Representations from contact geometry

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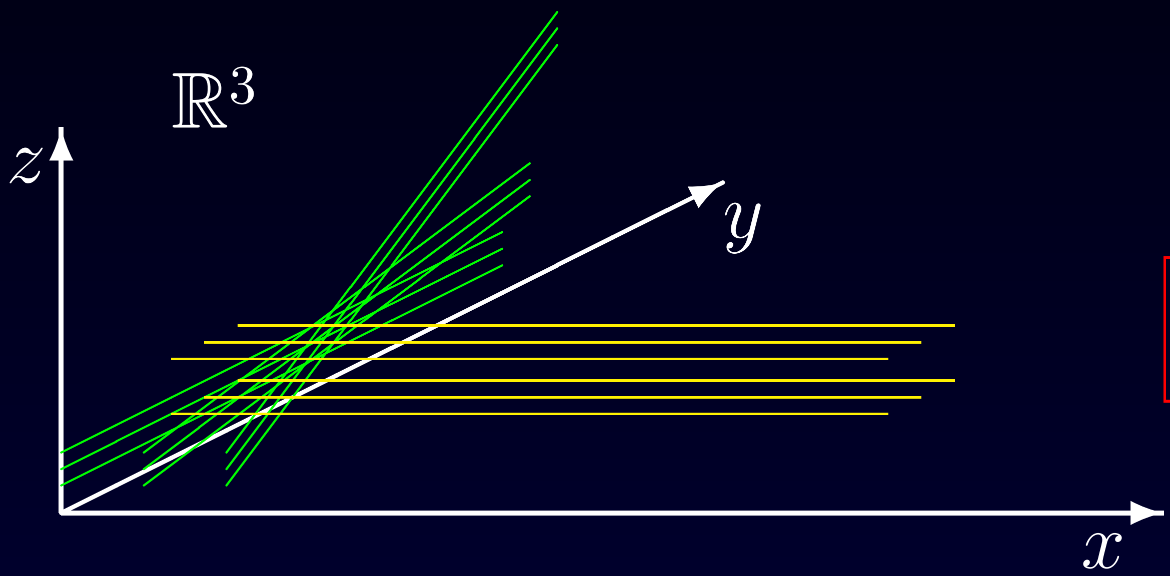
Double foliation in 3 dimensions

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$



Symmetries?

$$[X, Y] = \frac{\partial}{\partial z} \equiv Z \quad [X, Z] = 0 \quad [Y, Z] = 0$$

Symmetries of this geometry

$$\boxed{[X, Y] = Z \qquad [X, Z] = 0 \qquad [Y, Z] = 0}$$

Vector field K such that

- $\mathcal{L}_K X \propto X$
- $\mathcal{L}_K Y \propto Y$

i.e.

- $[X, K] \propto X$
- $[Y, K] \propto Y$

Write $K = f_+ Y - f_- X + gZ$. Then

- $Xg + f_+ = 0 \quad \& \quad Xf_+ = 0$
- $Yg + f_- = 0 \quad \& \quad Yf_- = 0$

Hence

$$\boxed{\boxed{X^2 g = 0 \quad \& \quad Y^2 g = 0}}$$

Warm-up exercise

$\boxed{Xg = 0 \ \& \ Yg = 0} \quad \Rightarrow Zg = 0 \quad \Rightarrow g$ is constant

$\boxed{X^2g = 0 \ \& \ Yg = 0}$

Introduce f and p by $Xg + f = 0$ and $Zg - p = 0$.

Recall $\quad [X, Y] = Z \quad [X, Z] = 0 \quad [Y, Z] = 0$.

