Approximation of spectral intervals and associated leading directions for linear DAEs via smooth SVDs*

Vu Hoang Linh

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Faculty of Mathematics, Mechanics and Informatics
VIETNAM NATIONAL UNIVERSITY - HANOI
*joint work with Volker Mehrmann, TU-Berlin

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Overview

1. Introduction
2. Lyapunov and Sacker-Sell Spectral Intervals for DAEs
3. Leading Directions and Solution Subspaces
4. SVD-based Methods
5. Numerical Experiments and Conclusions

Main references:

V.H.L. and V. Mehrmann, *J. Dyn. Diff. Equ.*, 2009;
V.H.L., V. Mehrmann, and E. Van Vleck, *Adv. Comp. Math.*, 2010;
V.H.L. and V. Mehrmann, *submitted for publication*, 2010.

Linear time-varying DAEs

- Linear differential-algebraic equations (DAEs) have the form

$$E(t)x' = A(t)x + f(t).$$

- They arise as linearization of nonlinear systems

$$F(t, x, x') = 0$$

  around reference solutions.
- $E(t)$ (or $\partial F/\partial x'$) is singular. Other names: implicit systems, generalized systems, or descriptor systems.

Applications

- Mechanical multibody systems
- Electrical circuit simulation
- Mechatronical systems
- Chemical reactions
- Semidiscretized PDEs (Stokes, Navier-Stokes)
- Automatically generated coupled systems

DAEs versus ODEs

- Initial conditions must be consistent.
- The solution may depend on the derivative(s) of the input, sensitive to perturbation.
- The existence and uniqueness theory is more complicated.
- Solving DAEs may involve both integration and differentiation (the latter is ill-posed problem !).

In fact, DAEs generalize ODEs !

Index concepts

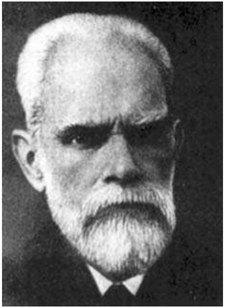
- ”Index” is a notion used in the theory of DAEs for measuring the distance from a DAE to its related ODE.
- The index is a nonnegative integer that provides useful information about how difficult a DAE is in both qualitative and numerical aspects. In general, the higher index a DAE has, the more difficult its analytical and numerical treatments are.
- There are different index definitions: Kronecker index (for linear constant coefficient DAEs), differentiation index (Brenan et al. 1996), perturbation index (Hairer et al. 1996), tractability index (Griepentrog et al. 1986), geometrical index (Rabier et.al. 2002, Riaza 2008), strangeness index (Kunkel et al. 2006), (for general DAEs). On simpler problems they are identical.

Stability results for DAEs

- Lyapunov theory for regular constant coeff. DAEs Stykel 2002.
- Index 1 systems, index 2 systems in semi-explicit form Ascher/Petzold 1993, Cao/Li/Petzold/Serban 2003.
- Mechanical systems Müller 1996.
- Systems of tractability index ≤ 2, Tischendorf 1994, Hanke/Macana/März 1998.
- Systems with properly stated leading term, Higueras/März/Tischendorf 2003, März 1998, März/Riazza 2002, Riazza 2002, Riazza/Tischendorf 2004, Balla/V.H.L. 2005.
- Exponential dichotomy in bound. val. problems, Lentini/März 1990.
- Lyapunov exponents and regularity, Cong/Nam 2003,2004.
- Exponential stability and Bohl exponents, Du/V.H.L. 2006, 2008.

Aleksandr Mikhailovich Lyapunov 1857 - 1918

The general problem of the stability of motion,
Ph.D. thesis (in Russian), 1892.



Lyapunov exponents

For the linear ODE $\dot{x} = A(t)x$ with bounded coefficient function $A(t)$ and nontrivial solution x we define the *upper and lower Lyapunov exponents*,

$$\lambda^u(x) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| , \quad \lambda^l(x) = \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t)\| .$$

Since A is bounded, the Lyapunov exponents are finite.

Theorem Lyapunov 1892/1907 If the maximal upper Lyapunov exponent for all solutions $\dot{x} = A(t)x$ is negative, then the system is *asymptotically stable*. For $A(t) \equiv A$, the Lyapunov exponents are exactly the real parts of the eigenvalues.

Bohl exponents, Piers Bohl 1865-1921

Definiton Bohl 1913 Let x be a nontrivial solution of $\dot{x} = A(t)x$. The *(upper) Bohl exponent* $\kappa_B^u(x)$ of this solution is the greatest lower bound of all those numbers ρ for which there exist numbers N_ρ such that

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|$$

for any $t \geq s \geq 0$. If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$.
Similarly, the *lower Bohl exponent* $\kappa_B^\ell(x)$ is the least upper bound of all those numbers ρ' for which there exist numbers N'_ρ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

The interval $[\kappa_B^\ell(x), \kappa_B^u(x)]$ is called the *Bohl interval* of the solution.

Mark Grigorievich Krein 1907 - 1989

Stability of solutions of differential equations in Banach spaces, AMS book (jointly with Daleckii), 1974.

