Contributions to Fully Matricial Function Theory

When a matrix of functions is a function of matricies

Paul S. Muhly[1] Baruch Solel[2]

[1]Department of Mathematics
University of Iowa

[2]Department of Mathematics
Technion

BIRS Workshop on Multivariate Operator Theory, August 17, 2010

The Problem

Sometimes an operator algebra can be viewed usefully as an algebra of operator-valued functions on its space of representations.

This possibility arises naturally in the theory of operator tensor algebras and certain relatives.

The problem is to determine what sort of functions they are and, to paraphrase Herman Weyl in *Die Idee der Riemannnschen Fläche*, what is the nature of the soil in which they grow?

The Problem

Sometimes an operator algebra can be viewed usefully as an algebra of operator-valued functions on its space of representations.
This possibility arises naturally in the theory of operator tensor algebras and certain relatives.
The problem is to determine what sort of functions they are and, to paraphrase Herman Weyl in *Die Idee der Riemannnschen Fläche*, what is the nature of the soil in which they grow?

The Problem

Sometimes an operator algebra can be viewed usefully as an algebra of operator-valued functions on its space of representations.
This possibility arises naturally in the theory of operator tensor algebras and certain relatives.
The problem is to determine what sort of functions they are and, to paraphrase Herman Weyl in *Die Idee der Riemannnschen Fläche*, what is the nature of the soil in which they grow?

Two Solutions

The first solution is suggested by work of Joe Taylor in *A general framework for a multi-operator functional calculus*, Advances in Math. **9** (1972), 183–252.

In more recent times, Taylor's ideas have been pursued by Dan Voiculescu in his work on free analysis questions and "fully matricial function theory", by Dmitry Kalyuzhnyi-Verbovetzkyi and Victor Vinnikov in their work, by Bill Helton, Igor Klepp, Scott McCullough and Nick Slinglend on noncommutative ball maps and dimension free inequalities, and by others.

The second solution comes from our work (and the work of others) on generalizations of the Nevanlinna-Pick interpolation theorem.

I want to describe here these solutions in the context of the work that Baruch and I have been doing.

Two Solutions

The first solution is suggested by work of Joe Taylor in *A general framework for a multi-operator functional calculus*, Advances in Math. **9** (1972), 183–252.

In more recent times, Taylor's ideas have been pursued by Dan Voiculescu in his work on free analysis questions and "fully matricial function theory", by Dmitry Kalyuzhnyi-Verbovetzkyi and Victor Vinnikov in their work, by Bill Helton, Igor Klepp, Scott McCullough and Nick Slinglend on noncommutative ball maps and dimension free inequalities, and by others.

The second solution comes from our work (and the work of others) on generalizations of the Nevanlinna-Pick interpolation theorem.

I want to describe here these solutions in the context of the work that Baruch and I have been doing.

Two Solutions

The first solution is suggested by work of Joe Taylor in *A general framework for a multi-operator functional calculus*, Advances in Math. **9** (1972), 183–252.

In more recent times, Taylor's ideas have been pursued by Dan Voiculescu in his work on free analysis questions and "fully matricial function theory", by Dmitry Kalyuzhnyi-Verbovetzkyi and Victor Vinnikov in their work, by Bill Helton, Igor Klepp, Scott McCullough and Nick Slinglend on noncommutative ball maps and dimension free inequalities, and by others.

The second solution comes from our work (and the work of others) on generalizations of the Nevanlinna-Pick interpolation theorem.

I want to describe here these solutions in the context of the work that Baruch and I have been doing.

Two Solutions

The first solution is suggested by work of Joe Taylor in *A general framework for a multi-operator functional calculus*, Advances in Math. **9** (1972), 183–252.
In more recent times, Taylor's ideas have been pursued by Dan Voiculescu in his work on free analysis questions and "fully matricial function theory", by Dmitry Kalyuzhnyi-Verbovetzkyi and Victor Vinnikov in their work, by Bill Helton, Igor Klepp, Scott McCullough and Nick Slinglend on noncommutative ball maps and dimension free inequalities, and by others.
The second solution comes from our work (and the work of others) on generalizations of the Nevanlinna-Pick interpolation theorem.
I want to describe here these solutions in the context of the work that Baruch and I have been doing.

