Quantum Sheaf Cohomology and Brute Force Techniques

Josh Guffin

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A Kähler manifold $X$

A hermetian holomorphic bundle $\mathcal{E}$ satisfying
- $\text{ch}_2(\mathcal{E}) = \text{ch}_2(\mathcal{T}_X)$
- $\det \mathcal{E}^\vee \cong \omega_X$
- $\text{rk} \mathcal{E} \leq 8$ (if $\mathcal{E}$ is not a deformation of $\mathcal{T}_X$)

Quantum Sheaf Cohomology

$$QH(X, \mathcal{E}) = \bigoplus_{p,q} H^p(X, \Lambda^q \mathcal{E}^\vee)$$

along with a “quantum product”
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along with a “quantum product”
- Comes from a subset of operators in the $g = 0$ twisted NLSM
- The (0,2) *chiral ring* or (0,2) *topological ring*.
- Arises in analogy with the (2,2) chiral ring
- Use the arguments of [ADE06]
- Comes from a subset of operators in the $g = 0$ twisted NLSM
- The $(0,2)$ chiral ring or $(0,2)$ topological ring.
- Arises in analogy with the $(2,2)$ chiral ring
- Use the arguments of [ADE06]
(2,2) NLSM, topologically A-twisted (an SCFT)

- Two scalar supersymmetry charges $Q, \overline{Q}$
- BPS bounds on operators: for $\mathcal{O}$ of conformal weight $(h, \overline{h})$,

\[
\begin{align*}
    h &\geq 0 \\
    \overline{h} &\geq 0
\end{align*}
\]

- Saturated when $\mathcal{O}$ is in the kernel of $Q$ or $\overline{Q}$:

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\begin{align*}
    Q\mathcal{O} = 0 \iff h = 0 \\
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• An operator $\mathcal{O}$ is \textit{chiral} if $\mathcal{O} \in \ker Q \cap \ker \overline{Q}$

• $Q$ and $\overline{Q}$ are linear and obey Liebniz

• Operator Product Expansion: in a basis for all operators

$$\mathcal{O}_a(z)\mathcal{O}_b(0) = \sum_c f_{abc} z^{h_c-h_a-h_b} \mathcal{O}_c(0)$$

• $\mathcal{O}_a, \mathcal{O}_b$ chiral $\Rightarrow$ $\mathcal{O}_a(z)\mathcal{O}_b(0) = \sum_c f_{abc} \mathcal{O}_c(0)$

• Independent of $z \Rightarrow$ the \textit{chiral ring} is \textit{topological}

• Equivalently, $\overline{Q}$-closed with $h = 0$. 

(2,2) topological rings
(0,2) topological rings
Summary
• An operator $\mathcal{O}$ is \textit{chiral} if $\mathcal{O} \in \ker Q \cap \ker \overline{Q}$

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(0,2) NLSM, topologically $\frac{A}{2}$-twisted (an SCFT)

* One scalar supersymmetry charge, $\tilde{Q}$
* Right-moving BPS bound on operators

$$\bar{h} \geq 0$$

* Saturated when $\mathcal{O}$ is in the kernel of $\tilde{Q}$:

$$\tilde{Q}\mathcal{O} = 0 \iff \bar{h} = 0$$

* Such operators are *half-chiral*
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• For half-chiral $h = 0$ operators,

$$\mathcal{O}_a(z) \mathcal{O}_b(0) = \sum_c f_{abc} z^{h_c} \mathcal{O}_c(0)$$

• On a compact Riemann surface, only problems come from $h_c < 0$. We can forbid these operators with very mild constraints.
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(0,2) NLSMs from deformations of $T_X$

- Family of half-chiral rings
- Parametrize the family by $\alpha$ with $\alpha = 0$ the (2,2) point.

$$\alpha \to 0 \Rightarrow \mathcal{E}(\alpha) \to T_X$$

- Half-chiral operator in the (0,2) NLSM $\mathcal{O}(\alpha)$
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Half-chiral operator in the (0,2) NLSM $\mathcal{O}(\alpha)$
All (0,2) NLSMs in the family conformal $\Rightarrow$ spin quantization

