\mathcal{F}(z) = 0$.

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COROLLARY

The operator $\mathcal{F}$ is a submersion in every $z \in \mathbb{Z}$ with $\mathcal{F}(z) = 0$.

In the nonlinear case, the derivative array equations have the form

$$F_\ell(t, z, \dot{z}, \ldots, z^{(\ell+1)}) = 0,$$

where

$$F_\ell(t, z, \dot{z}, \ldots, z^{(\ell+1)}) = \begin{bmatrix} F(t, z, \dot{z}) \\ \frac{d}{dt}F(t, z, \dot{z}) \\ (\frac{d}{dt})^2 F(t, z, \dot{z}) \\ \vdots \\ (\frac{d}{dt})^\ell F(t, z, \dot{z}) \end{bmatrix},$$

with

$$\tfrac{d}{dt}F(t, z, \dot{z}) = F_t(t, z, \dot{z}) + F_z(t, z, \dot{z})\dot{z} + F_{\dot{z}}(t, z, \dot{z})\ddot{z}$$

etc.

Hypothesis

There are integers $\mu, a, d \in \mathbb{N}_0$ with $a + d = n$ such that

$$\mathbb{L}_\mu = F_\mu^{-1}(\{0\}) \neq \emptyset$$

and (locally)

1. $\operatorname{rank} F_{\mu;\dot{z},\ldots,z^{(\mu+1)}} = (\mu+1)n - a$ on $\mathbb{L}_\mu$ and thus the existence of

   Z_2 smooth matrix function, max. rank a, orth. columns,
   $Z_2^T F_{\mu;\dot{z},\ldots,z^{(\mu+1)}} = 0$ on $\mathbb{L}_\mu$,

2. $\operatorname{rank} Z_2^T F_{\mu;z} = a$ on $\mathbb{L}_\mu$ and thus the existence of

   T_2 smooth matrix function, max. rank $d + l$, orth. columns,
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3. $\operatorname{rank} F_{\dot{z}} T_2 = d$ on $\mathbb{L}_\mu$ and thus the existence of

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Remarks

- Given a solution z of the DAE in the form of a path
$$(t, z, \mathcal{P}(t)) \in \mathbb{L}_\mu,$$
the matrix functions fixed by the above hypothesis can be defined globally.

- We can then (under some additional technical assumptions) extract a so-called reduced DAE
$$\widehat{F}(t, x, u, \dot{x}) = 0, \quad \widehat{F}(t, x, u, \dot{x}) = \begin{bmatrix} \widehat{F}_1(t, x, u, \dot{x}) \\ \widehat{F}_2(t, x, u) \end{bmatrix},$$
where
$$\widehat{F}_1(t, x, u, \dot{x}) = Z_1^T F(t, x, u, \dot{x})$$
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Remarks (cont)

- With some further technical assumptions, the above reduced DAE is equivalent to a DAE of the form

$$\dot{x}_1 = \mathcal{L}(t, x_1, u), \quad x_2 = \mathcal{R}(t, x_1, u),$$

  where $(x_1, x_2) = Qx$ with a pointwise orthogonal $Q \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$.

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by

$$\mathcal{F}(z) = (\dot{x}_1 - \mathcal{L}(t, x_1, u), x_2 - \mathcal{R}(t, x_1, u), x_1(\underline{t})),$$

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Theorem

The operator $\mathcal{F}$ is a submersion in every $z \in \mathbb{Z}$ with $\mathcal{F}(z) = 0$.

3 Necessary conditions

THEOREM

The necessary condition for a local minimum of the linear-quadratic optimal control problem is given by the boundary value problem

$$E\tfrac{d}{dt}(E^+Ex) = (A + E\tfrac{d}{dt}(E^+E))x + Bu + f,\ (E^+Ex)(\underline{t}) = \underline{x},$$

$$E^T\tfrac{d}{dt}(EE^+\lambda) = Wx + Su - (A + EE^+\dot{E})^T\lambda,\ (EE^+\lambda)(\bar{t}) = -(E^{+T}Mx)(\bar{t}),$$

$$0 = S^Tx + Ru - B^T\lambda.$$

REMARK

The Lagrange multiplier $\Lambda : \mathbb{Y} \to \mathbb{R}$ has the form

$$\Lambda(g, r) = \int_{\underline{t}}^{\bar{t}} \lambda^T g\, dt + \gamma^T r$$

with

$$\lambda \in C^1_{EE^+}(\mathbb{I}, \mathbb{R}^n), \quad \gamma \in \text{cokernel}\, E(\underline{t})$$

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Observations

- The necessary condition is a boundary value problem for a DAE even when we start with an ODE.
- It may happen that this DAE satisfies the above hypothesis only for non-vanishing μ.
- We can characterize the case $\mu = 0$.
- We can achieve $\mu = 0$ just by modifying the costs.
- The boundary value problem transforms covariantly with respect to feedback controls in the DAE.
- For every additional end condition on the state, we loose an initial condition on the Lagrangian function.