Conclude (prolongation)

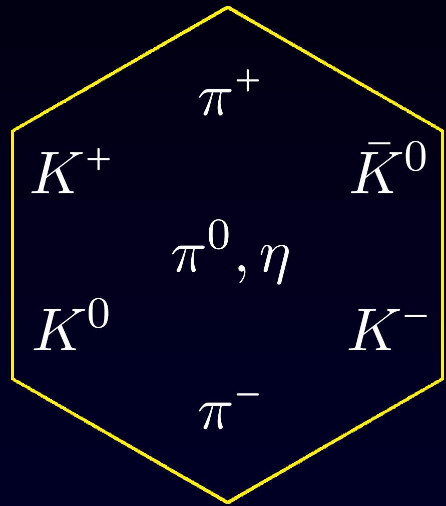
$$\nabla \begin{bmatrix} g \\ f \\ p \end{bmatrix} \equiv \begin{bmatrix} Xg + f & Yg & Zg - p \\ Xf & Yf - p & Zf \\ Xp & Yp & Zp \end{bmatrix} = 0$$

$\boxed{\text{flat connection}} \quad \Rightarrow \{g \text{ s.t. } X^2g = 0 = Yg\} \cong \mathbb{R}^3$

Symmetries cont'd

$\boxed{X^2 g = 0 \;\&\; Y^2 g = 0}$ $\rightsquigarrow\rightsquigarrow$prolongation$\rightsquigarrow\rightsquigarrow$

$$\nabla \begin{bmatrix} & g & \\ f_+ & & f_- \\ & p_+, p_- & \\ r_- & & r_+ \\ & a & \end{bmatrix} \equiv$$



$$\begin{bmatrix} & Xg + f_+ \;\; Yg + f_- \;\; Zg - p_+ - p_- & \\ Xf_+ \;\; Yf_+ - p_+ \;\; Zf_+ - r_+ & & *** \\ & Xp_+ - r_+ \;\; ** \;\; , \;\; ** \;\; Zp_- - a & \\ Xr_- + a \;\; Yf_- \;\; Zr_- & & *** \\ & Xa \;\; Ya \;\; Za & \end{bmatrix}$$

Eightfold way

The story so far

- $\boxed{\{g \text{ s.t. } Xg = 0 = Yg\} \cong \mathbb{R}}$
- $\boxed{\{g \text{ s.t. } X^2g = 0 = Yg\} \cong \mathbb{R}^3}$
- $\boxed{\{g \text{ s.t. } X^2g = 0 = Y^2g\} \cong \mathbb{R}^8}$

Symmetries

More generally,

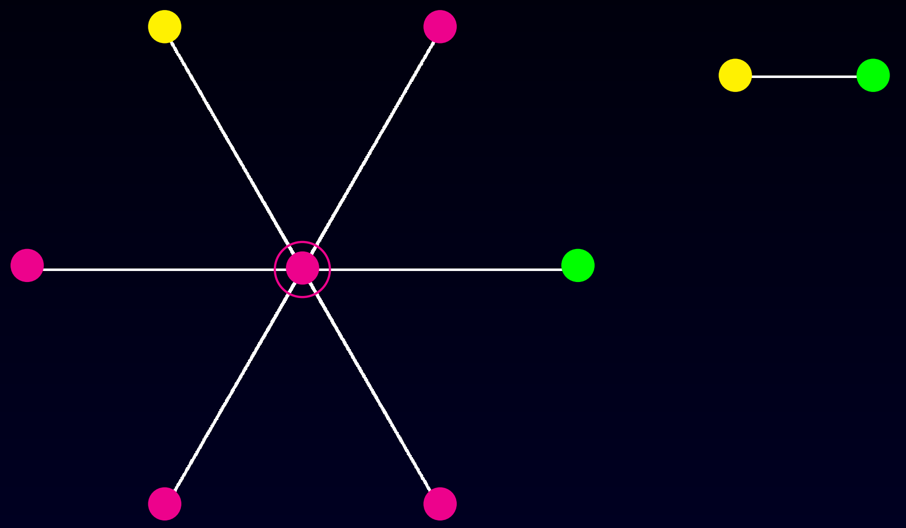
$$\boxed{\{g \text{ s.t. } X^{p+1}g = 0 = Y^{q+1}g\} \cong \mathbb{R}^{(p+1)(q+1)(p+q+2)/2}}$$

