Unitary Representations of Nilpotent Super Lie groups

Hadi Salmasian
University of Ottawa

July 6, 2010

The orbit method

Unitary representations of G $\longleftrightarrow$ Quantization of symplectic G – manifolds

The orbit method

$$\begin{array}{c}\text{Unitary representations}\\ \text{of } G\end{array} \longleftrightarrow \begin{array}{c}\text{Quantization of}\\ \text{symplectic } G-\text{manifolds}\end{array}$$

If G is a nilpotent simply connected Lie group, then there exists a bijective correspondence

$$\begin{array}{c}\text{Irreducible unitary}\\ \text{representations of } G\end{array} \leftrightsquigarrow \begin{array}{c}G-\text{orbits}\\ \text{in } \mathfrak{g}^*\end{array}$$

The orbit method

$$\begin{array}{c} \text{Unitary representations} \\ \text{of } G \end{array} \longleftrightarrow \begin{array}{c} \text{Quantization of} \\ \text{symplectic } G-\text{manifolds} \end{array}$$

If G is a nilpotent simply connected Lie group, then there exists a bijective correspondence

$$\begin{array}{c} \text{Irreducible unitary} \\ \text{representations of } G \end{array} \leftrightsquigarrow \begin{array}{c} G-\text{orbits} \\ \text{in } \mathfrak{g}^* \end{array}$$

There is a dictionary :

Algebraic operation	Geometric operation
$\mathrm{Res}_H^G \pi$	$p(\mathcal{O})$ where $p : \mathfrak{g}^* \to \mathfrak{h}^*$
$\mathrm{Ind}_H^G \pi$	$p^{-1}(\mathcal{O})$ where $p : \mathfrak{g}^* \to \mathfrak{h}^*$
$\pi_1 \otimes \pi_2$	$\mathcal{O}_1 + \mathcal{O}_2$
...	...

Nilpotent Lie groups

G : nilpotent and simply connected $\qquad \mathfrak{g} = \mathrm{Lie}(G) \qquad \mathcal{O} \subset \mathfrak{g}^*$: a G-orbit.

Nilpotent Lie groups

G : nilpotent and simply connected $\quad \mathfrak{g} = \mathrm{Lie}(G) \quad \mathcal{O} \subset \mathfrak{g}^*$: a G-orbit.

Recipe to construct π from $\mathcal{O}$

1. Fix $\lambda \in \mathcal{O}$. Consider the skew-symmetric form

$$\Omega_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

defined by $\Omega_\lambda(X, Y) = \lambda([X, Y])$.

Nilpotent Lie groups

G : nilpotent and simply connected $\qquad \mathfrak{g} = \mathrm{Lie}(G) \qquad \mathcal{O} \subset \mathfrak{g}^*$: a G-orbit.

Recipe to construct π from $\mathcal{O}$

1. Fix $\lambda \in \mathcal{O}$. Consider the skew-symmetric form

$$\Omega_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

   defined by $\Omega_\lambda(X, Y) = \lambda([X, Y])$.

2. **Proposition.** There exists a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{m}$ is a maximal isotropic subspace of Ω_λ.

Nilpotent Lie groups

G : nilpotent and simply connected $\qquad \mathfrak{g} = \mathrm{Lie}(G) \qquad \mathcal{O} \subset \mathfrak{g}^*$: a G-orbit.

Recipe to construct π from $\mathcal{O}$

1. Fix $\lambda \in \mathcal{O}$. Consider the skew-symmetric form
$$\Omega_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$
defined by $\Omega_\lambda(X, Y) = \lambda([X, Y])$.
2. **Proposition.** There exists a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{m}$ is a maximal isotropic subspace of Ω_λ.
3. Set $M = \exp(\mathfrak{m})$ and define $\chi_\lambda : M \to \mathbb{C}^\times$ by
$$\chi_\lambda(\exp(X)) = e^{\lambda(X)\sqrt{-1}} \qquad \text{for every } X \in \mathfrak{m}.$$

Nilpotent Lie groups

G : nilpotent and simply connected $\qquad \mathfrak{g} = \mathrm{Lie}(G) \qquad \mathcal{O} \subset \mathfrak{g}^*$: a G-orbit.

Recipe to construct π from $\mathcal{O}$

1. Fix $\lambda \in \mathcal{O}$. Consider the skew-symmetric form
$$\Omega_\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$
defined by $\Omega_\lambda(X, Y) = \lambda([X, Y])$.
2. **Proposition.** There exists a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{m}$ is a maximal isotropic subspace of Ω_λ.
3. Set $M = \exp(\mathfrak{m})$ and define $\chi_\lambda : M \to \mathbb{C}^\times$ by
$$\chi_\lambda(\exp(X)) = e^{\lambda(X)\sqrt{-1}} \qquad \text{for every } X \in \mathfrak{m}.$$
4. Set $\pi = \mathrm{Ind}_M^G \chi_\lambda$.

Example : the Schrödinger model

- (W, Ω) : finite dimensional symplectic vector space, i.e.,
  - Ω is nondegenerate,
  - $\Omega(v, w) = -\Omega(w, v)$.

Example : the Schrödinger model

- (W, Ω) : finite dimensional symplectic vector space, i.e.,
  - Ω is nondegenerate,
  - $\Omega(v, w) = -\Omega(w, v)$.
- **The Heisenberg group :**

$$H_n = \{\, (v, s) \mid v \in W \text{ and } s \in \mathbb{R}\}$$

The group law is given by

$$(v_1, s_1) \bullet (v_2, s_2) = (v_1 + v_2, s_1 + s_2 + \frac{1}{2}\Omega(v_1, v_2)).$$

Example : the Schrödinger model

- (W, Ω) : finite dimensional symplectic vector space, i.e.,
  - Ω is nondegenerate,
  - $\Omega(v, w) = -\Omega(w, v)$.
- **The Heisenberg group :**

$$H_n = \{ (v, s) \mid v \in W \text{ and } s \in \mathbb{R} \}$$

  The group law is given by

$$(v_1, s_1) \bullet (v_2, s_2) = (v_1 + v_2, s_1 + s_2 + \frac{1}{2}\Omega(v_1, v_2)).$$

- $\dim \mathcal{Z}(H_n) = 1$ and $H_n/\mathcal{Z}(H_n)$ is commutative (i.e., H_n is two-step nilpotent).

Example : the Schrödinger model (cont.)

- Consider a polarization of (W, Ω), i.e., a direct sum decomposition

$$W = X \oplus Y \text{ such that } \Omega(X, X) = \Omega(Y, Y) = 0.$$

Example : the Schrödinger model (cont.)

- Consider a polarization of (W, Ω), i.e., a direct sum decomposition

$$W = X \oplus Y \text{ such that } \Omega(X, X) = \Omega(Y, Y) = 0.$$

- Set $\mathcal{H} := \mathrm{L}^2(Y) := \{ f : Y \to \mathbb{C} \mid \int_Y |f|^2 d\mu < \infty \}$.

Example : the Schrödinger model (cont.)

- Consider a polarization of (W, Ω), i.e., a direct sum decomposition

$$W = X \oplus Y \text{ such that } \Omega(X, X) = \Omega(Y, Y) = 0.$$

- Set $\mathcal{H} := \mathrm{L}^2(Y) := \{ f : Y \to \mathbb{C} \mid \int_Y |f|^2 d\mu < \infty \}$.
- Fix a nonzero $a \in \mathbb{R}$ and define a representation π_a of H_n on $\mathcal{H}$ via

$$\begin{aligned}
\Big(\pi_a(v, 0) f\Big)(y) &= e^{a\Omega(y,v)\sqrt{-1}} f(y) && \text{if } v \in X, \\
\Big(\pi_a(0, v) f\Big)(y) &= f(y + v) && \text{if } v \in Y, \\
\Big(\pi_a(0, s) f\Big)(y) &= e^{as\sqrt{-1}} f(y) && \text{otherwise.}
\end{aligned}$$

Example : the Schrödinger model (cont.)

