Where We Are At With p Harmonic Measure

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ODE TO THE P LAPLACIAN

I used to be in love with the Laplacian so worked hard to please her with beautiful theorems. However she often scorned me for the likes of Björn Dahlberg, Gene Fabes, Carlos Kenig, and Thomas Wolff. Gradually I became interested in her sister the p Laplacian, $1 < p < \infty, p \neq 2$. **I did not find her as pretty as the Laplacian and she was often difficult to handle because of her nonlinearity. However over many years I took a shine to her and eventually developed an understanding of her disposition. Today she is my girl and the Laplacian pales in comparison to her.**

Introduction

To begin, let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded domain (i.e., a connected open set) and p fixed, $1 < p < \infty$. Let N be an open neighborhood of $\partial\Omega$ and suppose that u is p harmonic in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0 \tag{1}$$

whenever $\theta \in C_0^\infty(\Omega \cap N)$. Here ∇u is the gradient of u. Note that if u had continuous second partials in $\Omega \cap N$ and $\nabla u \neq 0$ in $\Omega \cap N$, then (1) would imply that u is a classical solution to the p Laplace equation in $\Omega \cap N$:

$$\nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = 0,$$

where $\nabla\cdot$ denotes the divergence of u.

If u is positive on $\Omega \cap N$ with boundary value zero on $\partial\Omega$, in the $W^{1,p}$ Sobolev sense, one can extend u to a function in $W^{1,p}(N)$ by setting $u \equiv 0$ on $N \setminus \Omega$. Then there exists (see Heinonen, Kilpelainen, Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, ch 17, Dover Publications, 2006), a unique positive Borel measure μ on $\mathbf{R}^n$ with support $\subset \partial\Omega$, for which

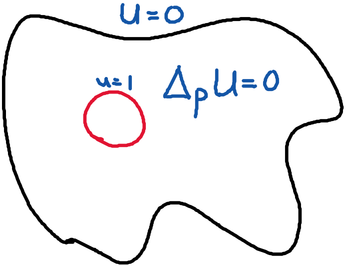
$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = - \int \phi \, d\mu \tag{2}$$

whenever $\phi \in C_0^\infty(N)$. In fact if $\partial\Omega, |\nabla u|$, are smooth

$$d\mu = |\nabla u|^{p-1} dH^{n-1} \text{ on } \partial\Omega,$$

where H^k denotes k dimensional Hausdorff measure in $\mathbf{R}^n$. If $p = 2$ and u is the Green's function with pole at $x_0 \in \Omega$, then $\mu = \omega(\cdot, x_0)$ is harmonic measure with respect to $x_0 \in \Omega$.

Green's functions can be defined for the p Laplacian when $1 < p < \infty$, but are not very useful due to the nonlinearity of the p Laplacian when $p \neq 2$. Instead we often study the measure associated with a p capacitary function, say u, in $\Omega \setminus \bar{B}(x_0, r)$, where $B(x_0, r) = \{y : |y - x_0| < r\}$ and $B(x_0, 4r) \subset \Omega$. That is, u is p harmonic in $\Omega \setminus \bar{B}(x_0, r)$ with continuous boundary values, $u \equiv 1$ on $\partial B(x_0, r)$ and $u \equiv 0$ on $\partial\Omega$.



Remark 1. μ as in (2) is different from the so called p harmonic measure introduced by Martio, which in fact is not a measure (see J. Llorente, J. Manfredi, J.M. Wu,
' p Harmonic Measure Is Not additive on Null Sets,' Ann. Sc. Norm. Super. Pisa Cl. Sci (5) 4 (2005), no. 2, 357-373).
Define the Hausdorff dimension of μ by

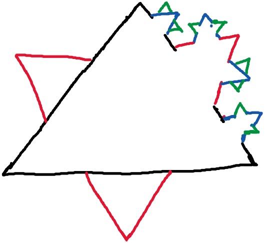
$$\text{H-dim } \mu = \inf\{k : \text{ there exists } E \text{ Borel} \subset \partial\Omega \text{ with } H^k(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}.$$

Remark 2. Today we discuss for a fixed $p, 1 < p < \infty$, what is known about H-dim μ when μ corresponds to a positive p harmonic function u in $\Omega \cap N$ with boundary value 0 in the $W^{1,p}$ Sobolev sense. It turns out that H-dim μ is independent of u as above. Thus we often refer to H-dim μ as the dimension of p harmonic measure in Ω.

For $p = 2$ and harmonic measure, Carleson in
On the support of harmonic measure for sets of Cantor type, Ann. Acad. Sci. Fenn. **10** (1985), 113 - 123.
used ideas from ergodic theory andboundary Harnack inequalities for harmonic functions to deduce H-dim $\omega = 1$ when Ω is a ‘snowflake’ type domain and H-dim $\omega \leq 1$ when Ω is the complement of a self similar Cantor set. He was also the first to recognize the importance of

$$\int_{\partial\Omega_n} |\nabla g_n| \log |\nabla g_n| \, dH^1$$

(g_n is Green’s function for Ω_n with pole at zero and (Ω_n) is an increasing sequence of domains whose union is Ω).



p Harmonic Measure in Quasi-Circles

Inspired by Carleson's work, Björn Bennewitz, and I, managed to obtain the following results in

On the Dimension of p Harmonic Measure (with Björn Bennewitz), Ann. Acad. Sci. Fenn. **30** (2005), 459-505.

Theorem A.

Fix $p, 1 < p < \infty$, and let $u > 0$ be p harmonic in $\Omega \cap N \subset \mathbf{R}^2$ with $u = 0$ continuously on $\partial\Omega$. If $\partial\Omega$ is a snowflake and $1 < p < 2$, then H-dim $\mu > 1$ while if $2 < p < \infty$, then H-dim $\mu < 1$.

Theorem B.

Let p, u, μ be as in Theorem A. If $\partial\Omega$ is a self similar Cantor set and $2 < p < \infty$, then H-dim $\mu < 1$.

Theorem C.