Formulas for Bohl exponents

Theorem Daleckii/Krein 1974

$$\kappa_B^\ell(x) \le \lambda^\ell(x) \le \lambda^u(x) \le \kappa_B^u(x).$$

The Bohl exponents are given by

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \quad \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}.$$

If $A(t)$ is *integrally bounded*, i.e., if

$$\sup_{t\ge 0} \int_t^{t+1} \|A(s)\|\, ds < \infty,$$

then the Bohl exponents are finite.

Relation between Lyapunov and Bohl exponents

- Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$.
- If the least upper bound of upper Lyapunov exponents for all solutions $\dot{x} = A(t)x$ is negative, then the system is asymptotically stable. If the same holds for the least upper bound of the upper Bohl exponents then the system is (uniformly) exponentially stable.
- Bohl exponents are stable without any extra assumption (which is not true in the case of Lyapunov exponents).

Sacker-Sell spectrum

Definition The fundamental matrix solution X of $\dot{X} = A(t)X$ is said to admit an *exponential dichotomy* if there exist a projector $P : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and constants $\alpha, \beta > 0$, as well as $K, L \geq 1$, such that

$$\begin{aligned} \left\|X(t)PX^{-1}(s)\right\| &\leq Ke^{-\alpha(t-s)}, & t \geq s, \\ \left\|X(t)(I-P)X^{-1}(s)\right\| &\leq Le^{\beta(t-s)}, & t \leq s. \end{aligned}$$

The *Sacker-Sell (or exponential-dichotomy) spectrum* Σ_S for is given by those values $\lambda \in \mathbb{R}$ such that the *shifted system*

$$\dot{x}_\lambda = [A(t) - \lambda I]x_\lambda$$

does not have exponential dichotomy. The complement of Σ_S is called the *resolvent set*.

Sacker-Sell spectrum

Theorem Sacker/Sell 1978

The property that a system posseses an exponential dichotomy as well as the exponential dichotomy spectrum are preserved under kinematic similarity transformations.

Σ_S is the union of at most n disjoint closed intervals, and it is stable.

Furthermore, the Sacker-Sell intervals contain the Lyapunov intervals, i.e.

$$\Sigma_L \subseteq \Sigma_S.$$

Numerical methods for computing spectra

Idea: find an orthogonal transformation to bring the system into triangular form.

- Discrete QR algorithm
- Continuous QR algorithm
- Others: SVD algorithms, Spatial integration and hybrid methods

References: Benettin et.al. 1980, Greene & Kim 1987, Geist et.al. 1990, Dieci & Van Vleck 1995-2009, Oliveira & Stewart 2000, Bridges & Reich 2001, Beyn & Lust 2009

A crash course in DAE Theory

We put $E(t)\dot{x} = A(t)x + f(t)$ and its derivatives up to order μ into a large DAE

$$M_k(t)\dot{z}_k = N_k(t)z_k + g_k(t), \quad k \in \mathbb{N}_0$$

for $z_k = (x, \dot{x}, \dots, x^{(k)})$.

$$M_2 = \begin{bmatrix} E & 0 & 0 \\ A - \dot{E} & E & 0 \\ \dot{A} - 2\ddot{E} & A - \dot{E} & E \end{bmatrix}, \; N_2 = \begin{bmatrix} A & 0 & 0 \\ \dot{A} & 0 & 0 \\ \ddot{A} & 0 & 0 \end{bmatrix}, \; z_2 = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}.$$

Theorem, Kunkel/Mehrmann 1996

Under some constant rank assumptions, for a square regular linear DAE there exist integers μ, a, d such that:

1. $\mathrm{rank}\, M_\mu(t) = (\mu+1)n - a$ on $\mathbb{I}$, and there exists a smooth matrix function Z_2 with $Z_2^T M_\mu(t) = 0$.
2. The first block column $\hat{A}_2$ of $Z_2^* N_\mu(t)$ has full rank a so that there exists a smooth matrix function T_2 such that $\hat{A}_2 T_2 = 0$.
3. $\mathrm{rank}\, E(t) T_2 = d = n - a$ and there exists a smooth matrix function Z_1 of size (n, d) with $\mathrm{rank}\, \hat{E}_1 = d$, where $\hat{E}_1 = Z_1^T E$.

Reduced problem - Strangeness-free form

- The quantity μ is called the strangeness-index. It describes the smoothness requirements for the inhomogeneity.
- It generalizes the usual differentiation index to general DAEs (and counts slightly differently).
- We obtain a numerically computable **strangeness-free formulation** of the equation with the same solution.

$$\begin{array}{rcll} \hat{E}_1(t)\dot{x} & = & \hat{A}_1(t)x + \hat{f}_1(t), & d \text{ differential equations} \\ 0 & = & \hat{A}_2(t)x + \hat{f}_2(t), & a \text{ algebraic equations} \end{array} \tag{1}$$

where $\hat{A}_1 = Z_1^T A$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_2 = Z_2^T g_\mu$.