The 2 dim reps of $\mathbb{C}[X]$

Clearly, each 2 dimensional representation of $\mathbb{C}[X]$ is given by sending X to some prescribed 2×2 matrix.

When this is done, we see that $\mathbb{C}[X]$ is represented as polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$. The problem is to identify which polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ come from polynomials in $\mathbb{C}[X]$.

In very concrete terms . . .

The 2 dim reps of $\mathbb{C}[X]$

Clearly, each 2 dimensional representation of $\mathbb{C}[X]$ is given by sending X to some prescribed 2×2 matrix.
When this is done, we see that $\mathbb{C}[X]$ is represented as polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$. The problem is to identify which polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ come from polynomials in $\mathbb{C}[X]$.
In very concrete terms . . .

The 2 dim reps of $\mathbb{C}[X]$

Clearly, each 2 dimensional representation of $\mathbb{C}[X]$ is given by sending X to some prescribed 2×2 matrix.
When this is done, we see that $\mathbb{C}[X]$ is represented as polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$. The problem is to identify which polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ come from polynomials in $\mathbb{C}[X]$.
In very concrete terms ...

The 2 dim reps of $\mathbb{C}[X]$

Clearly, each 2 dimensional representation of $\mathbb{C}[X]$ is given by sending X to some prescribed 2×2 matrix.
When this is done, we see that $\mathbb{C}[X]$ is represented as polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$. The problem is to identify which polynomial maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ come from polynomials in $\mathbb{C}[X]$.
In very concrete terms ...

When is a matrix of polynomials a polynomial of matricies?

Suppose $\mathbf{P}$ is a 2×2 matrix, whose entries are polynomials in 4 variables.

That is, suppose $\mathbf{P} \in M_2(\mathbb{C}[Z_{11}, Z_{12}, Z_{21}, Z_{22}])$.

When is there a polynomial p of *one* variable so that

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22}) = p\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\right)?$$

When is a matrix of polynomials a polynomial of matricies?

Suppose $\mathbf{P}$ is a 2×2 matrix, whose entries are polynomials in 4 variables.
That is, suppose $\mathbf{P} \in M_2(\mathbb{C}[Z_{11}, Z_{12}, Z_{21}, Z_{22}])$.
When is there a polynomial p of *one* variable so that

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22}) = p\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\right)?$$

When is a matrix of polynomials a polynomial of matricies?

Suppose $\mathbf{P}$ is a 2×2 matrix, whose entries are polynomials in 4 variables.
That is, suppose $\mathbf{P} \in M_2(\mathbb{C}[Z_{11}, Z_{12}, Z_{21}, Z_{22}])$.
When is there a polynomial p of *one* variable so that

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22}) = p\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\right)?$$

An Answer

Write $\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ and similarly for $\mathbf{W}$ and $\mathbf{R}$.

Then

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22}) = p\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\right)$$

if and only if

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22})\mathbf{R} = \mathbf{R}\mathbf{P}(W_{11}, W_{12}, W_{21}, W_{22})$$

whenever

$$\mathbf{Z}\mathbf{R} = \mathbf{R}\mathbf{W}.$$

An Answer

Write $\mathbf{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ and similarly for $\mathbf{W}$ and $\mathbf{R}$.

Then

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22}) = p\left(\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\right)$$

if and only if

$$\mathbf{P}(Z_{11}, Z_{12}, Z_{21}, Z_{22})\mathbf{R} = \mathbf{R}\mathbf{P}(W_{11}, W_{12}, W_{21}, W_{22})$$

whenever

$$\mathbf{ZR} = \mathbf{RW}.$$

Definitions

- M - a W^*-algebra, i.e. a C^*-algebra that is a dual space.

Focus on the case when M is finite dimensional. Even $M = \mathbb{C}$ is very interesting.

- E - a W^*-correspondence over M, i.e.

1. E is a (right) Hilbert C^*-module over M.
2. E is self-dual.
3. E has a left action of M given by a normal representation φ of M in $\mathscr{L}(E)$.

Definitions

- M - a W^*-algebra, i.e. a C^*-algebra that is a dual space.

Focus on the case when M is finite dimensional. Even $M = \mathbb{C}$ is very interesting.

- E - a W^*-correspondence over M, i.e.