Facts:

- For every $a \in \mathbb{R}$, π_a is an irreducible unitary rep. of H_n.

Example : the Schrödinger model (cont.)

Facts:

- For every $a \in \mathbb{R}$, π_a is an irreducible unitary rep. of H_n.
- If $a \neq b$, the representations π_a and π_b are *not* unitarily equivalent.

Example : the Schrödinger model (cont.)

Facts:

- For every $a \in \mathbb{R}$, π_a is an irreducible unitary rep. of H_n.
- If $a \neq b$, the representations π_a and π_b are *not* unitarily equivalent.

Theorem (Stone-von Neumann, 1930's)

Up to unitary equivalence, an irreducible unitary representation of H_n is one of the following :

1. A one-dimensional representation (which factors through $H_n/\mathcal{Z}(H_n)$),
2. π_a, for some $a \in \mathbb{R}^\times$.

Example : Schrödinger model and the orbit method

Recall that :

$$H_n = \{ (v, s) \mid v \in W \text{ and } s \in \mathbb{R} \}$$

Set $\mathfrak{h}_n = \mathrm{Lie}(H_n)$ and fix $Z \in \mathcal{Z}(\mathfrak{h}_n)$.

Example : Schrödinger model and the orbit method

Recall that :

$$H_n = \{ (v, s) \mid v \in W \text{ and } s \in \mathbb{R}\}$$

Set $\mathfrak{h}_n = \mathrm{Lie}(H_n)$ and fix $Z \in \mathcal{Z}(\mathfrak{h}_n)$.

H_n-orbits in $\mathfrak{h}_n^*$ are :

- $\{\lambda\}$ where $\lambda(Z) = 0$ $\leftrightsquigarrow$ one-dimensional representations of H_n.
- $\{\lambda \in \mathfrak{h}_n^* \mid \lambda(Z) = a\}$ $\leftrightsquigarrow$ the representation π_a.

Solvable and semisimple groups

Theorem (Auslander - Kostant)

Suppose G is a solvable, connected, simply connected, type I Lie group. Then

$$\widehat{G} = \bigsqcup_{\mathcal{O} \subset \mathfrak{g}^*} \mathcal{S}_{\mathcal{O}}$$

where each $\mathcal{S}_{\mathcal{O}}$ is a torus of dimension $b_1(\mathcal{O})$ = first betti number of $\mathcal{O}$.

Solvable and semisimple groups

Theorem (Auslander - Kostant)

Suppose G is a solvable, connected, simply connected, type I Lie group. Then

$$\widehat{G} = \bigsqcup_{\mathcal{O} \subset \mathfrak{g}^*} \mathcal{S}_{\mathcal{O}}$$

where each $\mathcal{S}_{\mathcal{O}}$ is a torus of dimension $b_1(\mathcal{O})$ = first betti number of $\mathcal{O}$.

Semisimple Groups

- Elliptic orbits $\leftrightsquigarrow$ Discrete series
- Nilpotent orbits $\leftrightsquigarrow$ associated varieties of unitary rep's
- ...

Crash course on Lie superalgebras

- Introduced by physicists – motivated by supersymmetry.

Crash course on Lie superalgebras

- Introduced by physicists – motivated by supersymmetry.
- A (nonassociative) superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ (i.e., $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j \pmod 2}$).

Crash course on Lie superalgebras

- Introduced by physicists – motivated by supersymmetry.
- A (nonassociative) superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ (i.e., $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j \pmod 2}$).
- A *Lie superalgebra* is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a "bracket"
$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
satisfying
$$[X, Y] = -(-1)^{|X|\cdot|Y|}[Y, X]$$
and
$$(-1)^{|X|\cdot|Z|}[X, [Y, Z]] + (-1)^{|Y|\cdot|X|}[Y, [Z, X]] + (-1)^{|Z|\cdot|Y|}[Z, [X, Y]] = 0$$

Crash course on Lie superalgebras (cont.)

Examples of Lie superalgebras

- $\mathfrak{gl}(m|n)$:
  $V = V_0 \oplus V_1$ and $\mathfrak{g} = \mathrm{End}(V) = \mathrm{End}_0(V) \oplus \mathrm{End}_1(V)$
  where

$$\mathrm{End}_i(V) = \left\{ T \in \mathrm{End}(V) \mid T(V_s) \subseteq V_{s+i \pmod 2} \text{ for any s} \in \mathbb{Z}/2\mathbb{Z} \right\}$$

and for homogeneous X and Y, the bracket is given by

$$[X, Y] = XY - (-1)^{|X|\cdot|Y|} YX$$

Crash course on Lie superalgebras (cont.)

Examples of Lie superalgebras

- $\mathfrak{gl}(m|n)$:
  $V = V_0 \oplus V_1$ and $\mathfrak{g} = \mathrm{End}(V) = \mathrm{End}_0(V) \oplus \mathrm{End}_1(V)$
  where

  $$\mathrm{End}_i(V) = \left\{ T \in \mathrm{End}(V) \mid T(V_s) \subseteq V_{s+i \pmod 2} \text{ for any s} \in \mathbb{Z}/2\mathbb{Z} \right\}$$

  and for homogeneous X and Y, the bracket is given by

  $$[X, Y] = XY - (-1)^{|X|\cdot|Y|} YX$$

- Simple Lie superalgebras:
  $\mathfrak{sl}(m|n), \mathfrak{osp}(m|2n), \mathfrak{f}(4), \mathfrak{g}(3), \mathfrak{p}(n), \mathfrak{q}(n), \ldots$

Crash course on Lie superalgebras (cont.)

Examples of Lie superalgebras

- $\mathfrak{gl}(m|n)$:
  $V = V_0 \oplus V_1$ and $\mathfrak{g} = \mathrm{End}(V) = \mathrm{End}_0(V) \oplus \mathrm{End}_1(V)$
  where

  $$\mathrm{End}_i(V) = \left\{ T \in \mathrm{End}(V) \mid T(V_s) \subseteq V_{s+i \pmod 2} \text{ for any } s \in \mathbb{Z}/2\mathbb{Z} \right\}$$

  and for homogeneous X and Y, the bracket is given by

  $$[X, Y] = XY - (-1)^{|X|\cdot|Y|} YX$$

- Simple Lie superalgebras:
  $\mathfrak{sl}(m|n), \mathfrak{osp}(m|2n), \mathfrak{f}(4), \mathfrak{g}(3), \mathfrak{p}(n), \mathfrak{q}(n), \ldots$
- Heisenberg-Clifford Lie superalgebras.

Heisenberg-Clifford Lie superalgebra

Let (W, Ω) be a *supersymplectic* space, i.e.,

- $W = W_0 \oplus W_1$.
- $\Omega : W \times W \to \mathbb{R}$ satisfies
  - $\Omega(W_0, W_1) = \Omega(W_1, W_0) = 0$
  - $\Omega_{|W_1 \times W_1}$ is a nondegenerate symmetric form.
  - $\Omega_{|W_0 \times W_0}$ is a symplectic form.

Heisenberg-Clifford Lie superalgebra

Let (W, Ω) be a *supersymplectic* space, i.e.,

- $W = W_0 \oplus W_1$.
- $\Omega : W \times W \to \mathbb{R}$ satisfies
  - $\Omega(W_0, W_1) = \Omega(W_1, W_0) = 0$
  - $\Omega_{|W_1 \times W_1}$ is a nondegenerate symmetric form.
  - $\Omega_{|W_0 \times W_0}$ is a symplectic form.

Set $\mathfrak{h}_W = W \oplus \mathbb{R}$ where

$$[(v_1, s_1), (v_2, s_2)] = (0, \Omega(v_1, v_2))$$

Heisenberg-Clifford Lie superalgebra

Let (W, Ω) be a *supersymplectic* space, i.e.,

- $W = W_0 \oplus W_1$.
- $\Omega : W \times W \to \mathbb{R}$ satisfies
  - $\Omega(W_0, W_1) = \Omega(W_1, W_0) = 0$
  - $\Omega_{|W_1 \times W_1}$ is a nondegenerate symmetric form.
  - $\Omega_{|W_0 \times W_0}$ is a symplectic form.