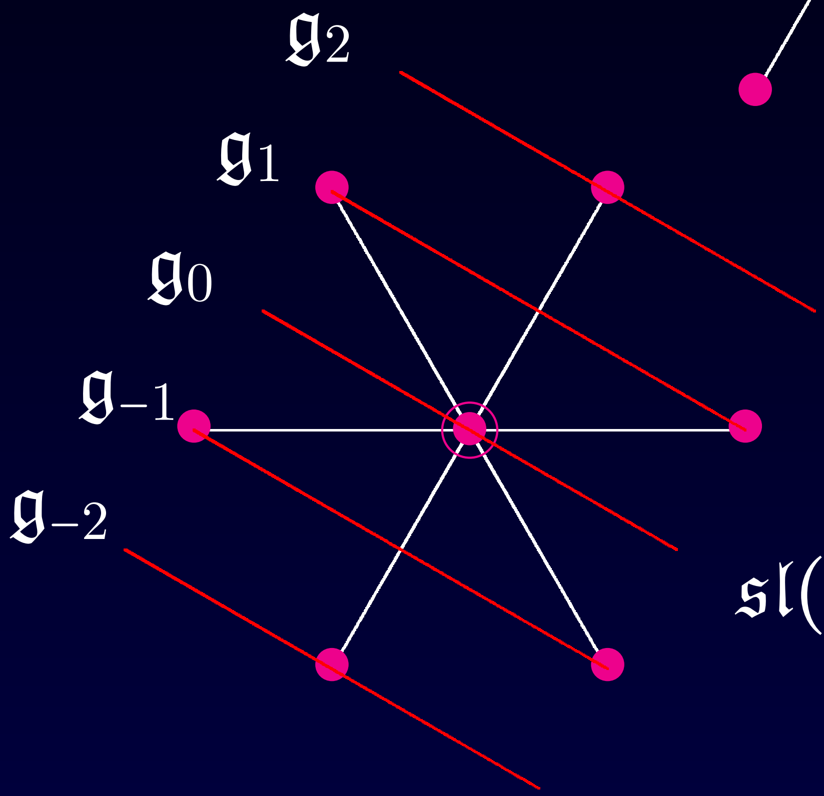
In fact,

- $\{\text{Symmetries}\} \cong \mathfrak{sl}(3, \mathbb{R})$
- $\{g \text{ s.t. } X^{p+1}g = 0 = Y^{q+1}g\} \cong \overset{p}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{q}{\bullet}$

Explain?