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The DAEs

$$E\tfrac{d}{dt}(E^{+}Ex) = (A + E\tfrac{d}{dt}(E^{+}E))x, \quad E^{T}\tfrac{d}{dt}(EE^{+}\lambda) = -(A + EE^{+}\dot{E})^{T}\lambda$$

are the correct formulations of the problems

$$E\dot{x} = Ax, \quad E^{T}\dot{\lambda} = -(A + \dot{E})^{T}\lambda.$$

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THEOREM

The necessary condition for a local minimum of the nonlinear optimal control problem is given by the boundary value problem

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\begin{aligned}
\dot{x}_1 &= \mathcal{L}(t, x_1, u), \;\; x_1(\underline{t}) = \underline{x}_1, \\
x_2 &= \mathcal{R}(t, x_1, u), \\
\dot{\lambda}_1 &= \mathcal{K}_{x_1}(t, x_1, x_2, u)^T - \mathcal{L}_{x_1}(t, x_1, x_2, u)^T \lambda_1 - \mathcal{R}_{x_1}(t, x_1, u)^T \lambda_2, \\
&\qquad\qquad\qquad\qquad \lambda_1(\bar{t}) = -\mathcal{M}_{x_1}(x_1(\bar{t}), x_2(\bar{t}))^T \\
0 &= \mathcal{K}_{x_2}(t, x_1, x_2, u)^T + \lambda_2, \\
0 &= \mathcal{K}_u(t, x_1, x_2, u)^T - \mathcal{L}_u(t, x_1, u)^T \lambda_1 - \mathcal{R}_u(t, x_1, u)^T \lambda_2, \\
\gamma &= \lambda_1(\underline{t})
\end{aligned}
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4 Numerical treatment

Returning to the original data, we must deal in the linear case with the DAE

$$\begin{aligned} Z_1^T E\dot{x} &= Z_1^T Ax + Z_1^T Bu + Z_1^T f, \\ 0 &= Z_2^T \hat{N}_\mu [\, I_n \;\; 0\,]^T x + Z_2^T \hat{N}_\mu [\, 0 \;\; I_l\,]^T u + Z_2^T g_\mu, \\ \tfrac{d}{dt}(E^T Z_1 \lambda_1) &= Wx + Su - A^T Z_1 \lambda_1 - [\, I_n \;\; 0\,] \hat{N}_\mu^T Z_2 \lambda_2, \\ 0 &= S^T x + Ru - B^T Z_1 \lambda_1 - [\, 0 \;\; I_l\,] \hat{N}_\mu^T Z_2 \lambda_2, \end{aligned}$$

where

$$\hat{N}_\mu = N_\mu [\, I_{n+l} \;\; 0 \;\; \cdots \;\; 0\,]^T.$$

Problem

We know that there exist smooth functions Z_1 and Z_2, but numerically we only can determine $Z_1 U_1$ and $Z_2 U_2$ with pointwise orthogonal but in general non-smooth U_1 and U_2.

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Observe that

$$E^T Z_1 \lambda_1 = E^T Z_1 Z_1^T Z_1 \lambda_1 = E^T Z_1 Z_1^T \hat{\lambda}_1,$$

with $\hat{\lambda}_1 = Z_1 \lambda_1 \in \operatorname{range} Z_1$.

Assuming that $\hat{Z}_1$ gives a pointwise orthogonal $[\, Z_1 \;\; \hat{Z}_1 \,]$, we can write the property $\hat{\lambda}_1 \in \operatorname{range} Z_1$ as $\hat{Z}_1^T \hat{\lambda}_1 = 0$.

The projector $Z_1 Z_1^T$ is unique and thus smooth.

We can proceed in a similar way for $\hat{N}_\mu^T Z_2 \lambda_2$ which gives $\hat{\lambda}_2 = Z_2 \lambda_2 \in \operatorname{range} Z_2$.

