Computing regular subalgebras of simple Lie algebras

Todor Milev, Jacobs University Bremen

July 8, 2010

Motivation for study

- $\mathfrak{g}$ - finite-dimensional semisimple Lie algebra. Field: $\mathbb{C}$.
- M - $\mathfrak{g}$-module.

Motivation for study

- $\mathfrak{g}$ - finite-dimensional semisimple Lie algebra. Field: $\mathbb{C}$.
- M - $\mathfrak{g}$-module.
- *The Fernando-Kac subalgebra* $\mathfrak{g}[M] \subset \mathfrak{g}$ *associated to* M: the set of elements acting locally finitely.

Motivation for study

- $\mathfrak{g}$ - finite-dimensional semisimple Lie algebra. Field: $\mathbb{C}$.
- M - $\mathfrak{g}$-module.
- *The Fernando-Kac subalgebra* $\mathfrak{g}[M] \subset \mathfrak{g}$ *associated to* M: the set of elements acting locally finitely.
- The set $\mathfrak{g}[M]$ is indeed a Lie subalgebra, (Kac, [Kac85], Fernando [Fer90]).

Motivation for study

- $\mathfrak{g}$ - finite-dimensional semisimple Lie algebra. Field: $\mathbb{C}$.
- M - $\mathfrak{g}$-module.
- *The Fernando-Kac subalgebra* $\mathfrak{g}[M] \subset \mathfrak{g}$ *associated to* M: the set of elements acting locally finitely.
- The set $\mathfrak{g}[M]$ is indeed a Lie subalgebra, (Kac, [Kac85], Fernando [Fer90]).
- $\mathfrak{l} \subset \mathfrak{g}$ is a *Fernando-Kac subalgebra* if there exists an **irreducible** module M for which $\mathfrak{l} = \mathfrak{g}[M]$.

Motivation for study

- $\mathfrak{g}$ - finite-dimensional semisimple Lie algebra. Field: $\mathbb{C}$.
- M - $\mathfrak{g}$-module.
- *The Fernando-Kac subalgebra* $\mathfrak{g}[M] \subset \mathfrak{g}$ *associated to* M: the set of elements acting locally finitely.
- The set $\mathfrak{g}[M]$ is indeed a Lie subalgebra, (Kac, [Kac85], Fernando [Fer90]).
- $\mathfrak{l} \subset \mathfrak{g}$ is a *Fernando-Kac subalgebra* if there exists an **irreducible** module M for which $\mathfrak{l} = \mathfrak{g}[M]$.
- M is a $(\mathfrak{g}, \mathfrak{l})$-module if $\mathfrak{l} \subset \mathfrak{g}[M]$.

Motivation for study

- $\mathfrak{g}$ - finite-dimensional semisimple Lie algebra. Field: $\mathbb{C}$.
- M - $\mathfrak{g}$-module.
- *The Fernando-Kac subalgebra* $\mathfrak{g}[M] \subset \mathfrak{g}$ *associated to* M: the set of elements acting locally finitely.
- The set $\mathfrak{g}[M]$ is indeed a Lie subalgebra, (Kac, [Kac85], Fernando [Fer90]).
- $\mathfrak{l} \subset \mathfrak{g}$ is a *Fernando-Kac subalgebra* if there exists an **irreducible** module M for which $\mathfrak{l} = \mathfrak{g}[M]$.
- M is a $(\mathfrak{g}, \mathfrak{l})$-module if $\mathfrak{l} \subset \mathfrak{g}[M]$.
- A $(\mathfrak{g}, \mathfrak{l})$-module M is of *finite type* if for any fixed irreducible finite-dimensional $\mathfrak{l}$-module V the Jordan-Hölder multiplicities of V in all finite-dimensional $\mathfrak{l}$-submodules of M are bounded.

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel.

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel.$\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$.

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel. $\mathfrak{l} = \mathfrak{k} \ni\!\!\!+ \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$. $\mathfrak{l} = \underbrace{\mathfrak{k}}_{\text{reductive part}} \ni\!\!\!+ \underbrace{\mathfrak{n}}_{\text{nilradical}}$ - semidirect sum.