Set $\mathfrak{h}_W = W \oplus \mathbb{R}$ where

$$[(v_1, s_1), (v_2, s_2)] = (0, \Omega(v_1, v_2))$$

- $\mathfrak{h}_W$ is two-step nilpotent and $\dim \left(\mathcal{Z}(\mathfrak{h}_W) \right) = 1$.

Towards unitary representations : super Lie groups

- A *super Lie group* is a group object in the category of supermanifolds.

Towards unitary representations : super Lie groups

- A *super Lie group* is a group object in the category of supermanifolds.

Proposition

The category of Super Lie groups is equivalent to a category of *Harish-Chandra pairs*, i.e., pairs $(G_0, \mathfrak{g})$ such that :

1. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra over $\mathbb{R}$.
2. G_0 is a real Lie group with Lie algebra $\mathfrak{g}_0$ which acts on $\mathfrak{g}$ smoothly via $\mathbb{R}$-linear automorphisms.
3. The action of G_0 on $\mathfrak{g}_0$ is the adjoint action. The adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$ is the differential of the action of G_0 on $\mathfrak{g}$.

Towards unitary representations : super Lie groups

- A *super Lie group* is a group object in the category of supermanifolds.

Proposition

The category of Super Lie groups is equivalent to a category of *Harish-Chandra pairs*, i.e., pairs $(G_0, \mathfrak{g})$ such that :

1. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra over $\mathbb{R}$.
2. G_0 is a real Lie group with Lie algebra $\mathfrak{g}_0$ which acts on $\mathfrak{g}$ smoothly via $\mathbb{R}$-linear automorphisms.
3. The action of G_0 on $\mathfrak{g}_0$ is the adjoint action. The adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}$ is the differential of the action of G_0 on $\mathfrak{g}$.

- For simplicity, from now on we assume that G_0 is connected and simply connected.

Super Hilbert spaces

"Wrong" definition : A super Hilbert space is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are closed subspaces and $\mathcal{H}_0 \perp \mathcal{H}_1$.

"Right" definition : Indeed $\mathcal{H}$ is endowed with an even super Hermitian form:

$$\langle x, y\rangle_{super} = \begin{cases} 0 & \text{if } x, y \text{ are of opposite parity,} \\ \langle x, y\rangle_{\mathcal{H}_0} & \text{if } x, y \in \mathcal{H}_0, \\ \sqrt{-1}\langle x, y\rangle_{\mathcal{H}_1} & \text{if } x, y \in \mathcal{H}_1. \end{cases}$$

We have:

$$\langle y, x\rangle_{super} = (-1)^{|x|\cdot|y|}\overline{\langle x, y\rangle}_{super}$$

$$\langle x, x\rangle_{super} > 0 \text{ for } x \in \mathcal{H}_0, x \neq 0$$

$$\sqrt{-1}\langle x, x\rangle_{super} < 0 \text{ for } x \in \mathcal{H}_1, x \neq 0$$

Unitary representations of super Lie groups

- Let $(G_0, \mathfrak{g})$ be a super Lie group. We want to consider unitary representations of $(G_0, \mathfrak{g})$ on super Hilbert spaces, i.e.,

$$\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}).$$

Unitary representations of super Lie groups

• Let $(G_0, \mathfrak{g})$ be a super Lie group. We want to consider unitary representations of $(G_0, \mathfrak{g})$ on super Hilbert spaces, i.e.,

$$\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}).$$

But if $X \in \mathfrak{g}_1$, then

$$\pi([X, X]) = \pi(X)\pi(X) + \pi(X)\pi(X) = 2\pi(X)^2$$

and $\pi([X, X])$ is an *unbounded*, (only) densely defined operator.

Unitary representations of super Lie groups

• Let $(G_0, \mathfrak{g})$ be a super Lie group. We want to consider unitary representations of $(G_0, \mathfrak{g})$ on super Hilbert spaces, i.e.,

$$\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}).$$

But if $X \in \mathfrak{g}_1$, then

$$\pi([X, X]) = \pi(X)\pi(X) + \pi(X)\pi(X) = 2\pi(X)^2$$

and $\pi([X, X])$ is an *unbounded*, (only) densely defined operator.

• A natural choice of representation space is $\mathcal{H}^\infty$ (the subspace of smooth vectors) defined as

$$\mathcal{H}^\infty = \left\{ v \mid v \in \mathcal{H} \text{ and the map } g \mapsto \pi(g)v \text{ is smooth} \right\}$$

But then one needs to know that $\pi(X)\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$.

Unitary representaions of super Lie groups (cont.)

Definition ([Carmeli, Cassinelli, Toigo, Varadarajan])

A *unitary representation* of $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ such that :

Unitary representaions of super Lie groups (cont.)

Definition ([Carmeli, Cassinelli, Toigo, Varadarajan])

A *unitary representation* of $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ such that :

- $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a super Hilbert space.

Unitary representaions of super Lie groups (cont.)

Definition ([Carmeli, Cassinelli, Toigo, Varadarajan])

A *unitary representation* of $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ such that :

- $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a super Hilbert space.
- $\pi : G_0 \to \mathrm{U}(\mathcal{H})$ is a unitary representation of G_0 (in the usual sense).

Unitary representaions of super Lie groups (cont.)

Definition ([Carmeli, Cassinelli, Toigo, Varadarajan])

A *unitary representation* of $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ such that :

- $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a super Hilbert space.
- $\pi : G_0 \to \mathrm{U}(\mathcal{H})$ is a unitary representation of G_0 (in the usual sense).
- $\rho^\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ is a super skew-Hermitian representation which satisfies

$$\rho^\pi([X, Y]) = \rho^\pi(X)\rho^\pi(Y) - (-1)^{|X|\cdot|Y|}\rho^\pi(Y)\rho^\pi(X).$$

Unitary representaions of super Lie groups (cont.)

Definition ([Carmeli, Cassinelli, Toigo, Varadarajan])

A *unitary representation* of $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ such that :

- $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a super Hilbert space.
- $\pi : G_0 \to \mathrm{U}(\mathcal{H})$ is a unitary representation of G_0 (in the usual sense).
- $\rho^\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ is a super skew-Hermitian representation which satisfies

$$\rho^\pi([X, Y]) = \rho^\pi(X)\rho^\pi(Y) - (-1)^{|X|\cdot|Y|}\rho^\pi(Y)\rho^\pi(X).$$

  - Here $\mathcal{H}^\infty$ is the space of smooth vectors of $(\pi, \mathcal{H})$.

Unitary representaions of super Lie groups (cont.)

Definition ([Carmeli, Cassinelli, Toigo, Varadarajan])

A *unitary representation* of $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ such that :

- $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a super Hilbert space.
- $\pi : G_0 \to \mathrm{U}(\mathcal{H})$ is a unitary representation of G_0 (in the usual sense).
- $\rho^\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ is a super skew-Hermitian representation which satisfies

$$\rho^\pi([X, Y]) = \rho^\pi(X)\rho^\pi(Y) - (-1)^{|X|\cdot|Y|}\rho^\pi(Y)\rho^\pi(X).$$

  - Here $\mathcal{H}^\infty$ is the space of smooth vectors of $(\pi, \mathcal{H})$.
- $\rho^\pi_{|\mathfrak{g}_0} = \pi^\infty$ and $\rho^\pi(\mathrm{Ad}(g)(X)) = \pi(g)\rho^\pi(X)\pi(g^{-1})$.

Restriction and induction

Let $(H_0, \mathfrak{h})$ be a sub super Lie group of $(G_0, \mathfrak{g})$. One can formally define restriction and induction functors.