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Therefore, we actually treat

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Z_1^T E\dot{x} &= Z_1^T A x + Z_1^T B u + Z_1^T f, \\
0 &= Z_2^T \hat{N}_\mu [\, I_n \;\; 0 \,]^T x + Z_2^T \hat{N}_\mu [\, 0 \;\; I_l \,]^T u + Z_2^T g_\mu, \\
\tfrac{d}{dt}(E^T Z_1 Z_1^T \hat{\lambda}_1) &= W x + S u - A^T \hat{\lambda}_1 - [\, I_n \;\; 0 \,] \hat{N}_\mu^T \hat{\lambda}_2, \\
0 &= S^T x + R u - B^T \hat{\lambda}_1 - [\, 0 \;\; I_l \,] \hat{N}_\mu^T \hat{\lambda}_2, \\
0 &= \hat{Z}_1^T \hat{\lambda}_1, \\
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\end{aligned}
$$

Remarks

- Numerical solutions do not depend on non-smooth scalings from the left.
- In the case that $\mu = 0$ for the DAE of the boundary value problem, we may use symmetric DAE collocation methods for discretization.

Therefore, we actually treat

$$\begin{aligned}
Z_1^T E\dot{x} &= Z_1^T Ax + Z_1^T Bu + Z_1^T f,\\
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In the nonlinear case, a corresponding approach yields

$$
\begin{aligned}
Z_1^T F &= 0, \\
F_\mu &= 0, \\
\tfrac{d}{dt}(F_{\dot{x}}^T Z_1 Z_1^T \hat{\lambda}_1) &= \mathcal{K}_x^T + F_x^T \hat{\lambda}_1 + F_{\mu;x}^T \hat{\lambda}_2, \\
0 &= \mathcal{K}_u^T + F_u^T \hat{\lambda}_1 + F_{\mu;u}^T \hat{\lambda}_2, \\
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$$

with the boundary conditions

$$
(\hat{E}_1^+ \hat{E}_1 x)(\underline{t}) = \underline{x}, \quad (Z_1^T \hat{\lambda}_1)(\bar{t}) = -\hat{E}_1^+(\bar{t})^T \mathcal{M}_x(x(\bar{t}))^T.
$$

A model problem for a motor controlled pendulum to be driven into its equilibrium with minimal costs is given by

$$
\begin{array}{llll}
 & J(x,u) = \int_0^3 u(t)^2 \, dt = \min! & & \\
\text{s.t.} & \dot{x}_1 = x_3, & x_1(0) = \tfrac{1}{2}\sqrt{2}, & g = 9.81, \\
 & \dot{x}_2 = x_4, & x_2(0) = -\tfrac{1}{2}\sqrt{2}, & \\
 & \dot{x}_3 = -2x_1x_5 + x_2u, & x_3(0) = 0, & \\
 & \dot{x}_4 = -g - 2x_2x_5 - x_1u, & x_4(0) = 0, & \\
 & 0 = x_1^2 + x_2^2 - 1, & x_5(0) = -\tfrac{1}{2}gx_2(0). &
\end{array}
$$

It is known that the differential-algebraic equation in the constraint satisfies the above hypothesis with $\mu = 2$, $a = 3$, and $d = 2$.

Hence, only two scalar initial values are sufficient to describe the initial state, e. g.

$$x_2(0) = -\tfrac{1}{2}\sqrt{2}, \quad x_3(0) = 0.$$

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We used a simple symmetric discretization of order two.

As initial trajectory we took

$$x_1(t) = \tfrac{1}{2}\sqrt{2} - \tfrac{1}{6}\sqrt{2}t, \qquad x_3(t) = 0,$$
$$x_2(t) = -\sqrt{1 - x_1(t)^2}, \qquad x_4(t) = 0, \qquad x_5(t) = -\tfrac{1}{2} g x_2(t),$$

with all other unknowns set to zero on an equidistant grid of 60 intervals. The required tolerance for the Gauß-Newton method was 10^{-7}.

Denoting the Euclidian norm of the corresponding Gauß-Newton correction by $\|\Delta w_k\|_2$, the course of the iteration was as follows.

k	$\|\Delta w_k\|_2$	k	$\|\Delta w_k\|_2$
1	0.140D+03	17	0.103D+01
2	0.223D+03	18	0.610D-02
⋮	⋮	19	0.318D-06
16	0.561D+01	20	0.966D-11

The bad convergence behavior in the initial phase is due to the bad initial guess, especially for the Lagrange parameter.

In the final phase, we see quadratic convergence.

The obtained final value of the cost function was $J_{\text{opt}} = 3.82$.

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Open Problems

- How can we treat the case when we have $\mu \neq 0$ for the DAE of the boundary value problem?
- How can we utilize the underlying structure of self-adjointness?
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- In order to derive necessary conditions for an optimal control, we must apply techniques of index reduction.
- We are allowed to regularize by feedback for simplification.
- The Lagrangian functional possesses an integral representation via a Lagrangian function.
- The necessary conditions have the form of a boundary value problem for a DAE in state, control and Lagrangian function involving the index reduction of the constraint DAE.
- The DAE of the boundary value problem may again have a higher index.
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