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel. $\mathfrak{l} = \mathfrak{k} \ni\!\!\!+ \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$. $\mathfrak{l} = \underbrace{\mathfrak{k}}_{\text{reductive part}} \ni\!\!\!+ \underbrace{\mathfrak{n}}_{\text{nilradical}}$ - semidirect sum. $\Delta(\mathfrak{g})$ - root system notation.

$$\underbrace{\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}^{\alpha}}_{\text{subalgebra of interest}} \quad \mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \in \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha}; \quad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \notin \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha} \quad .$$

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel. $\mathfrak{l} = \mathfrak{k} \ni\!\!\!\!+ \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$. $\mathfrak{l} = \underbrace{\mathfrak{k}}_{\text{reductive part}} \ni\!\!\!\!+ \underbrace{\mathfrak{n}}_{\text{nilradical}}$ - semidirect sum. $\Delta(\mathfrak{g})$ - root system notation.

$$\underbrace{\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}^{\alpha}}_{\text{subalgebra of interest}} \quad \mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \in \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha}; \quad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \notin \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha} \quad .$$

- $\Delta(\mathfrak{k})$ - root subsystem of $\Delta(\mathfrak{g})$.

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel. $\mathfrak{l} = \mathfrak{k} \ni\!\!\!\!+ \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$. $\mathfrak{l} = \underbrace{\mathfrak{k}}_{\text{reductive part}} \ni\!\!\!\!+ \underbrace{\mathfrak{n}}_{\text{nilradical}}$ - semidirect sum. $\Delta(\mathfrak{g})$ - root system notation.

$$\underbrace{\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}^\alpha}_{\text{subalgebra of interest}} \quad \mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \in \Delta(\mathfrak{l})}} \mathfrak{g}^\alpha; \quad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \notin \Delta(\mathfrak{l})}} \mathfrak{g}^\alpha .$$

- $\Delta(\mathfrak{k})$ - root subsystem of $\Delta(\mathfrak{g})$.
- A regular semisimple subalgebra is defined as one that can be written as $[\mathfrak{k}, \mathfrak{k}]$ (kills central piece of $\mathfrak{h}$).

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel. $\mathfrak{l} = \mathfrak{k} \niplus \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$. $\mathfrak{l} = \underbrace{\mathfrak{k}}_{\text{reductive part}} \niplus \underbrace{\mathfrak{n}}_{\text{nilradical}}$ - semidirect sum. $\Delta(\mathfrak{g})$ - root system notation.

$$\underbrace{\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}^{\alpha}}_{\text{subalgebra of interest}} \quad \mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \in \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha}; \quad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \notin \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha} \quad .$$

- $\Delta(\mathfrak{k})$ - root subsystem of $\Delta(\mathfrak{g})$.
- A regular semisimple subalgebra is defined as one that can be written as $[\mathfrak{k}, \mathfrak{k}]$ (kills central piece of $\mathfrak{h}$).
- The root subsystems of $\Delta(\mathfrak{g})$ classify all regular subalgebras.

Subalgebras containing a Cartan subalgebra

$\mathfrak{h}$ - fixed Cartan subaglebra; $\mathfrak{b}$ - fixed Borel. $\mathfrak{l} = \mathfrak{k} \ni \mathfrak{n}$ - subalgebra containing $\mathfrak{h}$. $\mathfrak{l} = \underbrace{\mathfrak{k}}_{\text{reductive part}} \ni \underbrace{\mathfrak{n}}_{\text{nilradical}}$ - semidirect sum. $\Delta(\mathfrak{g})$ - root system notation.

$$\underbrace{\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}^{\alpha}}_{\text{subalgebra of interest}} \quad \mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \in \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha}; \quad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{l}): \\ -\alpha \notin \Delta(\mathfrak{l})}} \mathfrak{g}^{\alpha} .$$

- $\Delta(\mathfrak{k})$ - root subsystem of $\Delta(\mathfrak{g})$.
- A regular semisimple subalgebra is defined as one that can be written as $[\mathfrak{k}, \mathfrak{k}]$ (kills central piece of $\mathfrak{h}$).
- The root subsystems of $\Delta(\mathfrak{g})$ classify all regular subalgebras.
- Done in "Semismiple Lie subalgebras of simple Lie algebras", Dynkin.

Penkov's conjecture

Definition (Penkov)

(a) *Cone condition.* $\mathfrak{l}$ satisfies the cone condition if $\mathrm{Cone}_{\mathbb{Q}}(\Delta(\mathfrak{n})) \cap \mathrm{Cone}_{\mathbb{Q}}(\mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})) = \{0\}$, where $\mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}$ stands for $\mathfrak{b}\cap\mathfrak{k}$-singular. Motivation: I. Penkov, V. Serganova, G. Zuckerman, 2004.

(b) *Centralizer condition.* $\mathfrak{l}$ satisfies the centralizer condition if $(C([\mathfrak{k},\mathfrak{k}]) \cap N(\mathfrak{n}))_{ss}$ has simple constituents of type A and C only. Motivation: S. Fernando, 1990.