- The reduced system is strangeness-free. This is a Remodeling! **For the theory we assume that this has been done.**

Essentially underlying implicit ODE

Lemma Consider a strangeness-free homogeneous DAE system of the form (1) with continuous coefficients E, A. Let $U \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$ be an arbitrary orthonormal basis of the solution subspace of (1). Then there exists a matrix function $P \in C(\mathbb{I}, \mathbb{R}^{n\times d})$ with pointwise orthonormal columns such that by the change of variables $x = Uz$ and multiplication of both sides of (1) from the left by P^T, one obtains the system

$$\mathcal{E}\dot{z} = \mathcal{A}z, \tag{2}$$

where $\mathcal{E} := P^T EU$, $\mathcal{A} := P^T AU - P^T E\dot{U}$ and $\mathcal{E}$ is upper triangular.

Relation between DAEs and EUODEs

- one-one relation between solutions: $z = U^T x$ and $Uz = UU^T x = x$, (UU^T is a projection onto the solution subspace!).
- The EUODEs possess the same spectral properties as the DAE.
- Different bases U and scaling functions P may give different EUODEs. However, the spectral properties are invariant.
- U, P, and the EUODE can be constructed numerically (QR and SVD algorithms for DAEs).

Fundamental solution matrices

Definition

A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times k})$, $\quad d \le k \le n$, is called *fundamental solution matrix of* $E(t)\dot{X} = A(t)X$ if each of its columns is a solution to $E(t)\dot{x} = A(t)x$ and $\operatorname{rank} X(t) = d$, for all $t \ge 0$.

A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively.

Every fundamental solution matrix has exactly d linearly independent columns and a minimal fundamental matrix solution can be easily made maximal by adding $n - d$ zero columns.

Lyapunov exponents for DAEs

Definition For a fundamental solution matrix X of a strangeness-free DAE system $E(t)\dot{x} = A(t)x$, and for $d \le k \le n$, we introduce

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\| \text{ and } \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|, \quad i = 1, 2, ..., k.$$

The columns of a minimal fundamental solution matrix form a *normal basis* if $\Sigma_{i=1}^d \lambda_i^u$ is minimal. The λ_i^u, $i = 1, 2, ..., d$, belonging to a normal basis are called *(upper) Lyapunov exponents* and the intervals $[\lambda_i^\ell, \lambda_i^u]$, $i = 1, 2, ..., d$, are called *Lyapunov spectral intervals*.

Normal basis and Lyapunov spectrum

Theorem Let Z be a minimal fundamental solution matrix for the strangeness-free DAE $E(t)\dot{x} = A(t)x$ such that the upper Lyapunov exponents of its columns are ordered decreasingly. Then there exists a nonsingular upper triangular matrix $C \in \mathbb{R}^{d\times d}$ such that the columns of $X(t) = Z(t)C$ form a normal basis.

Theorem Let X be a normal basis for (1). Then the Lyapunov spectrum of the DAE (1) and that of the ODE (2) are the same. If $\mathcal{E}, \mathcal{A}$ are as in (2) and if $\mathcal{E}^{-1}\mathcal{A}$ is bounded, then all the Lyapunov exponents of (1) are finite. Furthermore, the spectrum of (2) does not depend on the choice of the basis U and the scaling function P.

Further concepts and results

- Lyapunov regularity for DAEs and their adjoint DAEs.
- Perron's identity for the Lyapunov exponents of a DAE and those of its adjoint DAE.
- Essential differences between the spectral theory of DAEs and that of ODEs do exist. Spectral properties that are well-known for ODEs hold true for DAEs under more restrictive conditions (since the dynamics of DAEs is constrained).

Perturbed DAE systems

In order to study this sensitivity for DAEs, we consider the specially perturbed system

$$[E(t) + F(t)]\dot{x} = [A(t) + H(t)]x, \quad t \in \mathbb{I}, \tag{3}$$

where

$$F(t) = \begin{bmatrix} F_1(t) \\ 0 \end{bmatrix}, \; H(t) = \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix},$$

and where F_1 and H_1, H_2 are assumed to have the same order of smoothness as E_1 and A_1, A_2, respectively. Perturbations of this structure are called admissible.

Stability of Lyapunov exponents

The DAE (1) is said to be robustly strangeness-free if it stays strangeness-free under all sufficiently small admissible perturbations.

It happens iff $\bar{E} = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$ is *boundedly* invertible.

Definition The upper Lyapunov exponents $\lambda_1^u \geq \ldots \geq \lambda_d^u$ of (1) are said to be *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \|F(t)\| < \delta,\ \sup_t \|H(t)\| < \delta$, and $\sup_t \left\|\dot{H}_2(t)\right\| < \delta$ on the perturbations imply that the perturbed DAE system (3) is strangeness-free and

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \qquad \text{for all } i = 1, 2, \ldots, d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of the perturbed system (3).

Necessity of boundedness condition

Example Consider the system

$$\begin{array}{rcl} \dot{x}_1 & = & x_1 \\ 0 & = & x_2 \end{array} . \tag{4}$$

This DAE is robustly strangeness-free, Lyapunov regular, and it has only one Lyapunov exponent $\lambda = 1$. Now, we consider the perturbed DAE

$$\begin{array}{rcl} (1+\varepsilon^2 \sin 2nt)\,\dot{x}_1 - \varepsilon \cos nt\,\dot{x}_2 & = & x_1\,, \\ 0 & = & -2\varepsilon \sin nt\, x_1 + x_2\,, \end{array} \tag{5}$$

where ε is a small perturbation parameter and n is a given integer. We have

$$\dot{x}_1 = (1 + n\varepsilon^2 + n\varepsilon^2 \cos 2nt)\, x_1\,.$$

The only Lyapunov exponent $\hat{\lambda} = 1 + n\varepsilon^2$ is calculated.