1. E is a (right) Hilbert C^*-module over M.
2. E is self-dual.
3. E has a left action of M given by a normal representation φ of M in $\mathscr{L}(E)$.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $1 \leq d \leq \infty$. $(\mathbb{C}^\infty := \ell^2(\mathbb{N}))$
- $G = (G^0, G^1, r, s)$ - finite directed graph. $M := \ell^\infty(G^0)$ (so that M is simply $\mathbb{C}^n$, for some n, viewed as a W^*-algebra), $E := \ell^\infty(G^1)$ with structure given by

$$(\varphi(a)\xi b)(e) = a(r(e))\xi(e)b(s(e)), \quad a, b \in M, \ \xi \in E, \ e \in G^1$$

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \langle \xi(e), \eta(e) \rangle, \quad \xi, \eta \in E, \ v \in G^0.$$

Note: Every W^*-correspondence over a finite dimensional commutative W^*-algebra comes from a finite directed graph.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $1 \leq d \leq \infty$. $(\mathbb{C}^\infty := \ell^2(\mathbb{N}))$
- $G = (G^0, G^1, r, s)$ - finite directed graph. $M := \ell^\infty(G^0)$ (so that M is simply $\mathbb{C}^n$, for some n, viewed as a W^*-algebra), $E := \ell^\infty(G^1)$ with structure given by

$$(\varphi(a)\xi b)(e) = a(r(e))\xi(e)b(s(e))\,, \quad a, b \in M,\ \xi \in E\,,\ e \in G^1$$

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \langle \xi(e), \eta(e) \rangle\,, \quad \xi, \eta \in E\,,\ v \in G^0.$$

Note: Every W^*-correspondence over a finite dimensional commutative W^*-algebra comes from a finite directed graph.

Examples

- M - a W^*-algebra, $\Phi : M \to M$ - a normal, unital completely positive map. E - the GNS correspondence of Φ, $E = M \otimes_\Phi M$ - the completion of $M \otimes M$ in the inner product $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle := b_1^* \Phi(a_1^* a_2) b_2$. Obvious left and right actions of M.
- If Φ is a (not-necessarily-unital) normal endomorphism of M, then $M \otimes_\Phi M$ is naturally isomorphic to ${}_\Phi M$ - the identity Hilbert M-module with left action determined by Φ.

Examples

- M - a W^*-algebra, $\Phi : M \to M$ - a normal, unital completely positive map. E - the GNS correspondence of Φ, $E = M \otimes_\Phi M$ - the completion of $M \otimes M$ in the inner product $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle := b_1^* \Phi(a_1^* a_2) b_2$. Obvious left and right actions of M.
- If Φ is a (not-necessarily-unital) normal endomorphism of M, then $M \otimes_\Phi M$ is naturally isomorphic to ${}_\Phi M$ - the identity Hilbert M-module with left action determined by Φ.

More Definitions

- Every Hilbert C^*- module over M has a unique self-dual completion.
- $E^{\otimes n}$:=self-dual completion of the balanced C^*- tensor power of E.
- The Fock space of E, $\mathscr{F}(E) := \sum_{n \geq 0} E^{\otimes n}$ - the self-dual completion of the C^*-Hilbert module direct sum.

More Definitions

- Every Hilbert C^*- module over M has a unique self-dual completion.
- $E^{\otimes n}$:=self-dual completion of the balanced C^*- tensor power of E.
- The Fock space of E, $\mathscr{F}(E) := \sum_{n\geq 0} E^{\otimes n}$ - the self-dual completion of the C^*-Hilbert module direct sum.

More Definitions

- Every Hilbert C^*- module over M has a unique self-dual completion.
- $E^{\otimes n}$:=self-dual completion of the balanced C^*- tensor power of E.
- The Fock space of E, $\mathscr{F}(E) := \sum_{n \geq 0} E^{\otimes n}$ - the self-dual completion of the C^*-Hilbert module direct sum.