Theorem

Let $\mathfrak{l} = \mathfrak{k} \oplus\!\!\!\!\supset \mathfrak{n}$ be a subalgebra containing a Cartan subalgebra of the simple Lie algebra $\mathfrak{g} \simeq \mathrm{sl}(n)$, $\mathrm{so}(2n+1)$, $\mathrm{sp}(2n)$, $\mathrm{so}(2n)$, E_6, E_7, F_4 or G_2. Then $\mathfrak{l}$ is a Fernando-Kac subalgebra of finite type if and only if the cone condition and the centralizer condition are satisfied.

If the nilradical is zero, the cone condition is trivially satisfied. The theorem follows directly from [Fer90] (non-existence if centralizer condition fails) and the construction in [BL87] (if centralizer condition holds).

If the nilradical is zero, the cone condition is trivially satisfied. The theorem follows directly from [Fer90] (non-existence if centralizer condition fails) and the construction in [BL87] (if centralizer condition holds). The following proposition relates to the combinatorics involved in the proof of the theorem when the nilradical is non-zero.

Proposition

Suppose you can find a relation

$$a_1\alpha_1 + \cdots + a_l\alpha_l = b_1\beta_1 + \cdots + b_k\beta_k$$

where $\alpha \in \mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_i \in \Delta(\mathfrak{n})$, $a_i, b_j \in \mathbb{Z}_{>0}$, *and in addition*

If the nilradical is zero, the cone condition is trivially satisfied. The theorem follows directly from [Fer90] (non-existence if centralizer condition fails) and the construction in [BL87] (if centralizer condition holds). The following proposition relates to the combinatorics involved in the proof of the theorem when the nilradical is non-zero.

Proposition

Suppose you can find a relation

$$a_1\alpha_1 + \cdots + a_l\alpha_l = b_1\beta_1 + \cdots + b_k\beta_k$$

where $\alpha \in \mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_i \in \Delta(\mathfrak{n})$, $a_i, b_j \in \mathbb{Z}_{>0}$, *and in addition*

- $\beta_i \pm \beta_j$ *is not a root for all* i, j

If the nilradical is zero, the cone condition is trivially satisfied. The theorem follows directly from [Fer90] (non-existence if centralizer condition fails) and the construction in [BL87] (if centralizer condition holds). The following proposition relates to the combinatorics involved in the proof of the theorem when the nilradical is non-zero.

Proposition

Suppose you can find a relation

$$a_1\alpha_1 + \cdots + a_l\alpha_l = b_1\beta_1 + \cdots + b_k\beta_k$$

where $\alpha \in \mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_i \in \Delta(\mathfrak{n})$, $a_i, b_j \in \mathbb{Z}_{>0}$, *and in addition*

- $\beta_i \pm \beta_j$ *is not a root for all* i, j
- $[\mathfrak{g}^{\alpha_i}, \mathfrak{n}] \subset \mathfrak{n}$ *for all* i.

If the nilradical is zero, the cone condition is trivially satisfied. The theorem follows directly from [Fer90] (non-existence if centralizer condition fails) and the construction in [BL87] (if centralizer condition holds). The following proposition relates to the combinatorics involved in the proof of the theorem when the nilradical is non-zero.

Proposition

Suppose you can find a relation

$$a_1\alpha_1 + \cdots + a_l\alpha_l = b_1\beta_1 + \cdots + b_k\beta_k$$

where $\alpha \in \mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_i \in \Delta(\mathfrak{n})$, $a_i, b_j \in \mathbb{Z}_{>0}$, *and in addition*

- $\beta_i \pm \beta_j$ *is not a root for all* i, j
- $[\mathfrak{g}^{\alpha_i}, \mathfrak{n}] \subset \mathfrak{n}$ *for all* i.

Then $\mathfrak{l} = \mathfrak{k} \ni \mathfrak{n}$ *is not Fernando-Kac of finite type.*

If the nilradical is zero, the cone condition is trivially satisfied. The theorem follows directly from [Fer90] (non-existence if centralizer condition fails) and the construction in [BL87] (if centralizer condition holds). The following proposition relates to the combinatorics involved in the proof of the theorem when the nilradical is non-zero.

Proposition

Suppose you can find a relation

$$a_1\alpha_1 + \cdots + a_l\alpha_l = b_1\beta_1 + \cdots + b_k\beta_k$$

where $\alpha \in \mathrm{Sing}_{\mathfrak{b}\cap\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_i \in \Delta(\mathfrak{n})$, $a_i, b_j \in \mathbb{Z}_{>0}$, *and in addition*

- $\beta_i \pm \beta_j$ *is not a root for all* i, j
- $[\mathfrak{g}^{\alpha_i}, \mathfrak{n}] \subset \mathfrak{n}$ *for all* i.