Integral separation

Definition A minimal fundamental solution matrix X for a strangeness-free DAE is called *integrally separated* if for $i = 1, 2, ..., d-1$ there exist $b > 0$ and $c > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq ce^{b(t-s)},$$

for all t, s with $t \geq s \geq 0$.

If a DAE system has an integrally separated minimal fundamental solution matrix, then we say it has the *integral separation property*.

Criterion for the stability of exponents

By using a global kinematic equivalence transformation, (1) can always be transformed to a block-triangular form, where the block A_{21} becomes zero.

Theorem Consider (1) with $A_{21} = 0$. Suppose that the matrix $\bar{E}$ is boundedly invertible and that $E_{11}^{-1} A_{11}$, $A_{12} A_{22}^{-1}$ and the derivative of A_{22} are bounded on $[0, \infty)$. Then, the upper Lyapunov exponents of (1) are distinct and stable if and only if the system has the integral separation property.

Exponential Dichotomy

Definition The DAE (1) is said to have *exponential dichotomy* if for any minimal fundamental solution X there exist a projection $P \in \mathbb{R}^{d\times d}$ and positive constants K and α such that

$$\begin{aligned} \|X(t)PX^+(s)\| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \|X(t)(I_d - P)X^+(s)\| &\leq Ke^{\alpha(t-s)}, \quad s > t, \end{aligned} \tag{6}$$

where X^+ denotes the generalized Moore-Penrose inverse of X.

Since $X = UZ$ and $X^+ = Z^{-1}U^T$, we have
Theorem The DAE (1) has exponential dichotomy if and only if its corresponding EUODE (2) has exponential dichotomy.

Shifted DAE and shifted ODE

In order to extend the concept of exponential dichotomy spectrum to DAEs, we need *shifted DAE systems*

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \in \mathbb{I}, \tag{7}$$

where $\lambda \in \mathbb{R}$. By using the same transformation as for EUODE, we obtain the corresponding shifted EUODEs for (7)

$$\mathcal{E}\dot{z} = (\mathcal{A} - \lambda\mathcal{E})z. \tag{8}$$

Sacker-Sell spectrum

Definition

- The *Sacker-Sell (or exponential dichotomy) spectrum* of a strangeness-free DAE system is defined by

$$\Sigma_S := \{\lambda \in \mathbb{R}, \text{ the shifted DAE does not have an exponential dichotomy}\}.$$

- The complement of Σ_S is called the *resolvent set*, denoted by $\rho(E, A)$.
- The Sacker-Sell spectrum of a DAE system does not depend on the choice of the orthogonal basis and the corresponding EUODE system.

Sacker-Sell spectrum

Theorem The Sacker-Sell spectrum of (1) is exactly the Sacker-Sell spectrum of its EUODE (2). Further, the Sacker-Sell spectrum of (1) consists of at most d closed intervals.

Under the boundedness conditions of coefficient matrices, unlike the Lyapunov spectrum, the Sacker-Sell spectrum of the DAE (1) is stable with respect to admissible perturbations.
We can assume that Σ_S consists of $m \le d$ pairwise disjoint spectral intervals, i.e., $\Sigma_S = \cup_{i=1}^{d}[a_i, b_i]$, and $b_i < a_{i+1}$ for all $1 \le i \le m$. This can be easily achieved by combining overlapping spectral intervals to larger intervals.

Initial-condition subspace and solution subspace

- Let us denote by $\mathbb{S}_0$ the space of consistent initial vectors of (1). This is a d-dimensional subspace in $\mathbb{R}^n$.
- The solutions of (1) also form a d-dimensional subspace of functions in $C^1(\mathbb{I}, \mathbb{R}^n)$. We denote it by $\mathbb{S}(t)$.
- For $x_0 \in \mathbb{S}_0$ let us denote by $x(t; x_0)$ the (unique) solution of (1) that satisfies $x(0; x_0) = x_0$.

Filtration property

For $j = 1, \dots d$, define

$$W_j = \left\{ w \in \mathbb{S}_0 : \chi^u\left(x(\cdot; w)\right) \le \lambda_j^u \right\}, \quad j = 1, \dots, d. \tag{9}$$

Proposition Let d_j be the largest number of linearly independent solutions x of (1) such that $\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| = \lambda_j^u$. Then W_j is a d_j dimensional linear subspace of $\mathbb{S}_0$. Furthermore, the spaces W_j form a *filtration* of $\mathbb{S}_0$, i.e., if p is the number of distinct upper Lyapunov exponents of the system, then we have

$$\mathbb{S}_0 = W_1 \supset W_2 \supset \cdots \supset W_p \supset W_{p+1} = \{0\}.$$