Conformal weights satisfy $h(\alpha) - \overline{h}(\alpha) = s \in \mathbb{Z}$

\begin{align*}
\overline{h}(\alpha) &= 0 \\
\overline{h}(\alpha) &= h(\alpha) - \overline{h}(\alpha) = 4 < 0
\end{align*}

$0 \leftarrow \mathcal{O}(\alpha) \rightarrow \alpha$

$\overline{h}(0) = 0$

$h(0) = \overline{h}(0) = 4 < 0 \Rightarrow h(0) < 0$

Violates BPS Bound

Half-chiral ring is topological for deformations of $T_X$
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\end{align*}
\Rightarrow h(\alpha) < 0

\text{Violates BPS Bound}

Half-chiral ring is topological for deformations of $T_X$
- A unitary (0,2) SCFT with a left-moving $U(1)$ symmetry ($\det \mathcal{E}^\vee \cong K_X$)
  - CFT facts imply that $h \geq -\frac{r}{8}$
  - If $r < 8$, $h \geq 0$ and the topological ring exists.
A unitary (0,2) SCFT with a left-moving $U(1)$ symmetry
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A bundle $\mathcal{E}$ satisfying
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Set of $h = 0$ operators in $\ker \overline{Q}$ as a vector space is

$$\bigoplus_{p,q} H^p(X, \Lambda^q \mathcal{E}^\vee)$$

with product structure coming from the QFT
Would like to describe (0,2) topological rings
Techniques exist only for $X$ a toric variety or subvariety
- Brute-force method
  - Toric varieties
  - Bundle must be a deformation of the tangent bundle
- GLSM method
  - Subvarieties of a toric variety
  - Bundle is a deformation or the cohomology of a monad/kernel/cokernel
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GLSM method
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Goal: write down generators and find relations in

\[ \bigoplus_{p,q} H^p(X, \Lambda^q \mathcal{E}^\vee) \]

Compute correlation functions and deduce relations from them

\[ \langle O_1 \cdots O_s \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \langle O_1 \cdots O_s \rangle_\beta q^\beta \quad q^\beta := e^{i \int_\beta \omega} \]
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Compute $\langle O_1 \cdots O_s \rangle_\beta$ by

$$H^p(X, \Lambda^q E^\vee) \to H^p(\overline{M}_\beta, \Lambda^q F^\vee) \quad \text{(Eric's map)}$$

$$H^{p_1}(\overline{M}_\beta, \Lambda^{q_1} F^\vee_\beta) \otimes \cdots \otimes H^{p_s}(\overline{M}_\beta, \Lambda^{q_s} F^\vee_\beta) \to H^{n_\beta}(\overline{M}_\beta, \Lambda^{n_\beta} F^\vee_\beta)$$

Here $n_\beta = \dim \overline{M}_\beta$, $F_\beta$ is the induced sheaf on $\overline{M}_\beta$, and

$$H^{n_\beta}(\overline{M}_\beta, \Lambda^{n_\beta} F^\vee_\beta) \cong H^{n_\beta}(\overline{M}_\beta, \omega_{\overline{M}_\beta}) \cong \mathbb{C}$$

"The trace"
We require:

- explicit cohomology theory
- generators \( \mathcal{O}_a \in H^* (X, \Lambda^* \mathcal{E}^\vee) \)
- \( \overline{M}_\beta \)
- \( \mathcal{F}_\beta \)
- images \( \tilde{\mathcal{O}}_a \in H^p (\overline{M}_\beta, \Lambda^q \mathcal{F}_\beta^\vee) \)
- \( /\text{trace} \)

\( \rightarrow \) \( \check{C}ech \) complex
\( \rightarrow \) Euler sequence on \( X \)
\( \rightarrow \) Morrison/Plesser [MRP95]
\( \rightarrow \) Katz/Sharpe [KS06]
\( \rightarrow \) Euler sequence on \( \overline{M}_\beta \)
\( \rightarrow \) Lots of computer time