Then $\mathfrak{l} = \mathfrak{k} \ni \mathfrak{n}$ *is not Fernando-Kac of finite type.*
The failure of the cone condition alone is sufficient for the existence of such a relation in types A_n, B_n, D_n, E_6, E_7, G_2.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.
- An algorithm to decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.
- An algorithm to decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module.
- Tables for E_8 or any other rank≤ 8 simple Lie algebra on demand.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.
- An algorithm to decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module.
- Tables for E_8 or any other rank≤ 8 simple Lie algebra on demand.
- An algorithm to enumerate all nilradicals $\mathfrak{n}$ up to isomorphism that can be attached to a reductive subalgebra $\mathfrak{k} \supset \mathfrak{h}$.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.
- An algorithm to decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module.
- Tables for E_8 or any other rank≤ 8 simple Lie algebra on demand.
- An algorithm to enumerate all nilradicals $\mathfrak{n}$ up to isomorphism that can be attached to a reductive subalgebra $\mathfrak{k} \supset \mathfrak{h}$.
- An algorithm to generate all $\mathrm{sl}(2)$-subalgebras of a simple Lie algebra following Dynkin.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.
- An algorithm to decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module.
- Tables for E_8 or any other rank$\leq$ 8 simple Lie algebra on demand.
- An algorithm to enumerate all nilradicals $\mathfrak{n}$ up to isomorphism that can be attached to a reductive subalgebra $\mathfrak{k} \supset \mathfrak{h}$.
- An algorithm to generate all $\mathrm{sl}(2)$-subalgebras of a simple Lie algebra following Dynkin.
- Tables for E_8 or any other rank$\leq$ 8 simple Lie algebra on demand.

Plan for the rest of the talk

- An algorithm to generate all root subsystems of $\mathfrak{g}$ explicitly.
- An algorithm to decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module.
- Tables for E_8 or any other rank$\leq$ 8 simple Lie algebra on demand.
- An algorithm to enumerate all nilradicals $\mathfrak{n}$ up to isomorphism that can be attached to a reductive subalgebra $\mathfrak{k} \supset \mathfrak{h}$.
- An algorithm to generate all $\mathrm{sl}(2)$-subalgebras of a simple Lie algebra following Dynkin.
- Tables for E_8 or any other rank$\leq$ 8 simple Lie algebra on demand.
- Rant about C++, the "vector partition" program, etc. `http://vector-partition.jacobs-university.de/cgi-bin/vector_partition_linux_cgi?rootSAs`

Generating root subsystems up to isomorphism

Δ parametrizes the elements of a root subsystem and Δ' a simple basis of Δ; R is a list of root subsystems.

Generating root subsystems up to isomorphism

Δ parametrizes the elements of a root subsystem and Δ' a simple basis of Δ; R is a list of root subsystems.

- Step 0. Initialize Δ and Δ' to be the empty, and R to be the empty list.

Generating root subsystems up to isomorphism

Δ parametrizes the elements of a root subsystem and Δ' a simple basis of Δ; R is a list of root subsystems.

- Step 0. Initialize Δ and Δ' to be the empty, and R to be the empty list.
- Step 1. For each root α in $\Delta(\mathfrak{g})\setminus\Delta$ set $\Delta = \Delta \cup \{\alpha\}$. Corresponds to a "for loop".

Δ parametrizes the elements of a root subsystem and Δ' a simple basis of Δ; R is a list of root subsystems.

- Step 0. Initialize Δ and Δ' to be the empty, and R to be the empty list.
- Step 1. For each root α in $\Delta(\mathfrak{g})\backslash\Delta$ set $\Delta = \Delta \cup \{\alpha\}$. Corresponds to a "for loop".
- Step 2. Compute the simple basis Δ' of Δ.

Generating root subsystems up to isomorphism

Δ parametrizes the elements of a root subsystem and Δ' a simple basis of Δ; R is a list of root subsystems.

- Step 0. Initialize Δ and Δ' to be the empty, and R to be the empty list.
- Step 1. For each root α in $\Delta(\mathfrak{g})\setminus\Delta$ set $\Delta = \Delta \cup \{\alpha\}$. Corresponds to a "for loop".
- Step 2. Compute the simple basis Δ' of Δ.
- Step 3. Check whether Δ' is isomorphic via a root system isomorphism of $\Delta(\mathfrak{g})$ to a simple basis of an element already present in R. If so, terminate the current branch of computation. Otherwise, add Δ to R and go to Step 1.

Note: step 3 - not needed in the sense that one can compare only two pairs of Dynkin diagrams.