More Definitions

- $\mathscr{F}(E)$ is a W^*-correspondence over M, with left action denoted φ_∞. Thus $\varphi_\infty : M \to \mathscr{L}(\mathscr{F}(E))$ is given by

$$\varphi_\infty(a) = \begin{bmatrix} a & & & & \\ & \varphi(a) & & & \\ & & \varphi_2(a) & & \\ & & & \varphi_3(a) & \\ & & & & \ddots \end{bmatrix}$$

$$\varphi_n(a)(\xi_1 \otimes \xi_2 \otimes \cdots) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots .$$

More Definitions

- For $\xi \in E$,

$$T_\xi := \begin{bmatrix} 0 & & & & \\ T_\xi^{(1)} & 0 & & & \\ & T_\xi^{(2)} & 0 & & \\ & & T_\xi^{(3)} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

$T_\xi^{(k)} : E^{\otimes k} \to E^{\otimes (k+1)}$ by
$T_\xi^{(k)}(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k.$

- $T_\xi \in \mathscr{L}(\mathscr{F}(E))$ and is called the (left) *creation operator* detemined by ξ.

More Definitions

- For $\xi \in E$,

$$T_\xi := \begin{bmatrix} 0 & & & & \\ T_\xi^{(1)} & 0 & & & \\ & T_\xi^{(2)} & 0 & & \\ & & T_\xi^{(3)} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

$T_\xi^{(k)} : E^{\otimes k} \to E^{\otimes (k+1)}$ by
$T_\xi^{(k)}(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_k$.

- $T_\xi \in \mathscr{L}(\mathscr{F}(E))$ and is called the (left) *creation operator* detemined by ξ.

The Tensor Algebra and the Hardy Algebra

Definition

The norm-closed subalgebra of $\mathscr{L}(\mathscr{F}(E))$ generated by $\varphi_\infty(M)$ and $\{T_\xi \mid \xi \in E\}$ is called the *tensor algebra* of E and is denoted $\mathscr{T}_+(E)$. The C^*-algebra generated by $\mathscr{T}_+(E)$ is called the Toeplitz algebra of E and is denoted $\mathscr{T}(E)$. The ultra-weak closure of $\mathscr{T}_+(E)$ in $\mathscr{L}(\mathscr{F}(E))$ is called *the Hardy algebra* of E and is denoted $H^\infty(E)$.

The Tensor Algebra and the Hardy Algebra

Examples

- (The Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathscr{T}_+(\mathbb{C}^d)$ is Popescu's *noncommutative disc algebra* $\mathscr{A}_d$ and $H^\infty(\mathbb{C}^d)$ is his *noncommutative Hardy space* F^∞; it is also Davidson and Pitts's *noncommutative analytic Toeplitz algebra* $\mathscr{L}_d$.
- When $E = E(G)$, $G = (G^0, G^1, r, s)$, then $\mathscr{T}_+(E)$ is the norm closure of (a faithful representation of) the path algebra of G.
- If $E = {}_\Phi M$, then $\mathscr{T}_+(E)$ and $H^\infty(E)$ are called *analytic crossed products* - first considered by Kadison and Singer, and then by Arveson. Every E is Morita equivalent to a ${}_\Phi M$.

The Tensor Algebra and the Hardy Algebra

Examples

- (The Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathscr{T}_+(\mathbb{C}^d)$ is Popescu's *noncommutative disc algebra* $\mathscr{A}_d$ and $H^\infty(\mathbb{C}^d)$ is his *noncommutative Hardy space* F^∞; it is also Davidson and Pitts's *noncommutative analytic Toeplitz algebra* $\mathscr{L}_d$.
- When $E = E(G)$, $G = (G^0, G^1, r, s)$, then $\mathscr{T}_+(E)$ is the norm closure of (a faithful representation of) the path algebra of G.
- If $E = {}_\Phi M$, then $\mathscr{T}_+(E)$ and $H^\infty(E)$ are called *analytic crossed products* - first considered by Kadison and Singer, and then by Arveson. Every E is Morita equivalent to a ${}_\Phi M$.

The Tensor Algebra and the Hardy Algebra

Examples

- (The Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathscr{T}_+(\mathbb{C}^d)$ is Popescu's *noncommutative disc algebra* $\mathscr{A}_d$ and $H^\infty(\mathbb{C}^d)$ is his *noncommutative Hardy space* F^∞; it is also Davidson and Pitts's *noncommutative analytic Toeplitz algebra* $\mathscr{L}_d$.
- When $E = E(G)$, $G = (G^0, G^1, r, s)$, then $\mathscr{T}_+(E)$ is the norm closure of (a faithful representation of) the path algebra of G.
- If $E = {}_\Phi M$, then $\mathscr{T}_+(E)$ and $H^\infty(E)$ are called *analytic crossed products* - first considered by Kadison and Singer, and then by Arveson. Every E is Morita equivalent to a ${}_\Phi M$.