A2 root system



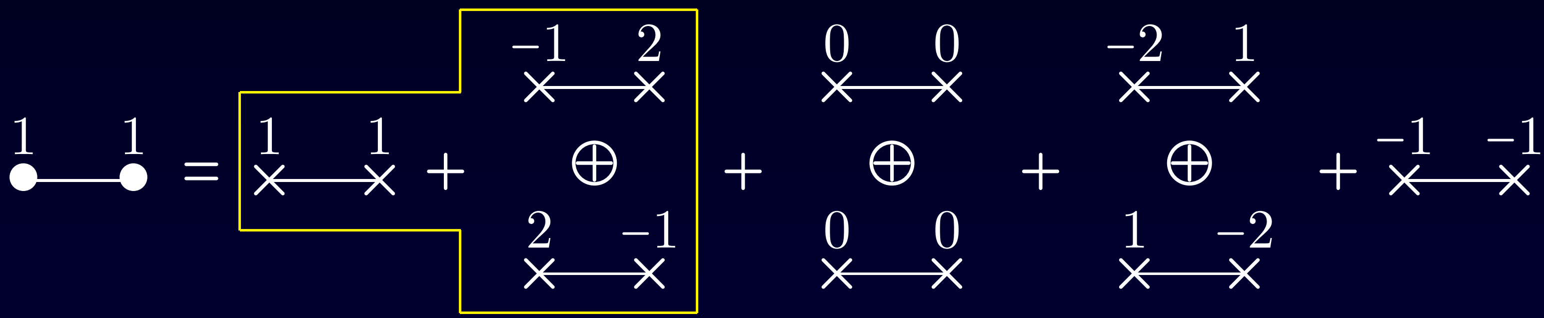


$$\mathfrak{sl}(3,\mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{p}}$$

Flag manifold

$$\mathrm{SL}(3,\mathbb{R})/P = \mathrm{SL}(3,\mathbb{R}) \Big/ \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} = \underline{\mathbb{F}_{1,2}(\mathbb{R}^3)}$$

$$\mathfrak{sl}(3,\mathbb{R}) = \underbrace{\mathfrak{g}_{-2} + \mathfrak{g}_{-1}}_{\mathfrak{sl}(3,\mathbb{R})/\mathfrak{p}} + \underbrace{\mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2}_{\mathfrak{p}}$$



↯ P-module

<u>Tangent Bundle</u>

Curved geometry

Flat

Model

$$\begin{array}{ccc} & \mathbb{F}_{1,2}(\mathbb{R}^3) & \\ \swarrow & & \searrow \\ \mathbb{F}_1(\mathbb{R}^3) & & \mathbb{F}_2(\mathbb{R}^3) \end{array}$$

$\mathbb{F}_{1,2}(\mathbb{R}^3) \supset \mathbb{R}^3$ (affine patch) where we started this talk

Curved Version

- M is a smooth real 3-manifold
- Line subbundles $H_+ \oplus H_- \subset TM$
- $[H_+, H_-] = TM$ (i.e. $H \equiv H_+ \oplus H_-$ is contact)

Theorem (Lie 1888, Tresse 1896, ≃Cartan 1924)

- $\dim\{$local symmetries of $M\} \leq 8$
- with equality iff locally flat

$$\boxed{y'' = f(x, y, y')}$$

Another curved geometry

Rephrase previous geometry on 3-dimensional M:–

- $H \subset TM$ a contact structure
- $J : H \to H$ s.t. $J^2 = \mathrm{Id}$ and $J \neq \pm\mathrm{Id}$

and now change a sign to define CR geometry

- $H \subset TM$ a contact structure
- $J : H \to H$ s.t. $J^2 = -\mathrm{Id}$ (complex structure)

Theorem (Poincaré 1907, Segre 1931, Cartan 1932)

- $\dim\{\text{local symmetries of } M\} \leq 8$
- with equality iff locally flat

Flat Model $\mathrm{SU}(2,1)/P = S^3 \subset \mathbb{C}^2$.

Existence of G2

Theorem (Engel 1893, Cartan 1893)

Killing's 1888 Lie algebra G2 exists.

Proof (Engel) G2 ≡ symmetries of $\dfrac{dz}{dx} = \left(\dfrac{d^2y}{dx^2}\right)^2$ $\square$

$\mathbb{R}^5 \ni (x, y, p, q, z)$ with 2-plane distribution defined by

$$dy - p\,dx \qquad dp - q\,dx \qquad dz - q^2\,dx$$

Curved geometry

- M is a smooth real 5-manifold
- rank 2 subbundle $H \subset TM$
- $[H, [H, H]] = TM$

Uniqueness of G2

Theorem (Cartan 1910 'five variables')