Proposition

For the two root subsystems Δ_1 and Δ_2 to be isomorphic via isomorphism of $\Delta(\mathfrak{g})$ it is necessary and sufficient that their Dynkin diagrams and the Dynkin diagrams of $\Delta_1^{\perp\!\!\!\!\perp}$ and $\Delta_2^{\perp\!\!\!\!\perp}$ are isomorphic, where $\perp\!\!\!\!\perp$ stands for strongly orthogonal ($\alpha \perp\!\!\!\!\perp \beta$ if $\alpha \pm \beta$ is not a root).

Note: step 3 - not needed in the sense that one can compare only two pairs of Dynkin diagrams.

Proposition

For the two root subsystems Δ_1 and Δ_2 to be isomorphic via isomorphism of $\Delta(\mathfrak{g})$ it is necessary and sufficient that their Dynkin diagrams and the Dynkin diagrams of $\Delta_1^{\perp\!\!\!\perp}$ and $\Delta_2^{\perp\!\!\!\perp}$ are isomorphic, where $\perp\!\!\!\perp$ stands for strongly orthogonal ($\alpha \perp\!\!\!\perp \beta$ if $\alpha \pm \beta$ is not a root).

Note: if Δ parametrizes a regular subalgebra $[\mathfrak{k}, \mathfrak{k}]$, then $\Delta^{\perp\!\!\!\perp}$ parametrizes the root system of the centralizer of $[\mathfrak{k}, \mathfrak{k}]$.

Note: step 3 - not needed in the sense that one can compare only two pairs of Dynkin diagrams.

Proposition

For the two root subsystems Δ_1 and Δ_2 to be isomorphic via isomorphism of $\Delta(\mathfrak{g})$ it is necessary and sufficient that their Dynkin diagrams and the Dynkin diagrams of $\Delta_1^{\perp\!\!\!\perp}$ and $\Delta_2^{\perp\!\!\!\perp}$ are isomorphic, where $\perp\!\!\!\perp$ stands for strongly orthogonal ($\alpha \perp\!\!\!\perp \beta$ if $\alpha \pm \beta$ is not a root).

Note: if Δ parametrizes a regular subalgebra $[\mathfrak{k}, \mathfrak{k}]$, then $\Delta^{\perp\!\!\!\perp}$ parametrizes the root system of the centralizer of $[\mathfrak{k}, \mathfrak{k}]$. The centralizer of a regular subalgebra consists of a regular subalgebra and a piece of the Cartan subalgebra.

Decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module

- $\mathfrak{k}$ contains a Cartan subalgebra and therefore separates root spaces. Therefore any $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is a direct sum of root spaces.

Decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module

- $\mathfrak{k}$ contains a Cartan subalgebra and therefore separates root spaces. Therefore any $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is a direct sum of root spaces.
- Any simple $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is uniquely identified by its $\mathfrak{b} \cap \mathfrak{k}$-singular weight vector.

Decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module

- $\mathfrak{k}$ contains a Cartan subalgebra and therefore separates root spaces. Therefore any $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is a direct sum of root spaces.
- Any simple $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is uniquely identified by its $\mathfrak{b}\cap\mathfrak{k}$-singular weight vector.
- $g^{\alpha}\in\mathfrak{g}^{\alpha}$ is $\mathfrak{b}\cap\mathfrak{k}$-singular if $\alpha+\gamma$ is not a root for any $\gamma\in\Delta(\mathfrak{b}\cap\mathfrak{k})$.