To find generators, we appeal to the GLSM description

- \( (0,2) \) GLSM \( \rightarrow \sigma \rightarrow H^1 (X, \mathcal{E}^\vee) \)
We require:

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- generators $\mathcal{O}_a \in H^*(X, \Lambda^* \mathcal{E}^\vee)$
- $\overline{M}_\beta$
- $\mathcal{F}_\beta$
- images $\tilde{\mathcal{O}}_a \in H^p(\overline{M}_\beta, \Lambda^q \mathcal{F}_\beta^\vee)$
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$\rightarrow$ Čech complex
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To find generators, we appeal to the GLSM description

$(0,2)$ GLSM $\rightarrow \sigma \rightarrow H^1(X, \mathcal{E}^\vee)$
For every toric variety $X$, the Euler sequence

$$0 \rightarrow \mathcal{O}_X^r \xrightarrow{E_0} \bigoplus_{\rho} \mathcal{O}_X(D_{\rho}) \rightarrow T_X \rightarrow 0$$

induces unobstructed deformations as

$$0 \rightarrow \mathcal{O}_X^r \xrightarrow{E} \bigoplus_{\rho} \mathcal{O}_X(D_{\rho}) \rightarrow \mathcal{E} \rightarrow 0$$
Dualizing,

$$0 \rightarrow \mathcal{E}^\vee \rightarrow \bigoplus_{\rho} O_X(-D_\rho) \xrightarrow{E_t} O_X^r \rightarrow 0$$

induces the long exact sequence containing

$$\cdots \rightarrow H^0(X, \bigoplus_{\rho} O_X(-D_\rho)) \rightarrow H^0(X, O_X^r) \rightarrow H^1(X, \mathcal{E}^\vee) \rightarrow H^1(X, \bigoplus_{\rho} O_X(-D_\rho)) \rightarrow \cdots$$

and when $\dim X \geq 2$,

$$H^1(X, \mathcal{E}^\vee) \cong H^0(X, O_X^r) \cong \mathbb{C}^r \cong H^1(X, \Omega_X^1)$$
To find $\widetilde{O}_a \in H^1(\overline{M}_\beta, \mathcal{F}^\vee)$, we have:

$$
0 \to \mathcal{F}^\vee \to \bigoplus_{\tilde{\rho}} \mathcal{O}_{\overline{M}_\beta}(-D_{\tilde{\rho}}) \xrightarrow{F^t} \mathcal{O}^r_{\overline{M}_\beta} \to 0
$$

leading via the induced long-exact sequence to

$$
H^1(\overline{M}_\beta, \mathcal{F}^\vee) \cong H^0(\overline{M}_\beta, \mathcal{O}^r_{\overline{M}_\beta}) \cong \mathbb{C}^r
$$

so compute by constructing the isomorphism on Čech cochains

$$
H^1(\overline{M}_\beta, \mathcal{F}^\vee) \cong \mathbb{C}^r \cong H^1(X, \mathcal{E}^\vee)
$$
• Explicitly construct generators as Čech cochains for each $\mathcal{M}_\beta$
• Teach a computer to cup/wedge and trace

$$H^{p_1}(\mathcal{M}_\beta, \wedge q_1 \mathcal{F}_\beta^\vee) \otimes \cdots \otimes H^{p_s}(\mathcal{M}_\beta, \wedge q_s \mathcal{F}_\beta^\vee) \rightarrow H^{n_\beta}(\mathcal{M}_\beta, \wedge n_\beta \mathcal{F}_\beta^\vee)$$

$$\mathbb{R} \downarrow \quad \mathbb{C}$$
Simplest example, $X = \mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}_X^2 \xrightarrow{E} \mathcal{O}_X(1,0)^2 \oplus \mathcal{O}_X(0,1)^2 \longrightarrow T_X \longrightarrow 0$$