Leading directions

- It follows that $\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t;w)\| = \lambda_j^u$ if and only if $w \in W_j \backslash W_{j+1}$. Moreover, if we have d distinct upper Lyapunov exponents, then each W_j, $j = 1, \dots, d$, has dimension $(d - j + 1)$.
- If V_j is defined as the orthogonal complement of W_{j+1} in W_j, i.e.,

$$W_j = W_{j+1} \oplus V_j, \quad V_j \perp W_{j+1},$$

then $\mathbb{S}_0 = V_1 \oplus V_2 \oplus \cdots \oplus V_p$, and

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t;w)\| = \lambda_j^u \text{ if and only if } w \in V_j.$$

It follows that if we have $m = d$ distinct Lyapunov exponents, then $\dim(V_j) = 1$ for all $j = 1, \dots, d$.

Stable and unstable sets

Consider now the resolvent set $\rho(E,A)$. For $\mu \in \rho(E,A)$, let us define the following *stable and unstable* sets, respectively.

$$\begin{aligned} \mathcal{S}_\mu &= \left\{ w \in \mathbb{S}_0 : \lim_{t\to\infty} e^{-\mu t} \|x(t;w)\| = 0 \right\}, \\ \mathcal{U}_\mu &= \left\{ w \in \mathbb{S}_0 : \lim_{t\to\infty} e^{-\mu t} \|x(t;w)\| = +\infty \right\} \cup \{0\}. \end{aligned} \tag{10}$$

By definition, it is clear that $\mathcal{S}_\mu \cap \mathcal{U}_\mu = \{0\}$ for $\mu \in \rho(E,A)$.
Furthermore, for $\mu_1, \mu_2 \in \rho(E,A),\ \mu_1 < \mu_2$, we have $\mathcal{S}_{\mu_1} \subseteq \mathcal{S}_{\mu_2}$ and $\mathcal{U}_{\mu_1} \supseteq \mathcal{U}_{\mu_2}$.
The sets (10) are related with

$$\begin{aligned} \mathcal{S}_\mu^d &= \left\{ v \in \mathbb{R}^d : \lim_{t\to\infty} e^{-\mu t} \|Z(t)v\| = 0 \right\}, \\ \mathcal{U}_\mu^d &= \left\{ v \in \mathbb{R}^d : \lim_{t\to\infty} e^{\mu t} \left\|Z(t)^{-T} v\right\| = 0 \right\}. \end{aligned} \tag{11}$$

which are stable and unstable sets associated with (2).

Characterization of stable and unstable sets

Now choose a set of values $\mu_0 < \mu_1 < \cdots < \mu_m$, such that $\mu_j \in \rho(E, A)$ and $\Sigma_S \cap (\mu_{j-1}, \mu_j) = [a_j, b_j]$ for $j = 1, \ldots, m$. In other words, we have

$$\mu_0 < a_1 \leq b_1 < \mu_1 < \cdots < \mu_{j-1} < a_j \leq b_j < \mu_j < \cdots < a_m \leq b_m < \mu_m.$$

Consider the intersections

$$\mathcal{N}_j^d = \mathcal{S}_{\mu_j}^d \cap \mathcal{U}_{\mu_{j-1}}^d, \; j = 1, \ldots, m. \tag{12}$$

Let U be an orthonormal basis of the solution subspace for (1) and introduce the sets

$$\mathcal{N}_j = U(0)\mathcal{N}_j^d = \left\{ w \in \mathbb{S}_0 : w = U(0)v, \; v \in \mathcal{N}_j^d \right\}, \; j = 1, \ldots, m. \tag{13}$$

Characterization of $\mathcal{N}_j$

Proposition Consider the EUODE (2) associated with (1), and the sets $\mathcal{N}_j$ defined in (13), $j = 1, \ldots, m$. If $w \in \mathcal{N}_j$ and

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t;w)\| = \chi^u, \quad \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t;w)\| = \chi^\ell,$$

then $\chi^\ell, \chi^u \in [a_j, b_j]$.

This means that $\mathcal{N}_j$ is the subspace of initial conditions associated with solutions whose upper and lower Lyapunov exponents are located inside $[a_j, b_j]$.

Uniform exponential growth of solutions

Theorem Consider the EUODE (2) associated with (1), and the sets $\mathcal{N}_j$ defined in (13), $j = 1, \ldots, m$. Then $w \in \mathcal{N}_j \setminus \{0\}$ if and only if

$$\frac{1}{K_{j-1}} e^{a_j(t-s)} \leq \frac{\|x(t;w)\|}{\|x(s;w)\|} \leq K_j e^{b_j(t-s)}, \quad \text{for all } t \geq s \geq 0, \tag{14}$$

and positive K_{j-1}, K_j.

Additional characterization of $\mathcal{N}_j$

Corollary Consider the EUODE (2) associated with (1) and the sets $\mathcal{N}_j$ defined in (13). Then for all $j = 1, \ldots, m$, we have

(i) $w \in \mathcal{N}_j \setminus \{0\}$ if and only if

$$a_j \le \kappa^{\ell}(x(\cdot; w)) \le \kappa^{u}(x(\cdot; w)) \le b_j,$$

where $\kappa^{\ell}, \kappa^{u}$ are the Bohl exponents.