Decompose $\mathfrak{g}$ as a $\mathfrak{k}$-module

- $\mathfrak{k}$ contains a Cartan subalgebra and therefore separates root spaces. Therefore any $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is a direct sum of root spaces.
- Any simple $\mathfrak{k}$-submodule of $\mathfrak{g}/\mathfrak{k}$ is uniquely identified by its $\mathfrak{b}\cap\mathfrak{k}$-singular weight vector.
- $g^\alpha \in \mathfrak{g}^\alpha$ is $\mathfrak{b}\cap\mathfrak{k}$-singular if $\alpha+\gamma$ is not a root for any $\gamma\in\Delta(\mathfrak{b}\cap\mathfrak{k})$.
- Take an arbitrary α' in $\Delta(\mathfrak{g})\backslash\Delta(\mathfrak{k})$. Start adding positive roots of $\mathfrak{k}$ to α' until possible. The root obtained in the end is the $\mathfrak{b}\cap\mathfrak{k}$-singular vector in the simple $\mathfrak{k}$-submodule containing $g^{\alpha'}$.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.
- Opposite modules. M_i is opposite to M_j if $\mathrm{Weights}(M_i) = -\mathrm{Weights}(M_j)$.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.
- Opposite modules. M_i is opposite to M_j if $\mathrm{Weights}(M_i) = -\mathrm{Weights}(M_j)$.
- Compute the group W' of all root system automorphisms of $\Delta(\mathfrak{g})$ that preserve $\Delta(\mathfrak{b} \cap \mathfrak{k})$.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.
- Opposite modules. M_i is opposite to M_j if $\mathrm{Weights}(M_i) = -\mathrm{Weights}(M_j)$.
- Compute the group W' of all root system automorphisms of $\Delta(\mathfrak{g})$ that preserve $\Delta(\mathfrak{b} \cap \mathfrak{k})$. Example: $\Delta(\mathfrak{k}) = 7A_1 \subset E_7$. Then W' has 168 elements.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.
- Opposite modules. M_i is opposite to M_j if $\mathrm{Weights}(M_i) = -\mathrm{Weights}(M_j)$.
- Compute the group W' of all root system automorphisms of $\Delta(\mathfrak{g})$ that preserve $\Delta(\mathfrak{b} \cap \mathfrak{k})$. Example: $\Delta(\mathfrak{k}) = 7A_1 \subset E_7$. Then W' has 168 elements.
- Choose an arbitrary order $\prec'$ on the set of all subsets of $\{M_1, \ldots, M_N\}$. Using W' and $\prec'$ induce a partial order $\prec$ on all subsets of $\{M_1, \ldots, M_N\}$.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.
- Opposite modules. M_i is opposite to M_j if $\mathrm{Weights}(M_i) = -\mathrm{Weights}(M_j)$.
- Compute the group W' of all root system automorphisms of $\Delta(\mathfrak{g})$ that preserve $\Delta(\mathfrak{b} \cap \mathfrak{k})$. Example: $\Delta(\mathfrak{k}) = 7A_1 \subset E_7$. Then W' has 168 elements.
- Choose an arbitrary order $\prec'$ on the set of all subsets of $\{M_1, \dots, M_N\}$. Using W' and $\prec'$ induce a partial order $\prec$ on all subsets of $\{M_1, \dots, M_N\}$.
- To enumerate all nilradicals up to isomorphism (getting one representative in each W'- class) one enumerates all subsets of the M_i's that respect the pairing table, have no opposite modules, and respect $\prec$.

Generate all nilradicals up to isomorphism that can be attached to $\mathfrak{k}$ (one representative per isomorphism class)

Let the $\mathfrak{k}$-module decomposition of $\mathfrak{g}/\mathfrak{k}$ be $M_1 \oplus \cdots \oplus M_N$.

- Pairing table. We say M_i and M_j pair to M_k if there exist $\alpha \in \mathrm{Weights}(M_i)$ and $\beta \in \mathrm{Weights}(M_j)$ such that $\alpha + \beta \in \mathrm{Weights}(M_k)$.
- Opposite modules. M_i is opposite to M_j if $\mathrm{Weights}(M_i) = -\mathrm{Weights}(M_j)$.
- Compute the group W' of all root system automorphisms of $\Delta(\mathfrak{g})$ that preserve $\Delta(\mathfrak{b} \cap \mathfrak{k})$. Example: $\Delta(\mathfrak{k}) = 7A_1 \subset E_7$. Then W' has 168 elements.
- Choose an arbitrary order $\prec'$ on the set of all subsets of $\{M_1, \dots, M_N\}$. Using W' and $\prec'$ induce a partial order $\prec$ on all subsets of $\{M_1, \dots, M_N\}$.
- To enumerate all nilradicals up to isomorphism (getting one representative in each W'- class) one enumerates all subsets of the M_i's that respect the pairing table, have no opposite modules, and respect $\prec$. Numerology: E_6: 64580 possibilities.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

- Let our $\mathrm{sl}(2)$ be given by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. h can be assumed to lie in a Cartan s.a. of $\mathfrak{g}$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

- Let our $\mathrm{sl}(2)$ be given by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. h can be assumed to lie in a Cartan s.a. of $\mathfrak{g}$.
- Suppose we know h. Compute a simple basis $\alpha_1, \dots, \alpha_n$ of $\Delta(\mathfrak{g})$ **with respect to** h. Define the characteristic of h as the n-tuple $(\alpha_1(h), \dots, \alpha_n(h))$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

- Let our $\mathrm{sl}(2)$ be given by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. h can be assumed to lie in a Cartan s.a. of $\mathfrak{g}$.
- Suppose we know h. Compute a simple basis $\alpha_1, \dots, \alpha_n$ of $\Delta(\mathfrak{g})$ **with respect to** h. Define the characteristic of h as the n-tuple $(\alpha_1(h), \dots, \alpha_n(h))$.
- The characteristic of h can consist of 0,1 and 2's only. Whenever characteristics of two $\mathrm{sl}(2)$'s coincide, the $\mathrm{sl}(2)$'s are conjugate.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