where

$$E = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & y_0 \\ 0 & y_1 \end{pmatrix}$$
$X = \mathbb{P}^1 \times \mathbb{P}^1$ unobstructed: parametrize the 6-dimensional family of deformations as

$$0 \longrightarrow \mathcal{O}_X^2 \xrightarrow{E} \mathcal{O}_X(1,0)^2 \oplus \mathcal{O}_X(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

where

$$E = \begin{pmatrix} x_0 & \epsilon_1 x_0 + \epsilon_2 x_1 \\ x_1 & \epsilon_3 x_0 \\ \gamma_1 y_0 + \gamma_2 y_1 & y_0 \\ \gamma_3 y_0 & y_1 \end{pmatrix}$$
\( H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) \cong \mathbb{C}^2 \), find Čech reps of \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \): \( \psi, \tilde{\psi} \)

Compute two-point functions in degree \((0,0)\) sector

\[
\langle \psi \tilde{\psi} \rangle = \langle \psi \psi \rangle_{0,0} = \frac{1}{\phi} (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3 )
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\[
\langle \psi \tilde{\psi} \rangle = \langle \psi \tilde{\psi} \rangle_{0,0} = \frac{1}{\phi} (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)
\]

\[
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Here

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\phi = (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1) (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3 ) - (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2
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\text{No other instanton sectors contribute}
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- $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) \cong \mathbb{C}^2$, find Čech reps of $(\frac{1}{0})$ and $(\frac{0}{1})$: $\psi, \tilde{\psi}$
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Here

$$
\phi = (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1) (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3) - (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2
$$

- No other instanton sectors contribute
Moduli space: $\overline{M}_{i,j} = \mathbb{P}^{2i+1} \times \mathbb{P}^{2j+1}$

On each $\overline{M}_{i,j}$, find Čech reps of image of $\psi, \tilde{\psi}$ in $H^1(\overline{M}_{i,j}, \mathcal{F}^\vee)$.

Four-point functions arise from total degree 1;

$$\langle \psi\psi\psi\psi \rangle = \langle \psi\psi\psi\psi \rangle_{1,0} q + \langle \psi\psi\psi\psi \rangle_{0,1} \tilde{q}$$
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\]
Quantum Sheaf Cohomology
Brute force computations
Relation to physical correlators

Idea
Toric simplifications
Example – \( \mathbb{P}^1 \times \mathbb{P}^1 \)

\[
\langle \psi \bar{\psi} \psi \bar{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left( \epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3 \right) \left[ \gamma_1 (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3) + 2(\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1) \right]
\]

\[
\langle \psi \bar{\psi} \psi \bar{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[ (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2 + \gamma_2 \gamma_3 (\epsilon_1 + \gamma_1 \epsilon_2 \epsilon_3)^2 \right]
\]

\[
\langle \psi \bar{\psi} \bar{\psi} \psi \rangle_{1,0} = \frac{1}{\phi^2} \left( \gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1 \right) \left[ 2 (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1) - \gamma_1 (1 - \gamma_2 \gamma_3 \epsilon_2 \epsilon_3) \right]
\]

\[
\langle \bar{\psi} \bar{\psi} \bar{\psi} \bar{\psi} \rangle_{1,0} = \frac{1}{\phi^2} \left[ (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1)^2 + \gamma_2 \gamma_3 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1)^2 \right]
\]

\[
\langle \bar{\psi} \psi \bar{\psi} \psi \rangle_{1,0} = -\frac{1}{\phi^2} \left( \gamma_1 + \epsilon_1 \gamma_2 \gamma_3 \right) \left[ \gamma_1 (\gamma_1 + \gamma_2 \gamma_3 \epsilon_1) - 2 \gamma_2 \gamma_3 (\gamma_2 \gamma_3 \epsilon_2 \epsilon_3 - 1) \right]
\]
Compute up to total degree 3

Deduce relations:

\[ \psi \ast \psi + \epsilon_1 (\psi \ast \tilde{\psi}) - \epsilon_2 \epsilon_3 (\tilde{\psi} \ast \tilde{\psi}) = q \]

\[ \tilde{\psi} \ast \tilde{\psi} + \gamma_1 (\psi \ast \tilde{\psi}) - \gamma_2 \gamma_3 (\psi \ast \psi) = \tilde{q}. \]