$(\pi, \rho^\pi, \mathcal{H})$ unitary rep. of $(G_0, \mathfrak{g})$ $\rightsquigarrow$ $\mathrm{Res}_{(H_0,\mathfrak{h})}^{(G_0,\mathfrak{g})}(\pi, \rho^\pi, \mathcal{H})$

$(\sigma, \rho^\sigma, \mathcal{K})$ unitary rep. of $(H_0, \mathfrak{h})$
$\mathfrak{g}_1 = \mathfrak{h}_1$ $\rightsquigarrow$ $\mathrm{Ind}_{(H_0,\mathfrak{h})}^{(G_0,\mathfrak{g})}(\sigma, \rho^\sigma, \mathcal{K})$

Not So Obvious Fact :

These functors are well defined.

Proof. Follows from [Carmeli, Cassinelli, Toigo, Varadarajan].

Unitary equivalence and parity

Unitary equivalence

Two unitary representations $(\pi, \rho^{\pi}, \mathcal{H})$ and $(\pi', \rho^{\pi'}, \mathcal{H}')$ are said to be unitarily equivalent if there exists a linear isometry $T : \mathcal{H} \to \mathcal{H}'$ such that :

- T preserves the $\mathbb{Z}/2\mathbb{Z}$-grading.

Unitary equivalence and parity

Unitary equivalence

Two unitary representations $(\pi, \rho^{\pi}, \mathcal{H})$ and $(\pi', \rho^{\pi'}, \mathcal{H}')$ are said to be unitarily equivalent if there exists a linear isometry $T : \mathcal{H} \to \mathcal{H}'$ such that :

- T preserves the $\mathbb{Z}/2\mathbb{Z}$-grading.
- For any $g \in G_0$, $\pi'(g) \circ T = T \circ \pi(g)$.
- For any $X \in \mathfrak{g}$, $\rho^{\pi'}(X) \circ T = T \circ \rho^{\pi}(X)$.

Unitary equivalence and parity

Unitary equivalence

Two unitary representations $(\pi, \rho^\pi, \mathcal{H})$ and $(\pi', \rho^{\pi'}, \mathcal{H}')$ are said to be unitarily equivalent if there exists a linear isometry $T : \mathcal{H} \to \mathcal{H}'$ such that :

- T preserves the $\mathbb{Z}/2\mathbb{Z}$-grading.
- For any $g \in G_0$, $\pi'(g) \circ T = T \circ \pi(g)$.
- For any $X \in \mathfrak{g}$, $\rho^{\pi'}(X) \circ T = T \circ \rho^\pi(X)$.

Parity change

Tensoring $(\pi, \rho^\pi, \mathcal{H})$ with the trivial representation on $\mathbb{C}^{0|1}$ yields $(\pi, \rho^\pi, {}^\Pi\mathcal{H})$ where ${}^\Pi\mathcal{H}_0 = \mathcal{H}_1$ and ${}^\Pi\mathcal{H}_1 = \mathcal{H}_0$.

Unitary equivalence and parity

Unitary equivalence

Two unitary representations $(\pi, \rho^{\pi}, \mathcal{H})$ and $(\pi', \rho^{\pi'}, \mathcal{H}')$ are said to be unitarily equivalent if there exists a linear isometry $T : \mathcal{H} \to \mathcal{H}'$ such that :

- T preserves the $\mathbb{Z}/2\mathbb{Z}$-grading.
- For any $g \in G_0$, $\pi'(g) \circ T = T \circ \pi(g)$.
- For any $X \in \mathfrak{g}$, $\rho^{\pi'}(X) \circ T = T \circ \rho^{\pi}(X)$.

Parity change

Tensoring $(\pi, \rho^{\pi}, \mathcal{H})$ with the trivial representation on $\mathbb{C}^{0|1}$ yields $(\pi, \rho^{\pi}, {}^{\Pi}\mathcal{H})$ where ${}^{\Pi}\mathcal{H}_0 = \mathcal{H}_1$ and ${}^{\Pi}\mathcal{H}_1 = \mathcal{H}_0$.

- $(\pi, \rho^{\pi}, \mathcal{H})$ and $(\pi, \rho^{\pi}, {}^{\Pi}\mathcal{H})$ are *not* necessarily unitarily equivalent.

Some of the difficulties . . .

1. General facts about rep's of locally compact groups (e.g., Mackey machinery) may not generalize.

Some of the difficulties . . .

1. General facts about rep's of locally compact groups (e.g., Mackey machinery) may not generalize.
2. If $\mathfrak{g}_0$ were reductive, we could work "infinitesimally" (as done by S. J. Cheng, H. Furutsu, K. Nishiyama, W. Wang, R. B. Zhang, . . .)

Some of the difficulties . . .

1. General facts about rep's of locally compact groups (e.g., Mackey machinery) may not generalize.
2. If $\mathfrak{g}_0$ were reductive, we could work "infinitesimally" (as done by S. J. Cheng, H. Furutsu, K. Nishiyama, W. Wang, R. B. Zhang, . . .)
3. One needs to define "super" polarizing subalgebras (and prove that they exist).

Nilpotent super Lie groups

- A super Lie group $(G_0, \mathfrak{g})$ is called *nilpotent* if the lower central series of $\mathfrak{g}$ has finitely many nonzero terms (equivalently, if $\mathfrak{g}$ appears in its own upper central series).

Nilpotent super Lie groups

- A super Lie group $(G_0, \mathfrak{g})$ is called *nilpotent* if the lower central series of $\mathfrak{g}$ has finitely many nonzero terms (equivalently, if $\mathfrak{g}$ appears in its own upper central series).
- Unlike Lie groups, certain super Lie groups do not have any faithful unitary representairons!

Nilpotent super Lie groups

- A super Lie group $(G_0, \mathfrak{g})$ is called *nilpotent* if the lower central series of $\mathfrak{g}$ has finitely many nonzero terms (equivalently, if $\mathfrak{g}$ appears in its own upper central series).
- Unlike Lie groups, certain super Lie groups do not have any faithful unitary representairons!

Lemma

If $X_1, ...X_m \in \mathfrak{g}_1$ such that

$$\sum_{i=1}^{m} [X_i, X_i] = 0$$

then for every unitary representation $(\pi, \rho^\pi, \mathcal{H})$ we have $\rho^\pi(X_1) = \cdots = \rho^\pi(X_m) = 0$.

Nilpotent super Lie groups

- A super Lie group $(G_0, \mathfrak{g})$ is called *nilpotent* if the lower central series of $\mathfrak{g}$ has finitely many nonzero terms (equivalently, if $\mathfrak{g}$ appears in its own upper central series).
- Unlike Lie groups, certain super Lie groups do not have any faithful unitary representairons!

Lemma

If $X_1, ...X_m \in \mathfrak{g}_1$ such that

$$\sum_{i=1}^{m} [X_i, X_i] = 0$$

then for every unitary representation $(\pi, \rho^\pi, \mathcal{H})$ we have $\rho^\pi(X_1) = \cdots = \rho^\pi(X_m) = 0$.

Proof. Observe that $\sum_{i=1}^m \rho^\pi(X_i)^2 = 0$ and for every i, the operator $e^{\frac{\pi}{4}\sqrt{-1}}\rho^\pi(X_i)$ is symmetric. For every $v \in \mathcal{H}^\infty$ we have :

$$\sum_{i=1}^{m} \langle e^{\frac{\pi}{4}\sqrt{-1}}\rho^\pi(X_i)v, e^{\frac{\pi}{4}\sqrt{-1}}\rho^\pi(X_i)v\rangle = \langle v, e^{\frac{\pi}{2}\sqrt{-1}} \sum_{i=1}^{m} \rho^\pi(X_i)^2 v\rangle = 0.$$

Reduced form

- Set $\mathfrak{a}^{(1)} = \langle X \in \mathfrak{g}_1 \mid [X, X] = 0 \rangle$. We call $\mathfrak{g}$ *reduced* if $\mathfrak{a}^{(1)} = \{0\}$.