- Let our $\mathrm{sl}(2)$ be given by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. h can be assumed to lie in a Cartan s.a. of $\mathfrak{g}$.
- Suppose we know h. Compute a simple basis $\alpha_1, \dots, \alpha_n$ of $\Delta(\mathfrak{g})$ **with respect to** h. Define the characteristic of h as the n-tuple $(\alpha_1(h), \dots, \alpha_n(h))$.
- The characteristic of h can consist of 0,1 and 2's only. Whenever characteristics of two $\mathrm{sl}(2)$'s coincide, the $\mathrm{sl}(2)$'s are conjugate.
- If there is a 1 in the characteristic, then our $\mathrm{sl}(2)$ lies inside a regular subalgebra of $\mathfrak{g}$ with rank smaller than $\mathrm{rk}\mathfrak{g}$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

- Let our $\mathrm{sl}(2)$ be given by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. h can be assumed to lie in a Cartan s.a. of $\mathfrak{g}$.
- Suppose we know h. Compute a simple basis $\alpha_1, \dots, \alpha_n$ of $\Delta(\mathfrak{g})$ **with respect to** h. Define the characteristic of h as the n-tuple $(\alpha_1(h), \dots, \alpha_n(h))$.
- The characteristic of h can consist of 0,1 and 2's only. Whenever characteristics of two $\mathrm{sl}(2)$'s coincide, the $\mathrm{sl}(2)$'s are conjugate.
- If there is a 1 in the characteristic, then our $\mathrm{sl}(2)$ lies inside a regular subalgebra of $\mathfrak{g}$ with rank smaller than $\mathrm{rk}\mathfrak{g}$.
- The element e is a linear combination of elements of $\mathfrak{g}^{\alpha}$, where α runs over all roots with $\alpha(h) = 2$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: starting facts form Dynkin

Some facts from "Semisimple Lie subalgebras of semisimple Lie algebras".

- Let our $\mathrm{sl}(2)$ be given by h, e, f with $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. h can be assumed to lie in a Cartan s.a. of $\mathfrak{g}$.
- Suppose we know h. Compute a simple basis $\alpha_1, \dots, \alpha_n$ of $\Delta(\mathfrak{g})$ **with respect to** h. Define the characteristic of h as the n-tuple $(\alpha_1(h), \dots, \alpha_n(h))$.
- The characteristic of h can consist of 0,1 and 2's only. Whenever characteristics of two $\mathrm{sl}(2)$'s coincide, the $\mathrm{sl}(2)$'s are conjugate.
- If there is a 1 in the characteristic, then our $\mathrm{sl}(2)$ lies inside a regular subalgebra of $\mathfrak{g}$ with rank smaller than $\mathrm{rk}\mathfrak{g}$.
- The element e is a linear combination of elements of $\mathfrak{g}^\alpha$, where α runs over all roots with $\alpha(h) = 2$.
- The element f is a linear combination of elements of $\mathfrak{g}^\alpha$, where α runs over all roots with $\alpha(h) = -2$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.
- Throw out h's that would imply a non-trivial centralizer of the $\mathrm{sl}(2)$ in $[\mathfrak{k}, \mathfrak{k}]$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.
- Throw out h's that would imply a non-trivial centralizer of the $\mathrm{sl}(2)$ in $[\mathfrak{k}, \mathfrak{k}]$.
- Write $e = \sum_{\alpha_i(h)=2} a_i g^{\alpha_i}$, $f = \sum_{\alpha_i(h)=-2} b_i g^{-\alpha_i}$, where a_i, b_i - unknowns.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.
- Throw out h's that would imply a non-trivial centralizer of the $\mathrm{sl}(2)$ in $[\mathfrak{k}, \mathfrak{k}]$.
- Write $e = \sum_{\alpha_i(h)=2} a_i g^{\alpha_i}$, $f = \sum_{\alpha_i(h)=-2} b_i g^{-\alpha_i}$, where a_i, b_i - unknowns.
- Then $[e, f] = \sum_{i\neq j} n_{\alpha_i,-\alpha_j} a_i b_j g^{\alpha_i-\alpha_j} + \sum a_i b_i \frac{2}{\langle\alpha_i,\alpha_i\rangle} h_{\alpha_i}$, where $n_{\alpha_i,-\alpha_j}$ are the structure constants.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.
- Throw out h's that would imply a non-trivial centralizer of the $\mathrm{sl}(2)$ in $[\mathfrak{k}, \mathfrak{k}]$.
- Write $e = \sum_{\alpha_i(h)=2} a_i g^{\alpha_i}$, $f = \sum_{\alpha_i(h)=-2} b_i g^{-\alpha_i}$, where a_i, b_i - unknowns.
- Then $[e, f] = \sum_{i \neq j} n_{\alpha_i, -\alpha_j} a_i b_j g^{\alpha_i - \alpha_j} + \sum a_i b_i \frac{2}{\langle \alpha_i, \alpha_i \rangle} h_{\alpha_i}$, where $n_{\alpha_i, -\alpha_j}$ are the structure constants.
- This is a quadratic system of m equations, where m equals the sum of $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ and the number of roots of the form $\alpha_i - \alpha_j$.