Compare with (2,2) Relations

\[ \psi \ast \psi = q \]

\[ \tilde{\psi} \ast \tilde{\psi} = \tilde{q}. \]
• Compute up to total degree 3

• Deduce relations:

\[
\psi \ast \psi + \epsilon_1 (\psi \ast \tilde{\psi}) - \epsilon_2 \epsilon_3 (\tilde{\psi} \ast \tilde{\psi}) = q \\
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• Compare with (2,2) Relations

\[
\psi \ast \psi = q \\
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\]
• Compute up to total degree 3
• Deduce relations:

\[
\psi * \psi + \epsilon_1 (\psi * \tilde{\psi}) - \epsilon_2 \epsilon_3 (\tilde{\psi} * \tilde{\psi}) = q
\]
\[
\tilde{\psi} * \tilde{\psi} + \gamma_1 (\psi * \tilde{\psi}) - \gamma_2 \gamma_3 (\psi * \psi) = \tilde{q}.
\]

• Compare with (2,2) Relations

\[
\psi * \psi = q
\]
\[
\tilde{\psi} * \tilde{\psi} = \tilde{q}.
\]
Compare with ABS[ABS04] relations

\[ \psi \ast \psi + \epsilon_1 (\psi \ast \tilde{\psi}) - \epsilon_2 \epsilon_3 (\tilde{\psi} \ast \tilde{\psi}) = q \]
\[ \tilde{\psi} \ast \tilde{\psi} + \gamma_1 (\psi \ast \tilde{\psi}) - \gamma_2 \gamma_3 (\psi \ast \psi) = \tilde{q}. \]

\[ \psi \ast \psi - (\epsilon_1 - \epsilon_2) \psi \ast \tilde{\psi} = e^{it_1} \]
\[ \tilde{\psi} \ast \tilde{\psi} = e^{it_2}. \]
Consider a projective variety $X$, $\dim_{\mathbb{C}} X = 3$, with a $\mathcal{E}$ a generic deformation of $T_X$.

- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{\text{twisted}}$ gives the holomorphic dependence on bundle deformation parameters of the low-energy superpotential $W$.
- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{[\ell]}$ gives dependence of $W$ linear in $q$.
- If lines in $X$ are rigid $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{[\ell]} = 0$. 
Consider a projective variety $X$, $\dim_{\mathbb{C}} X = 3$, with a $\mathcal{E}$ a generic deformation of $T_X$.

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- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle[q]$ gives dependence of $W$ linear in $q$.
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- $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{\text{twisted}}$ gives the holomorphic dependence on bundle deformation parameters of the low-energy superpotential $W$.

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- If lines in $X$ are rigid, $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{[\ell]} = 0$. 

Consider a projective variety $X$, $\dim_{\mathbb{C}} X = 3$, with a $E$ a generic deformation of $T_X$.

$\langle O_1 O_2 O_3 \rangle_{\text{twisted}}$ gives the holomorphic dependence on bundle deformation parameters of the low-energy superpotential $W$.

$\langle O_1 O_2 O_3 \rangle_{[\ell]}$ gives dependence of $W$ linear in $q$.

If lines in $X$ are rigid $\langle O_1 O_2 O_3 \rangle_{[\ell]} = 0$.
Consider a generic quintic hypersurface $X \subset \mathbb{P}^4$

For all 2875 lines $\ell \subset X$, a generic deformation $\mathcal{E}$ has balanced splitting type:

$$\mathcal{E}|_\ell \cong \mathcal{O}_X^\oplus r$$

The sheaf $\mathcal{F}$ on $\overline{M}_{0,3}(X, [\ell])$ has no cohomology

$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{[\ell]} = 0$ on an open subset of the family of deformations, but is non-zero at the (2,2) point ($\mathcal{E} = T_X$)
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FIN
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