Let p, u, μ be as in Theorem A. If $\partial\Omega$ is a k quasicircle, then H-dim $\mu \leq 1$ for $2 < p < \infty$, while H-dim $\mu \geq 1$ for $1 < p < 2$.

p Harmonic Measure in Simply Connected Domains

Let

$$\gamma(r) = r\, exp\left(a\sqrt{log 1/r \log\log 1/r}\right) \text{ when } 0 < r < 1/100,$$

and let H^γ denote Hausdorff measure defined with respect to γ.
Recently in
p Harmonic Measure in Simply Connected Domains (with Pietro Poggi Corradini and Kaj Nyström), to appear Annals of the Institute Fourier, Grenoble.

we have proved the following theorem.

Theorem D

Fix $p, 1 < p < \infty$, and let $u > 0$ be p harmonic in $\Omega \cap N$, where Ω is simply connected, $\partial\Omega$ is compact, and N is a neighborhood of $\partial\Omega$. Suppose u has continuous boundary value 0 on $\partial\Omega$ and let μ be the measure associated with u as in (2).

(*a*) If $a = a(p) < -1$ is negative enough and $p > 2$, then μ is concentrated on a set of H^γ finite measure.

(*b*) If $a = a(p) > 1$ is large enough and $1 < p < 2$, then μ is absolutely continuous with respect to H^γ measure.

We note that Makarov proved for harmonic measure (i.e, $p = 2$) in Distortion of boundary sets under conformal mapping, Proc. London Math. Soc. **51** (1985), 369-384, the stronger theorem:

Theorem E

Let ω be harmonic measure with respect to a point in the simply connected domain Ω. Then

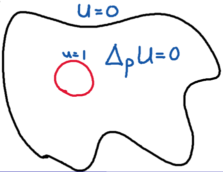
(a) ω is concentrated on a set of σ finite H^1 measure

(b) ω is absolutely continuous with respect to $H^{\hat{\gamma}}$ measure defined relative to $\hat{\gamma}(r) = r \exp[A\sqrt{\log 1/r \log\log\log 1/r}]$ for A sufficiently large.

The best known value of A in the definition of $\hat{\gamma}$ appears to be $A = 6\sqrt{\frac{\sqrt{24}-3}{5}}$ due to H. Hedenmalm and I. Kayamov in, On the Makarov law of the iterated logarithm, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2235-2248.

Basic Ingredients in the Proof of Theorems A-D

To outline the proofs of Theorems $A - D$ we switch to complex notation. So $z = x + iy$, or $x_1 + ix_2$, $B(z, \rho) = \{w : |w - z| < \rho\}$, and $d(E, F)$ denotes the distance between the sets E, F. Since all measures associated with p harmonic functions in Theorems A - E have the same dimension and since the p Laplacian is invariant under translations, dilations, and rotations, we assume as we may that $\bar{B}(0,1) \subset \Omega, d(0, \partial\Omega) = 4$, and that u is the p capacitary function for $D = \Omega \setminus \bar{B}(0,1)$. Thus $u \equiv 1$ on $\partial B(0,1), u \equiv 0$ on $\partial\Omega$ and u is p harmonic in D.



Fact A. u is real analytic in $D, \nabla u \neq 0$ in D, and $u_z = (1/2)(u_x - iu_y)$, is $k = k(p)$ quasi-regular in D. Consequently, $\log|\nabla u|$ is a weak solution to a divergence form PDE for which a Harnack inequality holds. That is, if $h \geq 0$ is a weak solution to this PDE in $B(\zeta, r) \subset D$, then $\max_{B(\zeta, r/2)} h \leq \tilde{c} \min_{B(\zeta, r/2)} h$, where $\tilde{c} = \tilde{c}(p)$.□

Next we note that if $\zeta = u_{x_1}, u_{x_2}$, or u, then for $z = x_1 + ix_2 \in D$,

$$L\zeta = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} [\, b_{ij}(z) \zeta_{x_j}(z) \,] = 0, \tag{3}$$

Here

$$b_{ij}(z) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](z), \tag{4}$$

for $1 \leq i, j \leq 2$, and δ_{ij} is the Kronecker δ. Observe that if $\xi = \xi_1 + i\xi_2$, then

$$\min\{p-1,1\}|\xi|^2\,|\nabla u(z)|^{p-2} \leq \sum_{i,k=1}^{2} b_{ik}(z)\,\xi_i\xi_k \tag{5}$$

$$\leq \max\{1,p-1\}|\nabla u(z)|^{p-2}\,|\xi|^2\,.$$

Also it turns out that $v = \log|\nabla u|$ satisfies for $p \neq 2, 1 < p < \infty$, that

$$\frac{Lv}{p-2} \approx \sum_{i,j=1}^{2} |\nabla u|^{p-4}(u_{x_i x_j})^2 \tag{6}$$

in D where $\approx$ means the two quantities are bounded above and below by constants depending on p. Thus $Lv \geq 0$ for $p > 2$ and $Lv \leq 0$ for $1 < p < 2$. Note that if

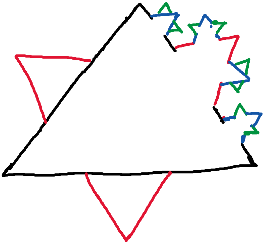
$$c^{-1}u(z)/d(z,\partial\Omega) \leq |\nabla u(z)| \leq cu(z)/d(z,\partial\Omega), \tag{7}$$

for some constant c and z near $\partial\Omega$. then Harnack's inequality for u and (5) imply that $(b_{ik}(z))$ are locally uniformly elliptic in Ω.

The righthand inequality in (7) can be proved using Fact A. However the lefthand inequality in (7) for simply connected domains, appears nontrivial, and is the main new result in Theorem E. That is, armed with (7) in the simply connected case, we were able to follow closely the proof of Theorem C. Note that if $p = 2$ and u is the Green's function with pole at 0, then (7) is an easy consequence of the area theorem for univalent functions.