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.
- Throw out h's that would imply a non-trivial centralizer of the $\mathrm{sl}(2)$ in $[\mathfrak{k}, \mathfrak{k}]$.
- Write $e = \sum_{\alpha_i(h)=2} a_i g^{\alpha_i}$, $f = \sum_{\alpha_i(h)=-2} b_i g^{-\alpha_i}$, where a_i, b_i - unknowns.
- Then $[e, f] = \sum_{i \neq j} n_{\alpha_i, -\alpha_j} a_i b_j g^{\alpha_i - \alpha_j} + \sum a_i b_i \frac{2}{\langle \alpha_i, \alpha_i \rangle} h_{\alpha_i}$, where $n_{\alpha_i, -\alpha_j}$ are the structure constants.
- This is a quadratic system of m equations, where m equals the sum of $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ and the number of roots of the form $\alpha_i - \alpha_j$. Solve it!

Generate all $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: algorithm

- Fix a root subsystem $\Delta(\mathfrak{k})$ and the corresponding subalgebra $\mathfrak{k} \supset \mathfrak{h}$. The corresponding regular subalgebra is $[\mathfrak{k}, \mathfrak{k}]$.
- Fix a simple basis of $\Delta(\mathfrak{k})$.
- Enumerate all possible characteristics of h of length $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ that do not contain 1's (total $2^{\mathrm{rk}[\mathfrak{k},\mathfrak{k}]}$).
- Generate the corresponding h's.
- Throw out h's that would imply a non-trivial centralizer of the $\mathrm{sl}(2)$ in $[\mathfrak{k}, \mathfrak{k}]$.
- Write $e = \sum_{\alpha_i(h)=2} a_i g^{\alpha_i}$, $f = \sum_{\alpha_i(h)=-2} b_i g^{-\alpha_i}$, where a_i, b_i - unknowns.
- Then $[e, f] = \sum_{i \neq j} n_{\alpha_i, -\alpha_j} a_i b_j g^{\alpha_i - \alpha_j} + \sum a_i b_i \frac{2}{\langle \alpha_i, \alpha_i \rangle} h_{\alpha_i}$, where $n_{\alpha_i, -\alpha_j}$ are the structure constants.
- This is a quadratic system of m equations, where m equals the sum of $\mathrm{rk}[\mathfrak{k}, \mathfrak{k}]$ and the number of roots of the form $\alpha_i - \alpha_j$. Solve it!
- Find a simple basis of $\mathfrak{g}$ with respect to h to recover the characteristic of h in $\mathfrak{g}$.

The "vector partition" program

- Project started December 2008.
- Can compute:
  - Everything described in the talk.
  - Algebraic expressions in closed form for the Kostant partition function. Can go up to A_6, D_4, B_4, C_4.
  - Hyperplane arrangements (needed to describe the combinatorial chambers for the Kostant partition function).
  - Weyl groups, Kazhdan-Lusztig coefficients, structure constants of simple Lie algebras.
  - Simplex algorithm (basic implementation).
  - Has its own large integer/rational number library, classes implementing quasipolynomials over $\mathbb{Q}$.
  - Uses its own classes for hashing arrays. No external packages!
- Current size of the mathematical part: $\sim$30 000 lines of code. Total project size >35 000 lines of code.
- 560+ commits in the public source code repository.

Thank you for Your attention!

D. J. Britten and F. W. Lemire.
A classification of simple Lie modules having a 1-dimensional weight space.
Transactions of the American Mathematical Society, 299:683–697, 1987.

E. Dynkin.
Semisimple subalgebras of semisimple Lie algebras.
Selected Papers of E. B. Dynkin with Commentary, pages 111– 312, 1972.

S. Fernando.
Lie algebra modules with finite-dimensional weight spaces.
Trans. Amer. Math. Soc., 322:2857–2869, 1990.

V. Kac.
Constructing groups associated to infinite-dimensional Lie algebras.
Infinite-dimensional groups with applications, 4:167–216, 1985.

I. Penkov, V. Serganova, and G. Zuckerman.
On the existence of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type.
Duke Math. Journ., 125:329–349, 2004.