(ii) $\mathcal{S}_{\mu_j} = U(0)\mathcal{S}^d_{\mu_j}, \mathcal{U}_{\mu_j} = U(0)\mathcal{U}^d_{\mu_j}$
$\mathcal{N}_j = \mathcal{S}_{\mu_j} \cap \mathcal{U}_{\mu_{j-1}}, \; j = 1, \ldots, m.$

Remarks

- An alternative way to characterize the unstable set $\mathcal{U}_\mu^d$ is as $\left\{v \in \mathbb{R}^d : \lim_{t\to\infty} e^{\mu t} \|Z(t)v\| = +\infty\right\} \cup \{0\}$ for $\mu \in \rho(E, A)$. Furthermore, we have

$$\begin{aligned} \mathcal{S}_{\mu_j} &= \{w \in \mathbb{S}_0 : \kappa^u(x(\cdot; w)) \le b_j\} \cup \{0\}, \\ \mathcal{U}_{\mu_{j-1}} &= \left\{w \in \mathbb{S}_0 : \kappa^\ell(x(\cdot; w)) \ge a_j\right\} \cup \{0\}, \end{aligned}$$

  as well.
- Under the integral separation assumption, Lyapunov and Sacker-Sell spectral intervals can be approximated by QR (L.-Mehrmann 09, L.-Mehrmann-Van Vleck 10) or SVD algorithms (L.-Mehrmann 10).
- The leading directions associated with the spectral intervals (the one dimensional V_j and $\mathcal{N}_j$) can be approximated as well by SVD algorithms.

Numerical methods

- Discrete and continuous methods based on smooth QR and SVD have been extended for DAEs.
- In the discrete methods the fundamental solution matrix X and its triangular factor R (or the diagonal factor Σ in the SVD) are indirectly evaluated by a reorthogonalized integration of the DAE system (1) via an appropriate QR factorization (and the product SVD algorithm).
- In the continuous counterparts, we derive DAEs for the factor Q (or U) and the scalar ODEs for the logarithms of the diagonal elements of R (or Σ) elementwise. (For integrating Q or U, use appropriate DAE orthogonal integrators.)
- Error and perturbation analysis has been carried out for the QR methods.
- In the continuous SVD method the (exponential) convergence rate of factor V is established.

Smooth SVDs

Let X be a minimal fundamental matrix solution, $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$. Suppose

$$X(t) = U(t)\Sigma(t)V^T(t), \quad t \geq 0, \tag{15}$$

where $U \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$, $V, \Sigma \in C^1(\mathbb{I}, \mathbb{R}^{d\times d})$,
$U^T(t)U(t) = V^T(t)V(t) = I_d$ for all $t \in \mathbb{I}$, and
$\Sigma(t) = \mathrm{diag}(\sigma_1(t), \dots, \sigma_d(t))$.
We assume that U, Σ, and V possess the same smoothness as X. This holds, e.g., if $X(t)$ is analytic or if the singular values of $X(t)$ are distinct for all t.
The adaptation of the discrete SVD algorithm to DAEs is straightforward. However, this method is less competitive because of the growth of the fundamental solution and the very large memory requirement of the product SVD algorithm.

Derivation of the continuous SVD

Differentiating X in (15), we obtain

$$E\dot{U}\Sigma V^T + EU\dot{\Sigma}V^T + EU\Sigma\dot{V}^T = AU\Sigma V^T,$$

or equivalently, using the orthogonality of V,

$$E\dot{U}\Sigma + EU\dot{\Sigma} + EU\Sigma\dot{V}^T V = AU\Sigma.$$

Using

$$A_2 U = 0, \qquad (16)$$

we obtain

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}\dot{U}\Sigma + \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}U\dot{\Sigma} + \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}U\Sigma\dot{V}^T V = \begin{bmatrix} A_1 \\ -\dot{A}_2 \end{bmatrix}U\Sigma. \qquad (17)$$

We define

$$\bar{E} = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A_1 \\ -\dot{A}_2 \end{bmatrix},$$

and the skew-symmetric matrix functions

$$H = U^T \dot{U}, \quad K = V^T \dot{V}.$$

Lemma There exists $P \in C(\mathbb{I}, \mathbb{R}^{n \times d})$, $P^T P = I_d$ such that

$$P^T \bar{E} = \mathcal{E} U^T, \tag{18}$$

where $\mathcal{E}$ is nonsingular and upper triangular with positive diagonal entries. Further, the pair $P, \mathcal{E}$ is unique.

Conditioning of the EUODE

Note that $P^T E U = P^T \bar{E} U = \mathcal{E}$ holds, so that P can be used to produce an EUODE of the form (2).
The conditioning of this EUODE is not worse than that of the original DAE.
Proposition Consider the matrix function P defined via (18). Then

$$\|\mathcal{E}\| \leq \|\bar{E}\|, \quad \|\mathcal{E}^{-1}\| \leq \|\bar{E}^{-1}\|.$$

Consequently, $\mathrm{cond}\,\mathcal{E} \leq \mathrm{cond}\,\bar{E}$.