Proof of Theorem A

To outline the proof of Theorem A suppose $\Omega_n, n = 1, 2, \ldots,$ is a sequence of approximating domains and for large n let u_n be the p capacitary function for $D_n = \Omega_n \setminus \bar{B}(0, 1)$.



Then one first proves:

Lemma 1. For fixed $p, 1 < p < \infty$,

$$\eta = \lim_{n \longrightarrow \infty} n^{-1} \int_{\partial \Omega_n} |\nabla u_n|^{p-1} \log |\nabla u_n| \, dH^1 z$$

exists. If $\eta > 0$ then H-dim $\mu < 1$ while if $\eta < 0$, then H-dim $\mu > 1$.□

To prove Lemma 1 we followed Carleson. Thus we proved a boundary Harnack inequality for the ratio of two positive p harmonic functions vanishing on $\partial \Omega$ which then enabled us to employ ergodic theorems of Birkhoff and Shannon, Mcmillan, Breiman, to eventually get Lemma 1. □

To prove Theorem A let u_n, Ω_n be as in Lemma 1. Using Fact A applied to u_n and (3) - (5) one deduces that if $v_n = \log |\nabla u_n|$, then

$$\int_{D_n} (u_n \, L v_n - v_n \, L u_n) \, dA = (p-1) \int_{\partial \Omega_n} |\nabla u_n|^{p-1} \log |\nabla u_n| \, dH^1 z + O(1). \tag{8}$$

In view of the estimate for Lv in (6) one concludes from Lemma 1 that it suffices to show

$$\liminf_{n \to \infty} \left(n^{-1} \int_{D_n} u_n \, |\nabla u_n|^{p-4} \sum_{i,j=1} (u_n)^2_{x_i x_j} \, dA \right) > 0. \tag{9}$$

To prove (9) we showed the existence of $c \geq 1$ and $\lambda \in (0, 1)$ such that if $z \in \Omega_n \setminus B(0, 2)$ and $d(z, \partial\Omega_n) \geq 3^{-n}$, then

$$c \int_{D_n \cap B(z, \lambda d(z, \partial\Omega_n))} u_n |\nabla u_n|^{p-4} \sum_{i,j=1}^{2} (u_n)^2_{y_i y_j} \, dA \geq \mu_n(B(z, 2d(z, \partial\Omega_n)) \tag{10}$$

where c depends on p and the k quasi-conformality of Ω. Covering $\{3^{-m-1} \leq d(z, \partial\Omega_n) \leq 3^{-m}\}$ by balls and summing over $1 \leq m \leq n-1$ in (10) we obtain first (9) and then Theorem A. $\square$

Proof of Theorem C

To prove Theorem C for quasi circles let $w(x) = \max(v - c, 0)$ when $1 < p < 2$ and $w(x) = \max(-v - c, 0)$ when $p > 2$. Here c is chosen so large that $|v| \leq c$ on $B(0, 2)$. Following Makarov we used Green's theorem, the coarea formula, (6), and (7) to prove that

Lemma 2. Let m be a nonnegative integer . There exists $c_+ = c_+(k, p) \geq 1$ such that for $0 < t < 1$,

$$\int_{\{z: u(z)=t\}} |\nabla u|^{p-1} \, w^{2m} \, dH^1 z \, \leq \, c_+^{m+1} \, m! \, [\log(2/t)]^m \, . \; \square$$

Dividing the above by $(2c_+)^m \, m! \, [\log(2/t)]^m$ and summing we obtained for $0 < t < 1$ that

$$\int_{\{z: u(z)=t\}} |\nabla u|^{p-1} \, \exp\left[\frac{w^2}{2c_+ \log(2/t)}\right] \, dH^1 z \, \leq \, 2 \, c_+. \qquad (11)$$

Using (11) and weak type estimates it follows that if

$$\lambda(t) = \sqrt{4\, c_+ \, \log(2/t)}\, \sqrt{\log(-\log t)} \text{ for } 0 < t < e^{-2},$$

$$F(t) = \{z : u(z) = t \text{ and } w(z) \geq \lambda(t)\}$$

then

$$\int_{F(t)} |\nabla u|^{p-1}\, dH^1 z \leq \frac{2c^+}{\log^2(1/t)} \tag{12}$$

Finally (12) combined with measure theoretic arguments and the inequality

$$\begin{aligned} c^{-1}\, r^{p-2}\, \mu[B(w, r/2)] &\leq \max_{B(w,r)} u^{p-1} \\ &\leq c\, r^{p-2}\, \mu[B(w, 2r)] \text{ whenever } w \in \partial\Omega,\ 0 < r \leq r_0, \end{aligned} \tag{13}$$

yield Theorem C. □

Proof of Theorem D

As mentioned earlier the major obstacle to proving Theorem E over Theorem C was that we could not prove the fundamental inequality in (6). That is, in our simply connected paper we prove

Theorem F

If u is the p capacitary function for D, then there exists $c = c(p) \geq 1$, such that

$$c|\nabla u|(z) \geq \frac{u(z)}{d(z, \partial\Omega)} \text{ whenever } z \in D.$$

To prove Theorem F we assume, as we may, that $\partial\Omega$ is a Jordan curve, since otherwise we can approximate Ω in the Hausdorff distance sense by Jordan domains and use the fact that the constant in Theorem F depends only on p to eventually get this theorem for Ω.

We also write $\rho(\cdot,\cdot)$ for the hyperbolic distance function in Ω.