For $H \subset TM$ a geometry as above

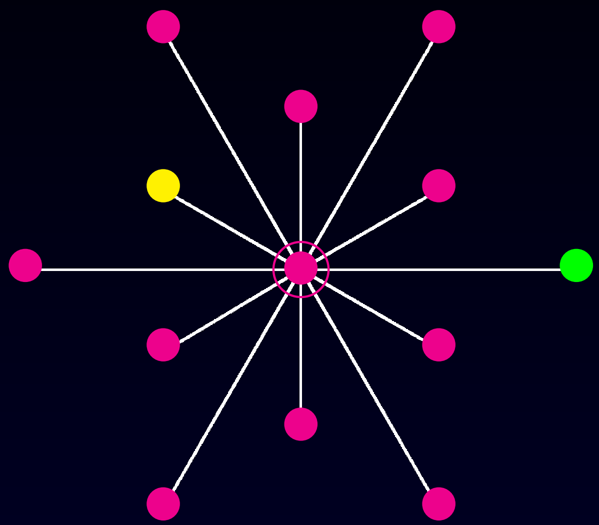
- $\dim\{\text{local symmetries of } M\} \leq 14$
- with equality iff locally flat . . .
- in which case $\{\text{symmetries}\} \cong \text{G2}$.

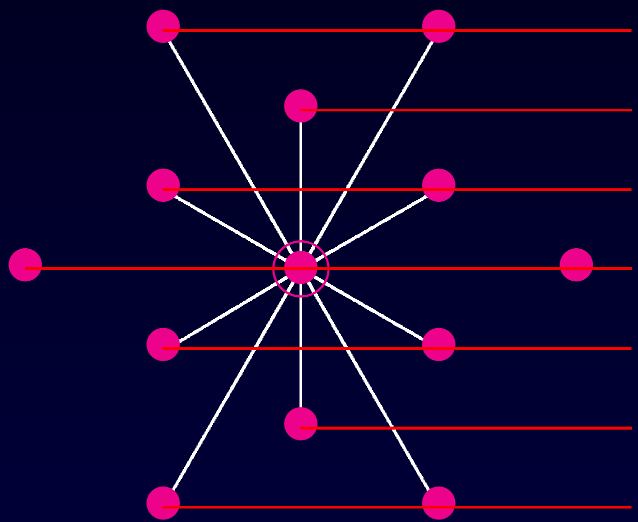
Proof is by rather difficult prolongation to obtain a