Reduced form

- Set $\mathfrak{a}^{(1)} = \langle X \in \mathfrak{g}_1 \mid [X, X] = 0 \rangle$. We call $\mathfrak{g}$ *reduced* if $\mathfrak{a}^{(1)} = \{0\}$.
- Set

$$\mathfrak{a}^{(2)} = \langle X \in \mathfrak{g}_1 \mid [X, X] \in \mathfrak{a}^{(1)} \rangle$$
$$\mathfrak{a}^{(3)} = \langle X \in \mathfrak{g}_1 \mid [X, X] \in \mathfrak{a}^{(2)} \rangle$$
$$\dots$$

Reduced form

- Set $\mathfrak{a}^{(1)} = \langle X \in \mathfrak{g}_1 \mid [X, X] = 0 \rangle$. We call $\mathfrak{g}$ *reduced* if $\mathfrak{a}^{(1)} = \{0\}$.
- Set

$$\mathfrak{a}^{(2)} = \langle X \in \mathfrak{g}_1 \mid [X, X] \in \mathfrak{a}^{(1)} \rangle$$
$$\mathfrak{a}^{(3)} = \langle X \in \mathfrak{g}_1 \mid [X, X] \in \mathfrak{a}^{(2)} \rangle$$
$$\ldots$$

We have

$$\mathfrak{a}^{(1)} \subset \mathfrak{a}^{(2)} \subset \mathfrak{a}^{(3)} \subset \cdots$$

Reduced form

- Set $\mathfrak{a}^{(1)} = \langle X \in \mathfrak{g}_1 \mid [X,X] = 0 \rangle$. We call $\mathfrak{g}$ *reduced* if $\mathfrak{a}^{(1)} = \{0\}$.
- Set
$$\mathfrak{a}^{(2)} = \langle X \in \mathfrak{g}_1 \mid [X,X] \in \mathfrak{a}^{(1)} \rangle$$
$$\mathfrak{a}^{(3)} = \langle X \in \mathfrak{g}_1 \mid [X,X] \in \mathfrak{a}^{(2)} \rangle$$
$$\ldots$$

  We have
$$\mathfrak{a}^{(1)} \subset \mathfrak{a}^{(2)} \subset \mathfrak{a}^{(3)} \subset \cdots$$

  Set $\mathfrak{a} = \bigcup_{j \geq 1} \mathfrak{a}^{(j)}$.

Observation

- $\rho^\pi(\mathfrak{a}) = 0$ for every unitary representation $(\pi, \rho^\pi, \mathcal{H})$.
- $\mathfrak{a}$ is $\mathbb{Z}/2\mathbb{Z}$-graded, hence corresponds to a sub super Lie group $(A_0, \mathfrak{a})$ of $(G_0, \mathfrak{g})$. The quotient $\mathfrak{g}/\mathfrak{a}$ is reduced.

Structure of nilpotent Lie superalgebras

Lemma (Kirillov ?)

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that $\mathfrak{g}$ is *reduced* and $\dim \mathcal{Z}(\mathfrak{g}) = 1$. Then exactly one of the following statements is true :

Structure of nilpotent Lie superalgebras

Lemma (Kirillov ?)

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that $\mathfrak{g}$ is *reduced* and $\dim \mathcal{Z}(\mathfrak{g}) = 1$. Then exactly one of the following statements is true :

- There exists a graded decomposition

$$\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w}$$

  such that $\mathrm{Span}\{X, Y, Z\}$ is a three-dimensional Heisenberg algebra, $Z \in \mathcal{Z}(\mathfrak{g})$,

$$\mathfrak{g}' := \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$$

  is a subalgebra, and $Y \in \mathcal{Z}(\mathfrak{g}')$.

Structure of nilpotent Lie superalgebras

Lemma (Kirillov ?)

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that $\mathfrak{g}$ is *reduced* and $\dim \mathcal{Z}(\mathfrak{g}) = 1$. Then exactly one of the following statements is true :

- There exists a graded decomposition

  $$\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w}$$

  such that $\mathrm{Span}\{X, Y, Z\}$ is a three-dimensional Heisenberg algebra, $Z \in \mathcal{Z}(\mathfrak{g})$,

  $$\mathfrak{g}' := \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$$

  is a subalgebra, and $Y \in \mathcal{Z}(\mathfrak{g}')$.
- $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_W$ where $W_0 = \{0\}$.

Structure of nilpotent Lie superalgebras

Lemma (Kirillov ?)

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that $\mathfrak{g}$ is *reduced* and $\dim \mathcal{Z}(\mathfrak{g}) = 1$. Then exactly one of the following statements is true :

- There exists a graded decomposition

$$\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}X \oplus \mathfrak{w}$$

  such that $\mathrm{Span}\{X, Y, Z\}$ is a three-dimensional Heisenberg algebra, $Z \in \mathcal{Z}(\mathfrak{g})$,

$$\mathfrak{g}' := \mathbb{R}Y \oplus \mathbb{R}Z \oplus \mathfrak{w}$$

  is a subalgebra, and $Y \in \mathcal{Z}(\mathfrak{g}')$.

- $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_W$ where $W_0 = \{0\}$.

Unitary representations as induced representations

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that

- $\mathfrak{g}$ is reduced,
- $\dim \mathcal{Z}(\mathfrak{g}) = 1$,
- $\mathfrak{g}$ is *not* isomorphic to $\mathfrak{h}_W$ with $W_0 = \{0\}$.

Unitary representations as induced representations

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that

- $\mathfrak{g}$ is reduced,
- $\dim \mathcal{Z}(\mathfrak{g}) = 1$,
- $\mathfrak{g}$ is *not* isomorphic to $\mathfrak{h}_W$ with $W_0 = \{0\}$.

Let $\mathfrak{g}'$ be as in Kirillov's lemma, and let $(G_0', \mathfrak{g}')$ be the sub super Lie group of $(G_0, \mathfrak{g})$ defined in the super version of Kirillov's lemma.

- Observe that $\dim \mathfrak{g}_1' = \dim \mathfrak{g}_1$, hence induction from $(G_0', \mathfrak{g}')$ to $(G_0, \mathfrak{g})$ yields unitary representaions.

Unitary representations as induced representations

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group such that

- $\mathfrak{g}$ is reduced,
- $\dim \mathcal{Z}(\mathfrak{g}) = 1$,
- $\mathfrak{g}$ is *not* isomorphic to $\mathfrak{h}_W$ with $W_0 = \{0\}$.

Let $\mathfrak{g}'$ be as in Kirillov's lemma, and let $(G_0', \mathfrak{g}')$ be the sub super Lie group of $(G_0, \mathfrak{g})$ defined in the super version of Kirillov's lemma.

- Observe that $\dim \mathfrak{g}_1' = \dim \mathfrak{g}_1$, hence induction from $(G_0', \mathfrak{g}')$ to $(G_0, \mathfrak{g})$ yields unitary representaions.

Proposition (codimension one induction)

Let $(\pi, \rho^\pi, \mathcal{H})$ be an irreducible unitary representation of $(G_0, \mathfrak{g})$ whose restriction to $\mathcal{Z}(G_0)$ is nontrivial. Then

$$(\pi, \rho^\pi, \mathcal{H}) = \mathrm{Ind}_{(G_0', \mathfrak{g}')}^{(G_0, \mathfrak{g})}(\pi', \rho^{\pi'}, \mathcal{H}')$$

for some irreducible unitary representation $(\pi', \rho^{\pi'}, \mathcal{H}')$ of $(G_0', \mathfrak{g}')$.

Unitary rep's of Heisenberg-Clifford super Lie groups

- Recall that $\mathfrak{h}_W = W \oplus \mathbb{R}$ where
$$[(v_1, s_1), (v_2, s_2)] = (0, \Omega(v, w))$$

Set $\mathfrak{g} = \mathfrak{h}_W$ and let $(G_0, \mathfrak{g})$ be the corresponding super Lie group.

Unitary rep's of Heisenberg-Clifford super Lie groups

- Recall that $\mathfrak{h}_W = W \oplus \mathbb{R}$ where

$$[(v_1, s_1), (v_2, s_2)] = (0, \Omega(v, w))$$

Set $\mathfrak{g} = \mathfrak{h}_W$ and let $(G_0, \mathfrak{g})$ be the corresponding super Lie group.