Lemma 3. There is a constant $c = c(p) \geq 1$ such that for any point $z_1 \in D \setminus B(0,2)$, there exists $z^\star \in D \setminus B(0,2)$ with $u(z^\star) = u(z_1)/2$ and $\rho(z_1, z^\star) \leq c$. $\square$

Assuming Lemma 3 one gets Theorem F from the following argument. Let Γ be the hyperbolic geodesic connecting z_1 to z^* and suppose that $\Gamma \subset D$. From properties of ρ one sees for some $c = c(p)$ that

$$H^1(\Gamma) \leq c d(z_1, \partial\Omega) \text{ and } d(\Gamma, \partial\Omega) \geq c^{-1} d(z_1, \partial\Omega). \tag{14}$$

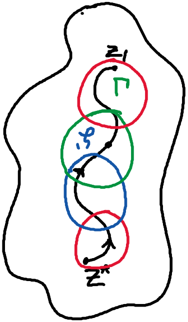
Thus

$$\frac{1}{2} u(z_1) \leq u(z_1) - u(z^\star) \leq \int_\Gamma |\nabla u(z)||dz| \\ \leq c H^1(\Gamma) \max_\Gamma |\nabla u| \leq c d(z_1, \partial\Omega) \max_\Gamma |\nabla u|.$$

So for some $\zeta \in \Gamma$ and $c^* = c^*(p) \geq 1$,

$$c^\star |\nabla u(\zeta)| \geq \frac{u(z_1)}{d(z_1, \partial\Omega)}. \tag{15}$$

Also from (14), we deduce the existence of Whitney balls $\{B(w_j, r_j\}$, with $w_j \in \Gamma$, $r_j \approx d(z_1, \partial\Omega)$, connecting ζ to z_1.



Using this deduction, the righthand inequality in (7), and Harnack's inequality applied to u we find

$$|\nabla u(z)| \leq c u(z_1)/d(z_1, \partial\Omega) \text{ when } z \in \bigcup_j B(w_j, r_j)\,. \qquad (16)$$

From (15), (16), we see that if $c = c(p)$ is large enough and

$$h(z) = \log\left(\frac{c\, u(z_1)}{d(z_1, \partial\Omega)\, |\nabla u(z)|}\right) \text{ for } z \in \bigcup_j B(w_i, r_i)$$

then $h > 0$ in $\cup_i B(w_i, r_i)$ and $h(\zeta) \leq c$. From Fact A we see that Harnack's inequality can be applied to h in successive balls of the form $B(w_i, r_i/2)$. Doing this we obtain $h(z_1) \leq c'$ where $c' = c'(p)$. Clearly, this inequality implies Theorem F.

We note that if $\partial\Omega$ is a quasicircle one can choose z^* to be a point on the line segment connecting z_1 to $w \in \partial\Omega$ where $|w - z_1| = d(z_1, \partial\Omega)$ The proof uses Hölder continuity of u near $\partial\Omega$ and the fact that for some $c = c(p, k),\ c u(z_1) \geq \max_{B(z_1, 2d(z_1, \partial\Omega))} u$.

This inequality need not hold in a Jordan domain and so we have to give a more complicated argument to get Lemma 3. To this end, we construct a Jordan arc

$\sigma : (-1,1) \to D$ with $\sigma(0) = z_1$, $\sigma(\pm 1) = \lim_{t \to \pm 1} \sigma(t) \in \partial\Omega$, and $\sigma(1) \neq \sigma(-1)$. Moreover, for some $c = c(p)$,

$$\begin{aligned} (\alpha)\ & H^1(\sigma) \leq c d(z_1, \partial\Omega) \\ (\beta)\ & u \leq c u(z_1) \text{ on } \sigma. \end{aligned} \tag{17}$$

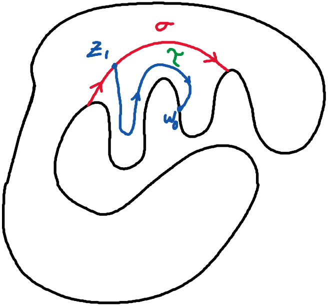
Let Ω_1 be the component of $\Omega \setminus \sigma$ not containing $B(0,1)$. Then we also require that there is a point w_0 on $\partial\Omega \cap \partial\Omega_1$ with

$$|w_o - z_1| \leq c d(z_1, \partial\Omega) \text{ and } d(w_0, \sigma) \geq c^{-1} d(z_1, \partial\Omega). \tag{18}$$

Finally we shall show the existence of a Lipschitz curve $\tau : (0,1) \to \Omega_1$ with $\tau(0) = z_1$, $\tau(1) = w_0$, satisfying the cigar condition:

$$\min\{H^1(\tau[0,t]), H^1(\tau[t,1])\} \leq \hat{c} d(\tau(t), \partial\Omega), \tag{19}$$

for $0 < t < 1$ and some absolute constant $\hat{c}$.

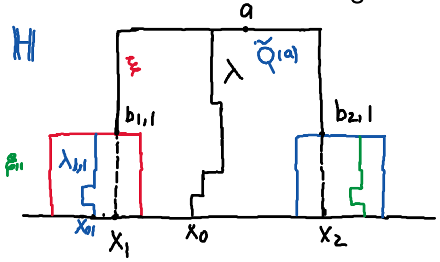


To get Lemma 3 from (17) - (19) let $u_1 = u$ in Ω_1 and $u_1 \equiv 0$ outside of Ω_1. From PDE estimates, (17) (β), and (18) one finds $\theta > 0, c < \infty$ such that

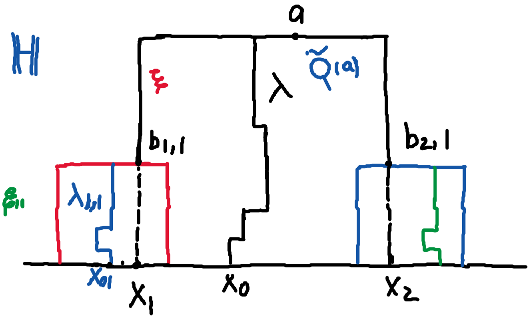
$$\max_{B(w_0,t)} u_1 \leq c u(z_1) \left(\frac{t}{d(z_1, \partial\Omega)} \right)^{\theta} \text{ for } 0 < t < d(w_0, \sigma). \qquad (20)$$

From (19), (20) we conclude the existence of z^* with $\rho(z_1, z^*) \leq c$ and $u(z^*) = 1/2$, which is Lemma 3.