Multiplying both sides of (17) with P^T from the left, we obtain

$$\mathcal{E}H\Sigma + \mathcal{E}\dot{\Sigma} + \mathcal{E}\Sigma K^T = P^T\bar{A}U\Sigma.$$

With

$$G = P^T\bar{A}U \text{ and } C = \mathcal{E}^{-1}G, \tag{19}$$

we then arrive at

$$H\Sigma + \dot{\Sigma} + \Sigma K^T = C\Sigma,$$

where

$$\begin{aligned} h_{i,j} &= \frac{c_{i,j}\sigma_j^2 + c_{j,i}\sigma_i^2}{\sigma_j^2 - \sigma_i^2}, \text{ for } i > j, \text{ and } \quad h_{i,j} = -h_{j,i} \text{ for } i < j; \\ k_{i,j} &= \frac{(c_{i,j} + c_{j,i})\sigma_i\sigma_j}{\sigma_j^2 - \sigma_i^2}, \text{ for } i > j, \text{ and } \quad k_{i,j} = -k_{j,i} \text{ for } i < j. \end{aligned} \tag{20}$$

Differential equations for the SVD's factors

We get immediately the differential equation for the diagonal elements of Σ

$$\dot{\sigma}_i = c_{i,i}\sigma_i, \quad i = 1, \dots, d, \tag{21}$$

and that for the V-factor,

$$\dot{V} = VK. \tag{22}$$

We also obtain the equation for the U-factor as

$$E\dot{U} = EU(H - C) + AU. \tag{23}$$

The latter (23) is a strangeness-free (non-linear) matrix DAE, that is linear with respect to the derivative. Furthermore, the algebraic constraint is also linear and the same as that of (1).

Assumptions and auxiliary results

Assume that C in (19) is uniformly bounded on $\mathbb{I}$. Furthermore, the functions σ_i are integrally separated, i.e., there exist constants $k_1 > 0$ and $k_2,\ 0 < k_2 \leq 1$, such that

$$\frac{\sigma_j(t)}{\sigma_j(s)} \frac{\sigma_{j+1}(s)}{\sigma_{j+1}(t)} \geq k_2 e^{k_1(t-s)}, \quad t \geq s \geq 0,\ j = 1, 2, \ldots, d-1. \qquad (24)$$

Proposition The following statements hold.

(a) There exists $\bar{t} \in \mathbb{I}$, such that for all $t \geq \bar{t}$,

$$\sigma_j(t) > \sigma_{j+1}(t), \quad j = 1, 2, \ldots, d-1.$$

(b) The skew-symmetric matrix function $K(t)$ converges exponentially to 0 as $t \to \infty$.

(c) The orthogonal matrix function $V(t)$ converges exponentially to a constant orthogonal matrix $\bar{V}$ as $t \to \infty$.

Obtaining spectral intervals

Theorem System (1) has distinct and stable Lyapunov exponents if and only if for any fundamental matrix solution X, the singular values of X are integrally separated. Moreover, if X is a fundamental solution, then

$$\lambda_j^u = \limsup_{t\to\infty} \frac{1}{t} \ln \sigma_j(t), \quad \lambda_j^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \sigma_j(t), \quad j = 1, 2, \ldots, d. \quad (25)$$

Theorem Suppose that (1) has distinct and stable Lyapunov exponents. Then, the Sacker-Sell spectrum of (1) is the same as that of the diagonal system

$$\dot{\Sigma}(t) = \mathrm{diag}(C(t))\Sigma(t).$$

Furthermore, this Sacker-Sell spectrum is given by the union of the Bohl intervals associated with the scalar equations $\dot{\sigma}_i(t) = c_{i,i}(t)\sigma_i(t)$, $i = 1, 2, \ldots, d$.

Obtaining leading directions

Theorem Suppose that (1) has distinct and stable Lyapunov exponents. Let $X(t) = U(t)\Sigma(t)V(t)^T$ be a smooth SVD of an arbitrary fundamental solution. Let $\bar{V} = [\bar{v}_1, \dots, \bar{v}_d]$ be the limit of the factor $V(t)$ as $t \to \infty$. Then

$$\chi^u(X(\cdot)\bar{v}_j) = \lambda_j^u, \quad \chi^\ell(X(\cdot)\bar{v}_j) = \lambda_j^\ell, \quad j = 1, 2, \dots, d.$$

Theorem Suppose that (1) has distinct and stable Lyapunov exponents and let $\Sigma_S = \bigcup_{j=1}^m [a_j, b_j]$. Then $\mathcal{N}_j = \hat{U}(0)\text{span}\{\bar{v}_k, \dots, \bar{v}_l\}$, where the integers k, l, $k < l$ are such that

$$\lambda_{l+1}^u < a_j \leq \lambda_l^\ell, \quad \lambda_k^u \leq b_j < \lambda_{k-1}^\ell.$$

Remark We also have $[a_j, b_j] = \bigcup_{i=k}^l [\kappa^\ell(X(\cdot)\bar{v}_i), \kappa^u(X(\cdot)\bar{v}_i)]$ for $j = 1, \dots, m$.