Theorem (generalized Stone-von Neumann)

Let $\chi : \mathbb{R} \to \mathbb{C}^\times$ be defined by $\chi(t) = e^{at\sqrt{-1}}$ where $a > 0$. (The case $a < 0$ is similar.)

- $\Omega_{|W_1 \times W_1}$ positive definite $\Rightarrow$ up to unitary equivalence and parity there exists a unique unitary representation with central character χ.
- $\Omega_{|W_1 \times W_1}$ not positive definite $\Rightarrow$ $(G_0, \mathfrak{g})$ does not have any unitary representations with central character χ.

Unitary rep's of Heisenberg-Clifford super Lie groups

- Recall that $\mathfrak{h}_W = W \oplus \mathbb{R}$ where

$$[(v_1, s_1), (v_2, s_2)] = (0, \Omega(v, w))$$

Set $\mathfrak{g} = \mathfrak{h}_W$ and let $(G_0, \mathfrak{g})$ be the corresponding super Lie group.

Theorem (generalized Stone-von Neumann)

Let $\chi : \mathbb{R} \to \mathbb{C}^\times$ be defined by $\chi(t) = e^{at\sqrt{-1}}$ where $a > 0$. (The case $a < 0$ is similar.)

- $\Omega_{|W_1 \times W_1}$ positive definite $\Rightarrow$ up to unitary equivalence and parity there exists a unique unitary representation with central character χ.
- $\Omega_{|W_1 \times W_1}$ not positive definite $\Rightarrow$ $(G_0, \mathfrak{g})$ does not have any unitary representations with central character χ.

Let $(\pi_\chi, \rho^{\pi_\chi}, \mathcal{H}_\chi)$ denote the unitary representation with central character χ.

$$\begin{aligned} \dim \mathfrak{g}_1 = 2k \quad &\Rightarrow \quad (\pi_\chi, \rho^{\pi_\chi}, \mathcal{H}_\chi) \not\simeq (\pi_\chi, \rho^{\pi_\chi}, {}^{\Pi}\mathcal{H}_\chi) \\ \dim \mathfrak{g}_1 = 2k+1 \quad &\Rightarrow \quad (\pi_\chi, \rho^{\pi_\chi}, \mathcal{H}_\chi) \simeq (\pi_\chi, \rho^{\pi_\chi}, \mathcal{H}_\chi) \end{aligned}$$

The general case

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

The general case

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

- For every $\lambda \in \mathfrak{g}_0^*$ one can define a symmetric bilinear form
$$\mathrm{B}_\lambda : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}$$
where $\mathrm{B}_\lambda(X, Y) = \lambda([X, Y])$.

The general case

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

- For every $\lambda \in \mathfrak{g}_0^*$ one can define a symmetric bilinear form
$$\mathrm{B}_\lambda : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}$$
where $\mathrm{B}_\lambda(X, Y) = \lambda([X, Y])$.
- Set
$$\mathfrak{g}_0^+ = \{\, \lambda \in \mathfrak{g}_0^* \mid \mathrm{B}_\lambda \text{ is nonnegative definite} \,\}$$

The general case

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

- For every $\lambda \in \mathfrak{g}_0^*$ one can define a symmetric bilinear form
$$\mathrm{B}_\lambda : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}$$
where $\mathrm{B}_\lambda(X, Y) = \lambda([X, Y])$.
- Set
$$\mathfrak{g}_0^+ = \{ \lambda \in \mathfrak{g}_0^* \mid \mathrm{B}_\lambda \text{ is nonnegative definite} \}$$
- Observe that $\mathfrak{g}_0^+$ is a G_0-invariant cone.

The general case

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

- For every $\lambda \in \mathfrak{g}_0^*$ one can define a symmetric bilinear form
$$\mathrm{B}_\lambda : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}$$
where $\mathrm{B}_\lambda(X, Y) = \lambda([X, Y])$.
- Set
$$\mathfrak{g}_0^+ = \{\, \lambda \in \mathfrak{g}_0^* \mid \mathrm{B}_\lambda \text{ is nonnegative definite} \,\}$$
- Observe that $\mathfrak{g}_0^+$ is a G_0-invariant cone.

THEOREM (S.)

There exists a bijective correspondence

$$\begin{array}{c} \text{Irreducible unitary} \\ \text{representations of } (G_0, \mathfrak{g}) \end{array} \quad \leftrightsquigarrow \quad \begin{array}{c} G_0 - \text{orbits} \\ \text{in } \mathfrak{g}_0^+ \end{array}$$

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

A polarizing system of $(G_0, \mathfrak{g})$ is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

such that :

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

A polarizing system of $(G_0, \mathfrak{g})$ is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

such that :

- $\dim \mathfrak{m}_1 = \dim \mathfrak{g}_1$.

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

A polarizing system of $(G_0, \mathfrak{g})$ is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

such that :

- $\dim \mathfrak{m}_1 = \dim \mathfrak{g}_1$.
- $\lambda \in \mathfrak{g}_0^*$ and $\mathfrak{m}_0$ is a maximally isotropic subalgebra of $\mathfrak{g}_0$ w.r.t. the skew symmetric form $\Omega_\lambda(X, Y) = \lambda([X, Y])$.

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

A polarizing system of $(G_0, \mathfrak{g})$ is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

such that :

- $\dim \mathfrak{m}_1 = \dim \mathfrak{g}_1$.
- $\lambda \in \mathfrak{g}_0^*$ and $\mathfrak{m}_0$ is a maximally isotropic subalgebra of $\mathfrak{g}_0$ w.r.t. the skew symmetric form $\Omega_\lambda(X, Y) = \lambda([X, Y])$.
- $(C_0, \mathfrak{c})$ is a Heisenberg-Clifford super Lie group such that $\dim C_0 = 1$.

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

A polarizing system of $(G_0, \mathfrak{g})$ is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

such that :

- $\dim \mathfrak{m}_1 = \dim \mathfrak{g}_1$.
- $\lambda \in \mathfrak{g}_0^*$ and $\mathfrak{m}_0$ is a maximally isotropic subalgebra of $\mathfrak{g}_0$ w.r.t. the skew symmetric form $\Omega_\lambda(X, Y) = \lambda([X, Y])$.
- $(C_0, \mathfrak{c})$ is a Heisenberg-Clifford super Lie group such that $\dim C_0 = 1$.
- $\Phi : (M_0, \mathfrak{m}) \to (C_0, \mathfrak{c})$ is an epimorphism.

Polarizing systems

Let $(G_0, \mathfrak{g})$ be a nilpotent super Lie group.

A polarizing system of $(G_0, \mathfrak{g})$ is a 6-tuple

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$$

such that :

- $\dim \mathfrak{m}_1 = \dim \mathfrak{g}_1$.
- $\lambda \in \mathfrak{g}_0^*$ and $\mathfrak{m}_0$ is a maximally isotropic subalgebra of $\mathfrak{g}_0$ w.r.t. the skew symmetric form $\Omega_\lambda(X, Y) = \lambda([X, Y])$.
- $(C_0, \mathfrak{c})$ is a Heisenberg-Clifford super Lie group such that $\dim C_0 = 1$.
- $\Phi : (M_0, \mathfrak{m}) \to (C_0, \mathfrak{c})$ is an epimorphism.
- $\mathfrak{m}_0 \cap \ker \Phi = \mathfrak{m}_0 \cap \ker \lambda$.

Proposition (everything is induced)

- Every irreducible rep $(\pi, \rho^\pi, \mathcal{H})$ of $(G_0, \mathfrak{g})$ is induced from a polarizing system $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$, i.e.,

$$(\pi, \rho^\pi, \mathcal{H}) = \mathrm{Ind}_{(M_0,\mathfrak{m})}^{(G_0,\mathfrak{g})}(\sigma \circ \Phi, \rho^{\sigma\circ\Phi}, \mathcal{K})$$

where for every $W \in \mathfrak{m}_0$, $\rho^{\sigma\circ\Phi}(W) = \rho^\sigma(\Phi(W)) = \lambda(W)$.