The most difficult part of the construction of σ, τ is to prove (17) (β). To briiefly outline this proof let f be the Riemann mapping function from the upper half plane, $\mathbb{H}$, onto Ω with $f(i) = 0$ and $f(a) = z_1$, where $a = is$ for some $s, 0 < s < 1$. In our paper we show the existence of $x_1 \in [-s, -s/2]$, $x_2 \in [s/2, s]$ with the following property. Let ξ be the curve consisting of the horizontal line segment joining $x_1 + is$ to $x_2 + is$, together with the vertical line segments joining x_j to $x_j + is$ for $j = 1, 2$. Let $\tilde{Q}(a)$ be the rectangle whose intersection with $\mathbb{H}$ is ξ. Then $f(\xi) = \sigma$ satisfies (17) (α). Moreover τ is the image of a curve λ, consisting of horizontal and vertical line segments, joining a to a point x_0.



It remains to prove $u \leq cu(z_1)$ on σ which is (17) (β). The proof is by contradiction. Suppose $u > Au(z_1)$ on σ. We shall obtain a contradiction if $A = A(p)$ is suitably large. The argument is based on the recurrence type scheme often attributed to Carleson - Domar in the complex world and Caffarelli et al in the PDE world. Given the rectangle $\tilde{Q}(a)$ one chooses points $b_{j,1} = x_j + i\delta s,\ j = 1, 2,$ on the vertical sides of $\tilde{Q}(a)$ with $df(b_{j,k}, \partial\Omega) \leq \frac{1}{2} df(a, \partial\Omega)$ and corresponding boxes $\tilde{Q}(b_{j,1}),\ \ j = 1, 2,$ similar to $\tilde{Q}(a)$. These boxes in turn will each spawn two more new boxes, and so on. Without loss of generality, we focus on $\tilde{Q}(b_{1,1})$.



In this box we construct a polygonal path $\lambda_{1,1}$ from $b_{1,1}$ to some point $x_{0,1} \in I(b_{1,1})$. λ_{11} is defined relative to $b_{1,1}$ in the same way that λ was defined relative to a. If A is large enough, one can show there exists a point far down on $\lambda_{1,1}$ where $U = u \circ f = Au(z_1)$. Using (20) with w_0, τ replaced by $f(x_{0,1}), f(\lambda_{1,1})$, we then get $U > A^2 u(z_1)$ on $\partial Q(b_{1,1})$. Continuing in this manner we get a sequence of points, tending to a point on $\mathbb{H}$, on which U tends to ∞. Since $u \equiv 0$ on $\partial\Omega$ we have reached a contradiction.

The above title is the name of a joint paper with B. Bennewitz, Kaj Nystrom and Andy Vogel, which should soon be on my webpage. First we state some positive results and after that discuss some examples. In order to state our results, we need a definition.

Definition A. Let $\Omega \subset \mathbf{R}^n$ be a domain and $0 < r \leq r_0$. Then Ω and $\partial\Omega$ are said to be (δ, r_0), Reifenberg flat provided that whenever $w \in \partial\Omega$, there exists a hyperplane, $P = P(w, r)$, containing w such that

$$(a)\, \Psi(\partial\Omega \cap B(w,r), P \cap B(w,r)) \leq \delta r$$

□

$$(b)\, \{x \in \Omega \cap B(w,r) : d(x, \partial\Omega) \geq 2\delta r\} \;\subset\; \text{one component of } \mathbf{R}^n \setminus P.$$

In Definition A, $\Psi(E, F)$ denotes the Hausdorff distance between the sets E and F defined by

$$\Psi(E,F) = \max(\sup\{d(y,E) : y \in F\}, \sup\{d(y,F) : y \in E\}).$$

We note that snowflakes in two dimensions and Wolff snowflakes in higher dimensions are Reifenberg flat. We prove

Theorem G

Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a (δ, r_0) Reifenberg flat domain, $w \in \partial\Omega$, and p fixed, $n \leq p < \infty$. Let $u > 0$ be p harmonic in Ω with $u = 0$ continuously on $\partial\Omega$. Let μ be the measure associated with u as in (2). There exists, $\hat{\delta} = \hat{\delta}(p, n) > 0$, such that if $0 < \delta \leq \hat{\delta}$, then μ is concentrated on a set of σ finite H^{n-1} measure.

Proof of Theorem G

To prove Theorem G define L relative to u as in (3), (4). That is,

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} [\, b_{ij}(z)\zeta_{x_j}(z) \,]$$

$$b_{ij}(z) = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2](z),$$

$$\text{and } |\xi|^2 \, |\nabla u(z)|^{p-2} \approx \sum_{i,j=1}^{n} b_{ij}(z)\, \xi_i \xi_j$$

whenever $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$.

Fact B: If $v = \log |\nabla u|$, then $Lv \geq 0$ in Ω when $p \geq n$.

We also need,

Lemma 4. Let Ω be (δ, r_0) Reifenberg flat, $1 < p < \infty$, and $u > 0$, a p harmonic function in Ω with $u \equiv 0$ on $\partial\Omega$. Then there exists, $\delta_0 > 0, c_1 \geq 1$, depending only on p, n, such that if $0 < \delta \leq \delta_0$ and $x \in \Omega$, then $u \in C^\infty(\Omega)$ and

$$(a)\, c_1^{-1} |\nabla u(x)| \leq \, u(x)/d(x, \partial\Omega) \, \leq c_1 |\nabla u(x)|, x \in \Omega,$$

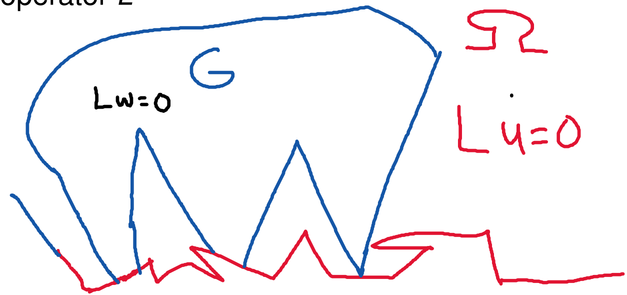
$$(b)\, |\nabla u|^{p-2} \text{ extends to an } A_2 \text{ weight on } \mathbf{R}^n \quad \text{with constant} \leq c_1.$$

From Lemma 4 we see that $(b_{ij}(x))$ as defined above are locally uniformly elliptic in Ω with ellipticity constants given in terms of an A_2 weight on $\mathbf{R}^n$. Armed with Fact B and Lemma 4 we can now essentially repeat an argument of Makarov who used Plessner's Theorem to prove (a) in Theorem E.