Representations, Bimodule Maps and Operators

Theorem

If $\rho : \mathscr{T}_+(E) \to B(H)$ is a completely contractive (c.c.) representation, then

$$\sigma(\cdot) := \rho \circ \varphi_\infty(\cdot)$$

is a C^*-representation of M, and

$$T(\xi) := \rho(T_\xi)$$

$\xi \in E$, is a completely contractive bimodule map of E on H, i.e., $T(\varphi(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$. Conversely, given such a pair (T, σ), these formulas define a completely contractive representation ρ of $\mathscr{T}_+(E)$ on H.

Representations, Bimodule Maps and Operators

Definition

The pair (T, σ) is called a *completely contractive covariant* (c^3) *representation* of (E, M) and ρ is called its *integrated form*. We write $\rho = T \times \sigma$.

Definition

(Rieffel) Given $\sigma : M \to B(H)$, there is a representation $\sigma^E : \mathscr{L}(E) \to B(E \otimes_\sigma H)$ defined by $\sigma^E(T) = T \otimes I_H$ and called the representation of $\mathscr{L}(E)$ *induced* by σ.

Representations, Bimodule Maps and Operators

Definition

The pair (T,σ) is called a *completely contractive covariant* (c^3) *representation* of (E,M) and ρ is called its *integrated form*. We write $\rho = T \times \sigma$.

Definition

(Rieffel) Given $\sigma : M \to B(H)$, there is a representation $\sigma^E : \mathscr{L}(E) \to B(E \otimes_\sigma H)$ defined by $\sigma^E(T) = T \otimes I_H$ and called the representation of $\mathscr{L}(E)$ *induced* by σ.

Representations, Bimodule Maps and Operators

Theorem

If (T, σ) *is a* c^3 *representation of* (E, M) *on* H *and if* $\widetilde{T} : E \otimes_\sigma H \to H$ *is defined by*

$$\widetilde{T}(\xi \otimes h) := T(\xi)h, \tag{1}$$

then $\|\widetilde{T}\| \leq 1$ *and*

$$\widetilde{T}(\sigma^E \circ \varphi(\cdot)) = \widetilde{T}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\widetilde{T}. \tag{2}$$

Conversely, if $\widetilde{T} : E \otimes H \to H$ *is contraction that satisfies (2), then* T*, defined by (1) is a c.c. bimodule map with respect to* σ *and* (T, σ) *is a* c^3 *representation.*

Key Observation

Once $\sigma : M \to B(H)$ is fixed, *all* the completely contractive representations ρ of $\mathscr{T}_+(E)$ on H such that $\rho \circ \varphi_\infty = \sigma$ are determined by contraction operators $\widetilde{T}$ from $E \otimes_\sigma H$ to H that $\widetilde{T}$ intertwines $\sigma^E \circ \varphi_\infty$ and σ.

Thus, all the completely contractive representations of $\mathscr{T}_+(E)$ are obtained by letting σ range over the C^*-representations of M and for each σ, letting $\widetilde{T}$ range over the contractive intertwiners.

Key Observation

Once $\sigma : M \to B(H)$ is fixed, *all* the completely contractive representations ρ of $\mathscr{T}_+(E)$ on H such that $\rho \circ \varphi_\infty = \sigma$ are determined by contraction operators $\widetilde{T}$ from $E \otimes_\sigma H$ to H that $\widetilde{T}$ intertwines $\sigma^E \circ \varphi_\infty$ and σ.
Thus, all the completely contractive representations of $\mathscr{T}_+(E)$ are obtained by letting σ range over the C^*-representations of M and for each σ, letting $\widetilde{T}$ range over the contractive intertwiners.

The σ-dual

Definition

Let $\sigma : M \to B(H)$ be a C^*-repesentation.
$E^\sigma := \{\mathfrak{z} : H \to E \otimes_\sigma H \mid \mathfrak{z}\sigma(\cdot) = \sigma \circ \varphi_\infty(\cdot)\mathfrak{z}\}$ is called the σ-*dual* of E.