Obtaining the IS fundamental solution

Theorem Suppose that the DAE system (1) has distinct and stable Lyapunov exponents. Let $X(t) = U(t)\Sigma(t)V(t)^T$ be a smooth SVD of an arbitrary fundamental solution and let $\bar{V} = [\bar{v}_1, \ldots \bar{v}_d]$ be the limit of $V(t)$ as $t \to \infty$. Then starting from $X(0)\bar{V}$ leads to an integral separated fundamental solution, i.e., $X(t)\bar{V}$ is integrally separated.

This improves a recent result of Dieci & Elia (2006).

Implementation issues

- Numerical computation of the scaling factor P.
- Computation of the singular values $\sigma_i(t)$ in a more stable manner (avoiding the risk of overflow).
- Efficient integration of the nonlinear *matrix* DAE for U. Half-explicit methods are useful.

Numerical experiment I

Lyapunov exponents computed via continuous SVD algorithm with half-explicit Euler integrator for a Lyapunov-regular DAE with $\lambda_1 = 1,\ \lambda_2 = -1$

T	h	λ_1	λ_2	CPU-time in s	CPU-time in s, $\ell = 1$
500	0.1	0.9539	-0.9579	3.0156	2.7344
500	0.05	0.9720	-0.9760	5.9375	5.4375
500	0.01	0.9850	-0.9890	29.5781	27.0625
1000	0.1	0.9591	-0.9592	5.9531	5.5000
1000	0.05	0.9772	-0.9773	11.7969	10.7969
1000	0.01	0.9902	-0.9903	58.7500	54.5000
2000	0.05	0.9801	-0.9805	23.4844	21.5938
2000	0.01	0.9932	-0.9936	117.4531	107.5156
5000	0.01	0.9952	-0.9955	294.1250	268.4531
10000	0.01	0.9960	-0.9962	586.9219	537.9375

Exponential convergence of V

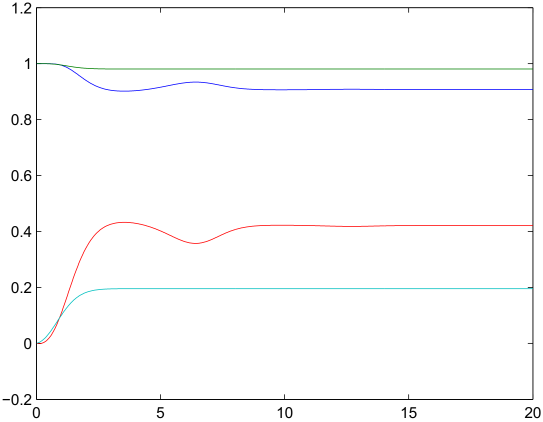


Figure: Graph of $V_{11}(t)$ and $V_{21}(t)$ for different λ_is.

Numerical experiment II

Lyapunov spectral intervals computed via continuous SVD algorithm with half-explicit Euler integrator for a DAE with the exact Lyapunov spectral intervals $[-1, 1]$ and $[-6, -4]$.

T	$\tilde{\tau}$	h	$[\lambda_1^l, \lambda_1^u]$	$[\lambda_2^l, \lambda_2^u]$	CPU-time in s
1000	100	0.10	[-1.0332 0.5704]	[-5.9311 -4.6909]	6.25
5000	100	0.10	[-1.0332 0.9851]	[-5.9311 -4.3592]	31.54
10000	100	0.10	[-1.0332 0.9851]	[-5.9311 -3.9980]	61.89
10000	100	0.05	[-1.0183 0.9946]	[-5.9421 -4.0107]	123.25
20000	100	0.10	[-1.0332 0.9851]	[-5.9311 -3.9746]	123.65
20000	100	0.05	[-1.0183 0.9946]	[-5.9421 -3.9882]	248.79
50000	100	0.05	[-1.0183 0.9946]	[-5.9421 -3.9882]	619.23
50000	500	0.05	[-0.9935 0.9946]	[-5.9421 -3.9882]	627.00
100000	100	0.05	[-1.0183 0.9946]	[-5.9421 -3.9882]	1283.3
100000	500	0.05	[-1.0087 0.9946]	[-5.9421 -3.9882]	1243.4

Conclusions

- The classical theory of Lyapunov/Bohl/Sacker-Sell has been extended to linear DAEs with variable coefficients.
- Strangeness-free formulations and EUODEs are the key tools.
- Leading directions and solution subspaces are characterized.
- Methods based on smooth QR and SVD are available for approximating spectral intervals and associated leading directions. They are **expensive !** However, in some cases, we need not all, but only few dominant exponents.

The exponents and the leading directions generalize the concepts of eigenvalues and eigenvectors from time-invariant systems (generalized eigenvalue problems) to time-varying systems.

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Work in progress and future work

- Pseudospectrum, conditioning, distance to non-exponential-dichotomy.
- Spectral analysis of nonlinear DAEs.
- Block version of algorithms
- Efficient implementation of QR and SVD algorithms.
- Real applications: semi-discretized Navier-Stokes system, etc.
- Extension to operator DAEs/PDEs.

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THANK YOU FOR YOUR ATTENTION !