Cartan connection

Nowadays use Kostant's Bott-Borel-Weil Theorem

G2 root system

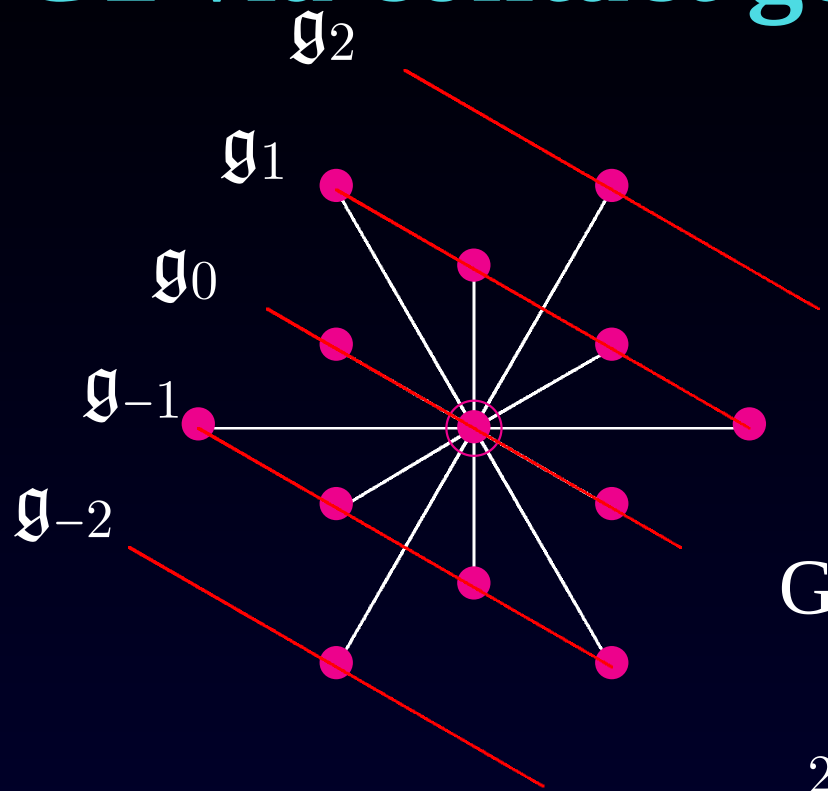




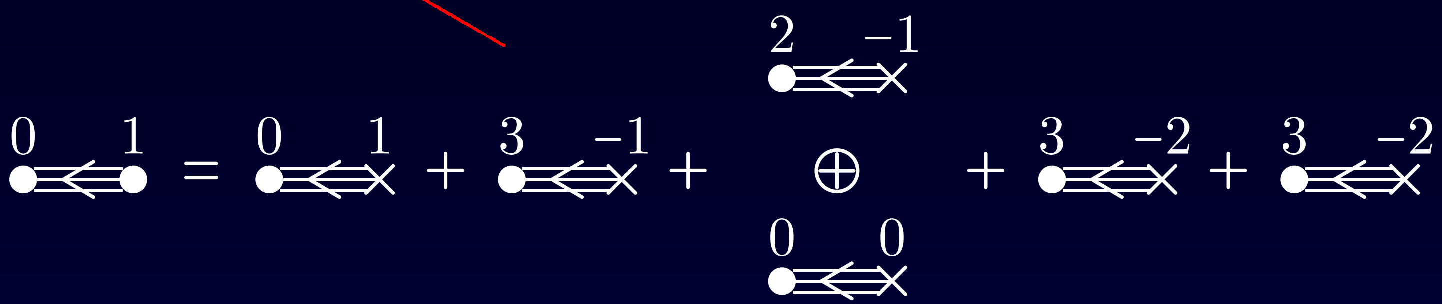
$$\left.\begin{array}{l}\mathfrak{g}_3\\ \mathfrak{g}_2\\ \mathfrak{g}_1\\ \mathfrak{g}_0\end{array}\right\}\mathfrak{p}$$

$$\left.\begin{array}{l}\mathfrak{g}_{-1}\\ \mathfrak{g}_{-2}\\ \mathfrak{g}_{-3}\end{array}\right\} H \subset [H,H] \subset [H,[H,H]] = TM$$

G2 via contact geometry



$$G2 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{p}}$$



<u>Curved geometry</u> • 5-dimensional contact manifold,
• reduction of structure group to $GL(2, \mathbb{R})$.

Contact parabolic geometry

- contact structure $H \subset TM$
- reduction of structure group of H to...

	G2	F4	E6	E7	E8
$\dim M$	5	15	21	33	57
	A1	C3	A5	D6	E7

Construction of representations

$$\overset{a}{\bullet}\!\Lleftarrow\!\overset{b}{\bullet} = \ker \overset{a}{\bullet}\!\Lleftarrow\!\overset{b}{\times} \xrightarrow{\nabla^{b+1}} \overset{a+2b+2}{\bullet}\!\Lleftarrow\!\overset{-b-2}{\times}$$

- Verma modules
- Bernstein-Gelfand-Gelfand resolution
- Jantzen-Zuckerman translation principle

Further reading

- D.N. Arnold, N. Douglas, and R.S. Falk, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. AMS **47** (2010) 281–354.
- A. Čap, M.G. Cowling, M.G. Eastwood, F. De Mari and R. McCallum, *The Heisenberg group*, $\mathrm{SL}(3,\mathbb{R})$, *and rigidity*, Harmonic Analysis, . . . in Honour of Roger Howe, Lect. Notes IMS Vol. 12, National University of Singapore 2007, pp. 41–52.
- M.G. Eastwood and A.R. Gover, *Prolongations on contact manifolds*, arXiv:0910.5519
- P. Nurowski and G.A.J. Sparling, *Three-dimensional Cauchy-Riemann structures and second-order ordinary differential equations*, Class. Quant. Grav. **20** (2003) 4995–5016.
- A. Čap and J. Slovák, *Parabolic Geometries 1*, AMS 2009.

THANK YOU

THE END