Moreover, the main step in proving Plessner's theorem is to show that if G is an open set with $G \subset B(0,1)$, ∂G is locally Lipschitz, and $H^1(\partial G \cap \partial B(0,1)) > 0$, then the harmonic measure of $\partial G \cap \partial B(0,1)$ with respect to some point in G is positive. In our situation we show, that if $G \subset \Omega$ is an NTA domain (in the sense of Jerion - Kenig), then

$$\mu(\partial G \cap \partial \Omega) > 0 \longrightarrow \omega(\partial G \cap \partial \Omega) > 0. \tag{21}$$

Here ω is elliptic measure defined with respect to a point in G and the operator L



To prove (21) we use results from

E. Fabes, C. Kenig, and R. Serapioni, The Local Regularity of Solutions to Degenerate Elliptic Equations, Comm. Partial Differential Equations, **7** (1982), no. 1, 77 - 116.

E. Fabes, D. Jerison, and C. Kenig, *Boundary Behavior of Solutions to Degenerate Elliptic Equations.* Conference on harmonic analysis in honor of Antonio Zygmund, Vol I, II Chicago, Ill, 1981, 577-589, Wadsworth Math. Ser, Wadsworth Belmont CA, 1983.

E. Fabes, D. Jerison, and C. Kenig, The Wiener Test for Degenerate Elliptic Equations, Ann. Inst. Fourier (Grenoble) **32** (1982), 151-182.

B. Dahlberg, D. Jerison, and C. Kenig, *Area integral estimates for elliptic differential operators with nonsmooth coefficients*, Ark. Mat. **22** (1984), no. 1, 97–108.

Wolff Snowflakes

T. Wolff in
Counterexamples with harmonic gradients in $\mathbf{R}^3$, Essays in honor of Elias M. Stein, Princeton Mathematical Series **42** (1995), 321-384.
used Carleson's ideas and brilliant ideas of his own to study the dimension of harmonic measure, ω, with respect to a point in domains bounded by 'Wolff snowflakes' $\subset \mathbf{R}^3$. He constructed snowflakes for which H-dim $\omega > 2$ and snowflakes for which H-dim $\omega < 2$. Verchota, Vogel, and Lewis in
Wolff Snowflakes, Pacific J. Math. 218 (2005), no.1, 139-166
used Wolff's method to construct a Wolff snowflake for which the harmonic measures on both sides of the snowflake were both of H-dim $< n - 1$ and also a snowflake for which the harmonic measures on both sides were of H-dim $> n - 1$.
Because of the above paper my original idea, when Bjorn Bennewitz became my student (about five years ago), was to use Wolff's method to construct snowflakes for which the dimension of p harmonic measure could be estimated.

This turned out to be difficult as his proof made important use of boundary Harnack inequalities. Only recently in a series of papers, mostly written with Kaj Nyström (listed later), we have developed the technology to make Wolff's method work.
Next we briefly describe the construction of Wolff snowflakes. Let $\Omega_0 = \{(x', x_n) : x' \in \mathbf{R}^{n-1}, x_n > 0\}$ be the upper half space and let

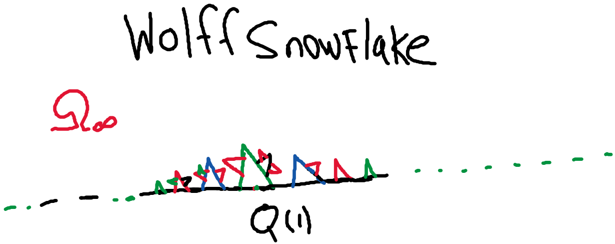
$$Q(1) = \{x' \in \mathbf{R}^{n-1} : \tfrac{1}{2} \leq x_i \leq \tfrac{1}{2} \text{ for } 1 \leq i \leq n-1\}.$$

Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, be a piecewise linear function with supp $\phi \subset \{x' : |x'| < 1/2\}$ and $\|\nabla\phi\|_\infty \leq \theta_0$ (small). For fixed N_0 large and $N \geq N_0$, set $\psi(x') = N^{-1}\phi(Nx')$. Let

$$\hat{\Omega} = \{x : x_n > \psi(x)\}, \quad \partial = \{x \in \mathbf{R}^n : x' \in Q(1), x_n = \psi(x')\}.$$

We say that $\hat{\Omega}$ is obtained from Ω_0 by adding a blip along $Q(1)$. By construction, ∂ consists of a finite number of faces.

Divide each face into Whitney $n-1$ cubes, whose side length is proportional to the distance from the nearest edge. We now add a blip to each of these Whitney cubes and in so doing get a domain Ω_1. More specifically, we map $Q(1)$ onto each Whitney cube by a conformal affine mapping (i.e, a composition of a rotation, translation, dilation). Then $T(\partial)$ is the boundary of the new blip. Each of the countable number of blips thus obtained inherits a natural subdivision into Whitney cubes. We then add a blip to each of the new Whitney cubes, which results in Ω_2. Continuing by induction we get $(\Omega_m)_1^\infty$. One can show that if N is large enough, then $\Omega_m \rightarrow \Omega_\infty$ in the Hausdorff distance sense. We call Ω_∞ a Wolff snowflake.



For fixed $p, 1 < p < \infty, p \neq 2$, let $u_\infty = u_\infty(\cdot, p)$ be the positive p harmonic function in Ω_∞ with continuous boundary value zero and $|x_n - u_\infty(x)| \to 0$ uniformly as $|x| \to \infty$. Let μ_∞ be the p harmonic measure associated with u_∞ and let μ'_∞ be the restriction of μ_∞ to $(Q(1) \times [-1, 1]) \cap \partial\Omega_\infty$.