Proposition (everything is induced)

- Every irreducible rep $(\pi, \rho^\pi, \mathcal{H})$ of $(G_0, \mathfrak{g})$ is induced from a polarizing system $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$, i.e.,

$$(\pi, \rho^\pi, \mathcal{H}) = \mathrm{Ind}_{(M_0,\mathfrak{m})}^{(G_0,\mathfrak{g})}(\sigma \circ \Phi, \rho^{\sigma\circ\Phi}, \mathcal{K})$$

where for every $W \in \mathfrak{m}_0$, $\rho^{\sigma\circ\Phi}(W) = \rho^\sigma(\Phi(W)) = \lambda(W)$.

$$(M_0, \mathfrak{m}) \xrightarrow{\Phi} (C_0, \mathfrak{c}) \dashrightarrow (\sigma, \rho^\sigma, \mathcal{K})$$

Proposition (everything is induced)

- Every irreducible rep $(\pi, \rho^\pi, \mathcal{H})$ of $(G_0, \mathfrak{g})$ is induced from a polarizing system $(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda)$, i.e.,

$$(\pi, \rho^\pi, \mathcal{H}) = \mathrm{Ind}_{(M_0,\mathfrak{m})}^{(G_0,\mathfrak{g})}(\sigma \circ \Phi, \rho^{\sigma\circ\Phi}, \mathcal{K})$$

  where for every $W \in \mathfrak{m}_0$, $\rho^{\sigma\circ\Phi}(W) = \rho^\sigma(\Phi(W)) = \lambda(W)$.

$$(M_0, \mathfrak{m}) \xrightarrow{\Phi} (C_0, \mathfrak{c}) \dashrightarrow (\sigma, \rho^\sigma, \mathcal{K})$$

- Moreover, if $(\pi, \rho^\pi, \mathcal{H})$ is induced from two different polarizing systems

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda) \text{ and } (M_0', \mathfrak{m}', \Phi, C_0', \mathfrak{c}', \lambda')$$

  then
  1. $(C_0, \mathfrak{c}) \simeq (C_0', \mathfrak{c}')$
  2. $\lambda' = \mathrm{Ad}^*(g)(\lambda)$ for some $g \in G_0$.

Nonnegativity condition

- Suppose $(\pi, \rho^{\pi}, \mathcal{H}) = \mathrm{Ind}_{(M_0,\mathfrak{m})}^{(G_0,\mathfrak{g})}(\sigma \circ \Phi, \rho^{\sigma\circ\Phi}, \mathcal{K})$.
  From $\lambda(W) = \rho^{\sigma} \circ \Phi(W)$ and properties of Clifford modules we have :

  for every $X \in \mathfrak{g}_1$,

$$\begin{aligned} \mathrm{B}_{\lambda}(X, X) = \lambda([X, X]) &= \rho^{\sigma} \circ \Phi([X, X]) \\ &= [\rho^{\sigma} \circ \Phi(X), \rho^{\sigma} \circ \Phi(X)] \geq 0 \end{aligned}$$

Nonnegativity condition

- Suppose $(\pi, \rho^{\pi}, \mathcal{H}) = \mathrm{Ind}_{(M_0,\mathfrak{m})}^{(G_0,\mathfrak{g})}(\sigma \circ \Phi, \rho^{\sigma\circ\Phi}, \mathcal{K})$.
  From $\lambda(W) = \rho^{\sigma} \circ \Phi(W)$ and properties of Clifford modules we have :

  for every $X \in \mathfrak{g}_1$,

  $$\begin{aligned} \mathrm{B}_{\lambda}(X,X) = \lambda([X,X]) &= \rho^{\sigma} \circ \Phi([X,X]) \\ &= [\rho^{\sigma} \circ \Phi(X), \rho^{\sigma} \circ \Phi(X)] \geq 0 \end{aligned}$$

  which implies that $\lambda \in \mathfrak{g}_0^{+}$.

Nonnegativity condition

- Suppose $(\pi, \rho^\pi, \mathcal{H}) = \mathrm{Ind}_{(M_0,\mathfrak{m})}^{(G_0,\mathfrak{g})}(\sigma \circ \Phi, \rho^{\sigma\circ\Phi}, \mathcal{K})$.
  From $\lambda(W) = \rho^\sigma \circ \Phi(W)$ and properties of Clifford modules we have :

  for every $X \in \mathfrak{g}_1$,
  $$\begin{aligned} \mathrm{B}_\lambda(X,X) = \lambda([X,X]) &= \rho^\sigma \circ \Phi([X,X]) \\ &= [\rho^\sigma \circ \Phi(X), \rho^\sigma \circ \Phi(X)] \geq 0 \end{aligned}$$

  which implies that $\lambda \in \mathfrak{g}_0^+$.
- Conversely, we should show that every $\lambda \in \mathfrak{g}_0^+$ fits into a polarizing system $(M_0, \mathfrak{m}, C_0, \mathfrak{c}, \Phi, \lambda)$.

Proposition

For every $\lambda \in \mathfrak{g}_0^+$ there exists a polarizing system

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda).$$

Proposition

For every $\lambda \in \mathfrak{g}_0^+$ there exists a polarizing system

$$(M_0, \mathfrak{m}, \Phi, C_0, \mathfrak{c}, \lambda).$$

The proof is based on the following lemma :

Lemma

Let $\lambda \in \mathfrak{g}_0^+$. Then there exists a subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ such that :

- $\mathfrak{p}_0$ is a maximal isotropic subalgebra for the skew symmetric form Ω_λ,
- $\mathfrak{p}_0 \supset [\mathfrak{g}_1, \mathfrak{g}_1]$.

Proof of the lemma

Lemma

There exists a subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ such that :

- $\mathfrak{p}_0$ is a maximal isotropic subalgebra for the skew symmetric form Ω_λ,
- $\mathfrak{p}_0 \supset [\mathfrak{g}_1, \mathfrak{g}_1]$.

Proof of the lemma

Lemma

There exists a subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ such that :

- $\mathfrak{p}_0$ is a maximal isotropic subalgebra for the skew symmetric form Ω_λ,
- $\mathfrak{p}_0 \supset [\mathfrak{g}_1, \mathfrak{g}_1]$.

1. $\mathfrak{i} = [\mathfrak{g}_1, \mathfrak{g}_1]$ is an ideal of $\mathfrak{g}_0$, hence there exists a sequence
$$\{0\} = \mathfrak{i}^0 \subset \mathfrak{i}^1 \subset \mathfrak{i}^2 \subset \cdots \subset \mathfrak{i}^s = \mathfrak{i} \subset \mathfrak{i}^{s+1} \subset \cdots \subset \mathfrak{i}^r = \mathfrak{g}_0$$
of ideals such that $\dim(\mathfrak{i}^k/\mathfrak{i}^{k-1}) = 1$ for every $k \geq 1$.

Proof of the lemma

Lemma

There exists a subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ such that :

- $\mathfrak{p}_0$ is a maximal isotropic subalgebra for the skew symmetric form Ω_λ,
- $\mathfrak{p}_0 \supset [\mathfrak{g}_1, \mathfrak{g}_1]$.

1. $\mathfrak{i} = [\mathfrak{g}_1, \mathfrak{g}_1]$ is an ideal of $\mathfrak{g}_0$, hence there exists a sequence
$$\{0\} = \mathfrak{i}^0 \subset \mathfrak{i}^1 \subset \mathfrak{i}^2 \subset \cdots \subset \mathfrak{i}^s = \mathfrak{i} \subset \mathfrak{i}^{s+1} \subset \cdots \subset \mathfrak{i}^r = \mathfrak{g}_0$$
of ideals such that $\dim(\mathfrak{i}^k/\mathfrak{i}^{k-1}) = 1$ for every $k \geq 1$.
2. **(M. Vergne)** Define $\mathfrak{p}_0$ to be
$$\mathfrak{p}_0 := \sum_{k=1}^{r} \mathrm{rad}(\Omega_{\lambda|\mathfrak{i}^k \times \mathfrak{i}^k}).$$
Then $\mathfrak{p}_0$ is a maximal isotropic subalgebra for Ω_λ.

Proof of the lemma

Lemma

There exists a subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$ such that :

- $\mathfrak{p}_0$ is a maximal isotropic subalgebra for the skew symmetric form Ω_λ,
- $\mathfrak{p}_0 \supset [\mathfrak{g}_1, \mathfrak{g}_1]$.

1. $\mathfrak{i} = [\mathfrak{g}_1, \mathfrak{g}_1]$ is an ideal of $\mathfrak{g}_0$, hence there exists a sequence
$$\{0\} = \mathfrak{i}^0 \subset \mathfrak{i}^1 \subset \mathfrak{i}^2 \subset \cdots \subset \mathfrak{i}^s = \mathfrak{i} \subset \mathfrak{i}^{s+1} \subset \cdots \subset \mathfrak{i}^r = \mathfrak{g}_0$$
of ideals such that $\dim(\mathfrak{i}^k/\mathfrak{i}^{k-1}) = 1$ for every $k \geq 1$.
2. **(M. Vergne)** Define $\mathfrak{p}_0$ to be
$$\mathfrak{p}_0 := \sum_{k=1}^{r} \mathrm{rad}(\Omega_{\lambda|\mathfrak{i}^k \times \mathfrak{i}^k}).$$
Then $\mathfrak{p}_0$ is a maximal isotropic subalgebra for Ω_λ.
3. One can show that $\Omega_\lambda([\mathfrak{g}_1, \mathfrak{g}_1], [\mathfrak{g}_1, \mathfrak{g}_1]) = 0$, which implies that $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{p}_0$.

Irreducibility

Proposition

If a unitary representation $(\pi, \rho^{\pi}, \mathcal{H})$ is induced from a polarizing system, then it is irreducibe.

Irreducibility

Proposition

If a unitary representation $(\pi, \rho^\pi, \mathcal{H})$ is induced from a polarizing system, then it is irreducibe.

Proof. By induction on $\dim \mathfrak{g}$.

Case I : $\mathfrak{g}$ is not reduced, or $\mathfrak{g}$ is reduced and $\mathcal{Z}(\mathfrak{g}) \cap \ker \lambda \neq \{0\}$.
Induction hypothesis.

Irreducibility

Proposition

If a unitary representation $(\pi, \rho^\pi, \mathcal{H})$ is induced from a polarizing system, then it is irreducibe.

Proof. By induction on $\dim \mathfrak{g}$.

Case I : $\mathfrak{g}$ is not reduced, or $\mathfrak{g}$ is reduced and $\mathcal{Z}(\mathfrak{g}) \cap \ker \lambda \neq \{0\}$.
Induction hypothesis.

Case II : $\mathfrak{g}$ is reduced and $\mathcal{Z}(\mathfrak{g}) \cap \ker \lambda \neq \{0\}$. Find a 3-dimensional Heisenberg subgroup and use explicit formulas for its action.

An observation

Corollary

For every unitary representation $(\pi, \rho^\pi, \mathcal{H})$ of $(G_0, \mathfrak{g})$ we have $\rho^\pi([\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]]) = 0$.

Proof. Get deep into the proof of classification!

An observation

Corollary

For every unitary representation $(\pi, \rho^\pi, \mathcal{H})$ of $(G_0, \mathfrak{g})$ we have $\rho^\pi([\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]]) = 0$.

Proof. Get deep into the proof of classification!

Observation (Neeb) :

- Suppose that $\bigcap_{(\pi, \rho^\pi, \mathcal{H})} \ker (\pi, \rho^\pi, \mathcal{H}) = \{0\}$.
- $\mathfrak{g}^c := [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \mathfrak{g}_1$.
- $C_\mathfrak{g}$:= closed convex cone in $\mathfrak{g}_0^c$ generated by $\{ [X, X] \mid X \in \mathfrak{g}_1 \}$.

An observation (cont.)

- $C_{\mathfrak{g}}$ is G_0-invariant, generating, and pointed.

An observation (cont.)

- $C_{\mathfrak{g}}$ is G_0-invariant, generating, and pointed.
- It follows that $\mathfrak{g}_0^c$ has a *compactly embedded* Cartan subalgebra $\mathfrak{t}$.

An observation (cont.)

- $C_{\mathfrak{g}}$ is G_0-invariant, generating, and pointed.
- It follows that $\mathfrak{g}_0^c$ has a *compactly embedded* Cartan subalgebra $\mathfrak{t}$.
- $\mathfrak{t}$ acts semisimply and nilpotently on $\mathfrak{g}^c$; hence $[\mathfrak{t}, \mathfrak{g}^c] = \{0\}$.

An observation (cont.)

- $C_{\mathfrak{g}}$ is G_0-invariant, generating, and pointed.
- It follows that $\mathfrak{g}_0^c$ has a *compactly embedded* Cartan subalgebra $\mathfrak{t}$.
- $\mathfrak{t}$ acts semisimply and nilpotently on $\mathfrak{g}^c$; hence $[\mathfrak{t}, \mathfrak{g}^c] = \{0\}$.
- It follows that $[\mathfrak{g}_0^c, \mathfrak{g}^c] = \{0\}$; in particular $[\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] = \{0\}$.

An observation (cont.)

- $C_{\mathfrak{g}}$ is G_0-invariant, generating, and pointed.
- It follows that $\mathfrak{g}_0^c$ has a *compactly embedded* Cartan subalgebra $\mathfrak{t}$.
- $\mathfrak{t}$ acts semisimply and nilpotently on $\mathfrak{g}^c$; hence $[\mathfrak{t}, \mathfrak{g}^c] = \{0\}$.
- It follows that $[\mathfrak{g}_0^c, \mathfrak{g}^c] = \{0\}$; in particular $[\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] = \{0\}$.

Problem. Classify solvable Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ for which $C_{\mathfrak{g}}$ is pointed.

Restriction of $(\pi, \rho^\pi, \mathcal{H})$ to G_0

Let $(\pi, \rho^\pi, \mathcal{H})$ be an irr. unitary rep. of $(G_0, \mathfrak{g})$ corresponding to $\mathcal{O}_\lambda := G_0 \cdot \lambda$. Then

$$(\pi, \rho^\pi, \mathcal{H})_{|G_0} = \underbrace{\pi_\lambda \oplus \cdots \oplus \pi_\lambda}_{2^l \text{ time}}$$

where π_λ is the irreducible unitary representation of G_0 corresponding to $\mathcal{O}_\lambda$.

Restriction of $(\pi, \rho^\pi, \mathcal{H})$ to G_0

Let $(\pi, \rho^\pi, \mathcal{H})$ be an irr. unitary rep. of $(G_0, \mathfrak{g})$ corresponding to $\mathcal{O}_\lambda := G_0 \cdot \lambda$. Then

$$(\pi, \rho^\pi, \mathcal{H})_{|G_0} = \underbrace{\pi_\lambda \oplus \cdots \oplus \pi_\lambda}_{2^l \text{ time}}$$

where π_λ is the irreducible unitary representation of G_0 corresponding to $\mathcal{O}_\lambda$.

When $(\pi, \rho^\pi, \mathcal{H}) \simeq (\pi, \rho^\pi, {}^\Pi\mathcal{H})$?

If $(\pi, \rho^\pi, \mathcal{H})$ is induced form a polarizing system

$$(M_0, \mathfrak{m}, C_0, \mathfrak{c}, \Phi, \lambda)$$

then

$$\dim \mathfrak{c} = \begin{cases} 2l & \text{if } (\pi, \rho^\pi, \mathcal{H}) \simeq (\pi, \rho^\pi, {}^\Pi\mathcal{H}) \\ 2l+1 & \text{otherwise.} \end{cases}$$

Thank you !