Theorem H.

Let $\Omega_\infty, u_\infty, \mu_\infty$, be as above. If $p \geq n \geq 3$, and θ_0, N_0^{-1} are small enough, then H-dim $\mu'_\infty < n - 1$. If $2 < p < n$, there is an Ω_∞, for which H-dim $\mu'_\infty < n - 1$ while if $1 < p < 2$ there is an Ω_∞ with H-dim $\mu'_\infty > n - 1$.

To outline Wolff's proof and also our proof for $1 < p < n$, let $\hat\Omega(\epsilon) = \{(x', x_n) : x_n > \epsilon\hat\theta(x'), x' \in \mathbf{R}^{n-1}\}$ where $\hat\theta$ is infinitely differentiable with support in $\{x' : |x'| < 1/2\}$. For fixed $p, 1 < p < \infty$, let $\hat u = \hat u(\cdot, \epsilon)$, be the unique p-harmonic function in $\hat\Omega$ with pole at ∞ defined by $\hat u \equiv 0$ on $\partial\hat\Omega$ and $|x_n - \hat u(x)| \to 0$ uniformly as $|x| \to \infty, x \in \hat\Omega$. Furthermore, let

$$I = I(\epsilon) = \int_{\partial\hat{\Omega}(\epsilon)} |\nabla \hat{u}(\cdot,\epsilon)|^{p-1} \log |\nabla \hat{u}(\cdot,\epsilon)| \, dH^{n-1}.$$

We prove existence and uniqueness of $\hat{u}$ and also existence and infinite differentiability of $I(\epsilon)$ with respect to ϵ. In fact we obtain

$$I(0) = 0, \;\; I'(0) = 0, \;\; I''(0) = \frac{p-2}{p-1} \int_{\mathbf{R}^{n-1}} |\nabla' \hat{\theta}|^2 \, dH^{n-1},$$

where ∇' denotes the gradient in the variables $x' \in \mathbf{R}^{n-1}$. From Taylor's theorem it follows that $I(\epsilon) > 0$ for $p > 2$ while $I(\epsilon) < 0$ when $1 < p < 2$, provided $0 < \epsilon \leq \epsilon_0$ and $\epsilon_0 = \epsilon_0(p, n, \hat{\theta})$ is small enough. We then show that we can approximate $\epsilon\hat{\theta}$ by a piecewise linear function, $\phi = \phi(\cdot, \epsilon)$ in such a way that the following is true.

Let $\tilde{\Omega} = \{x \in \mathbf{R}^n : x_n > \phi(x'), x' \in \mathbf{R}^{n-1}\}$ and let $\tilde{u}$ be the corresponding positive p harmonic function in $\tilde{\Omega}$ with continuous boundary value 0 and $x_n - \tilde{u}(x) \to 0$ uniformly as $|x| \to \infty$. Then

$$\tilde{I} = \int_{\partial\tilde{\Omega}} |\nabla \tilde{u}|^{p-1} \log |\nabla \tilde{u}| \, dH^{n-1} \text{ has the same sign as } I(\epsilon).$$

Next we construct a Wolff snowflake Ω_∞, as above, relative to $\hat{\Omega} = \{(x', x_n) : x_n > \psi(x'), x' \in \mathbf{R}^{n-1}\}$. Let $\Omega_\infty, u_\infty, \mu_\infty, \mu'_\infty$, be the corresponding domain, p harmonic function, measure, and restriction of the measure, as defined above Theorem H.

Theorem I.

There exists $\theta_0 \in (0, 1)$, N_0 large, depending on p, n, such that if $\|\nabla \phi\|_\infty \leq \theta_0, N \geq N_0$, and $\tilde{I} > 0$, then H-dim $\mu'_\infty < n - 1$ while if $\tilde{I} < 0$, then H-dim $\mu'_\infty > n - 1$.

From Theorem I and the above discussion we conclude for given $p, 1 < p < 2$, that there exist Wolff snowflakes for which H-dim $\mu'_\infty > n - 1$ while if $2 < p < n$, there exist snowflakes for which H-dim $\mu'_\infty < n - 1$. If $p \geq n$, then we can use the fact $\log |\nabla \tilde{u}|$ is a subsolution to the PDE listed earlier (with u replace by $\tilde{u}$) in order to show that $\tilde{I} > 0$ for any Lipschitz function ϕ with compact support in $\{x' : |x'| < 1/2\}$. Thus for $p \geq n$ one must always get using the Wolff method that H-dim $\mu'_\infty < n - 1$. Theorems G, H and I seem to indicate that the analogue of Theorem D should hold for sufficiently flat Reifenberg domains in space. However the situation is much more interesting as we show in Theorem J. To avoid confusion, for fixed $p, 1 < p < \infty$, we write $\mu_\infty(\cdot, p), \mu'_\infty(\cdot, p)$, for the above measures.

Theorem J.

There is a Wolff snowflake for which $\mu'_\infty(\cdot, p)$, p in an open interval containing 2, and $\mu'_\infty(\cdot, 2)$ both have H-dim either $> n - 1$ or $< n - 1$.

Open Problems for p Harmonic Measure

Note. In problems 1) - 8) the surrounding space is $\mathbf{R}^2$.

1) Can Theorem D for simply connected domains be generalized to:

(*a*) μ is concentrated on a set of σ finite H^1 measure whenever $p > 2$.

(*b*) If $a = a(p) > 1$ is large enough and $1 < p < 2$, then μ is absolutely continuous with respect to $H^{\hat{\gamma}}$ measure where $\hat{\gamma}$ is defined in Theorem E

2) Is H-dim μ concentrated on a set of σ finite H^1 measure when $p > 2$ and Ω is any planar domain.

For harmonic measure this result is due to P. Jones and T. Wolff: Hausdorff dimension of harmonic measures in the plane, Acta Math. **161** (1988), 131-144. and T. Wolff in Plane harmonic measures live on sets of σ finite length, Ark. Mat. **31** (1993), no. 1, 137-172.

3) What is the exact value of H-dim μ for a given p when $\partial\Omega$ is the Van Koch snowflake and $p \neq 2$?

4) For a given p, what is the supremum ($p < 2$) or infimum ($p > 2$) of H-dim μ taken over the class of quasi-circles and/or simply connected domains?.

5) Is H-dim μ continuous and/or decreasing as a function of p when $\partial\Omega$ is the Van Koch snowflake?

Regarding this question, the proof of Theorem A gives that H-dim $\mu = 1 + O(|p - 2|)$ as $p \to 2$ for a snowflake domain.

6) Are the p harmonic measures defined on each side of a snowflake mutually singular? The answer is yes when $p = 2$ as shown by C. Bishop, L. Carleson, J. Garnett, and P. Jones, in Harmonic Measures Supported on Curves, Pacific J. Math. **138** (1989), 233-236. One can ask a similar question for Wolff snowflakes in space. In this setting the answer is not known even for harmonic functions, although Kenig, Toro, and Preiss in Boundary structure and size in terms of interior and exterior hamonic measures in higher dimensions, J. Amer. Math. ww (2009), no. 3, 771-796
have made progress on this problem.

7) Is it always true for $1 < p < \infty$ that H-dim $\mu <$ Hausdorff dimension of $\partial\Omega$ when $\partial\Omega$ is a snowflake or a self similar Cantor set? The anwer is yes when $p = 2$ for the snowflake as shown by R. Kaufman and J.M. Wu in, On the Snowflake Domain, Ark. Mat. **23** (1985), 177-183. The answer is also yes for self similar Cantor sets when $p = 2$. This question and continuity questions for H-dim ω on certain four cornered Cantor sets are answered by Batakis in

Harmonic Measure of Some Cantor Type Sets, Ann. Acad. Sci. Fenn. **21** (1996), no 2, 255-270.

A Continuity Property of the Dimension of Harmonic Measure Under Perturbations, Ann. Inst. H. Poincaré Probab. Statist., **36** (1): 87-107, 2000.

Continuity of the Dimension of the Harmonic Measure of Some Cantor Sets Under Perturbations, Annales de l' Institut Fourier, **56**, no. 6 (2006), 1617-1631.

8) We noted in Remark 2 that H-dim μ was independent of the choice of u vanishing on $\partial\Omega$. However in more general scenarios we do not know whether H-dim μ is independent of u. For example, suppose $x_0 \in \partial\Omega$ and $u > 0$ is p harmonic in $\Omega \cap B(x_0, r)$ with $u = 0$ on $\partial\Omega \cap B(x_0, r)$ in the $W^{1,p}$ sense. If $\partial\Omega \cap B(x_0, r)$ has positive p capacity, then there exists a measure μ satisfying (2.1) with $\phi \in C_0^\infty(N)$ replaced by $\phi \in C_0^\infty(B(x_0, r))$. Is H-dim $\mu|_{B(x_0, r/2)}$ independent of u? If Ω is simply connected and $p = 2$, then I believe the answer to this question is yes.

In general this problem appears to be linked with boundary Harnack inequalities.

9) Is it true for $p \geq n$ that H-dim $\mu \leq 1$ whenever $\Omega \subset \mathbf{R}^n$? If not is there a more general class of domains than Reifenberg flat domains (see Theorem I) for which this inequality holds? Compare with problem 2.

10) What can be said about the dimension of p harmonic measure when $\Omega \subset \mathbf{R}^n$ and $1 < p < n$ (see Theorems G, H).

11) What can be said for the dimension of p harmonic measure, $p > \log 4/\log 3$, or even harmonic measure in $\Omega = \mathbf{R}^3 \setminus J$ where J is the Van Koch snowflake?

12) The existence of a measure μ, corresponding to a weak solution u with vanishing boundary values, as in (2), exists for a large class of divergence form partial differential equations. What can be said about analogues of Theorems $A - E, G - J$, for the measures corresponding to these solutions? What can be said about analogues of problems 1) - 11)?

Papers involving Boundary Harnack Inequalities for p Harmonic Functions

Boundary Behavior for *p* Harmonic Functions in Lipschitz and Starlike Lipschitz Ring Domains, (with Kaj Nyström) Ann. Sc. École Norm. Sup. (4) **40** (2007), no. 4, 765-813.

Boundary Behaviour and the Martin Boundary Problem for *p* Harmonic Functions in Lipschitz Domains (with Kaj Nyström), to appear Annals of Mathematics.

Boundary Behavior of *p* Harmonic Functions in Domains Beyond Lipschitz Domains (with Kaj Nyström), Advances in the Calculus of Variations, **1** (2008), 1 - 38.

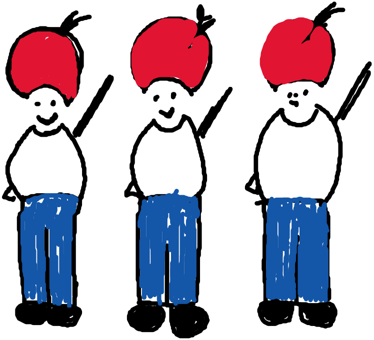
Boundary Harnack Inequalities for Operators of *p* Laplace Type in Reifenberg Flat Domains, (with Kaj Nyström and Niklas Lundström), Proceedings of Symposia in Pure Mathematics **79** (2008), 229-266.

Regularity and Free Boundary Regularity for the p Laplacian in Lipschitz and C^1 Domains (with Kaj Nyström), Ann. Acad. Sci. Fenn. **33** (2008), 1 - 26.

Regularity of Lipschitz Free Boundaries in Two Phase Problems for the p Laplace Operator (with Kaj Nyström), to appear Advances in Mathematics.

Regularity of Flat Free Boundaries in Two Phase Problems for the p Laplace Operator (with Kaj Nyström), submitted.

Thanks for Your Attention !!!



Thanks for not Snoring !!!