Theorem

E^σ is a W^*-correspondence over $\sigma(M)'$, where the bimodule actions are given by

$$a \cdot \mathfrak{z} \cdot b := (I_E \otimes a)\mathfrak{z}b,$$

$a, b \in \sigma(M)'$, and $\mathfrak{z} \in E^\sigma$, and the $\sigma(M)'$-valued inner product is given by the formula

$$\langle \mathfrak{z}_1, \mathfrak{z}_2 \rangle := \mathfrak{z}_1^* \mathfrak{z}_2.$$

The σ-dual

Definition

Let $\sigma : M \to B(H)$ be a C^*-repesentation.
$E^\sigma := \{\mathfrak{z} : H \to E \otimes_\sigma H \mid \mathfrak{z}\sigma(\cdot) = \sigma \circ \varphi_\infty(\cdot)\mathfrak{z}\}$ is called the σ-*dual* of E.

Theorem

E^σ is a W^*-correspondence over $\sigma(M)'$, where the bimodule actions are given by

$$a \cdot \mathfrak{z} \cdot b := (I_E \otimes a)\mathfrak{z}b,$$

$a, b \in \sigma(M)'$, and $\mathfrak{z} \in E^\sigma$, and the $\sigma(M)'$-valued inner product is given by the formula

$$\langle \mathfrak{z}_1, \mathfrak{z}_2 \rangle := \mathfrak{z}_1^* \mathfrak{z}_2.$$

Tensors as Functions

Definition

Write $\mathbb{D}(E^\sigma)^* := \{\mathfrak{z}^* \mid \mathfrak{z} \in E^\sigma, \|\mathfrak{z}\| < 1\}$. For $F \in \mathscr{T}_+(E)$, we define $\widehat{F} : \overline{\mathbb{D}(E^\sigma)^*} \to B(H_\sigma)$ by the formula

$$\widehat{F}(\mathfrak{z}^*) := (\mathfrak{z}^* \times \sigma)(F). \tag{3}$$

Thus, for each σ, $\mathscr{T}_+(E)$ may be represented as a space of $B(H_\sigma)$-valued functions on $\overline{\mathbb{D}(E^\sigma)^*}$.

Tensors as Functions

Problem

What sort of structure does $\mathbb{D}(E^\sigma)^*$ have, and what sort of function is an $\widehat{F}$?

$\mathbb{D}(E^\sigma)^*$ has a lot more structure than meets the eye.

How to describe the range of the homomorphism $F \to \widehat{F}$? Clearly, $\widehat{F}$ is continuous with respect to the operator norms on $\overline{\mathbb{D}(E^\sigma)^*}$ and $B(H_\sigma)$ and, with a little work, one can show that on $\mathbb{D}(E^\sigma)^*$, $\widehat{F}$ is holomorphic as a Banach space-valued function. However, these functions have much more structure.

Tensors as Functions

Problem

What sort of structure does $\mathbb{D}(E^\sigma)^*$ have, and what sort of function is an $\widehat{F}$?

$\mathbb{D}(E^\sigma)^*$ has a lot more structure than meets the eye.

How to describe the range of the homomorphism $F \to \widehat{F}$? Clearly, $\widehat{F}$ is continuous with respect to the operator norms on $\overline{\mathbb{D}(E^\sigma)^*}$ and $B(H_\sigma)$ and, with a little work, one can show that on $\mathbb{D}(E^\sigma)^*$, $\widehat{F}$ is holomorphic as a Banach space-valued function. However, these functions have much more structure.

Tensors as Functions

Problem

What sort of structure does $\mathbb{D}(E^{\sigma})^*$ have, and what sort of function is an $\widehat{F}$?
$\mathbb{D}(E^{\sigma})^*$ has a lot more structure than meets the eye.
How to describe the range of the homomorphism $F \to \widehat{F}$? Clearly, $\widehat{F}$ is continuous with respect to the operator norms on $\overline{\mathbb{D}(E^{\sigma})^*}$ and $B(H_{\sigma})$ and, with a little work, one can show that on $\mathbb{D}(E^{\sigma})^*$, $\widehat{F}$ is holomorphic as a Banach space-valued function. However, these functions have much more structure.

Tensors as Functions

Example

$M = \mathbb{C}$, $E = \mathbb{C}^d$, and assume σ represents $\mathbb{C}$ on $\mathbb{C}^n$. Then $\mathbb{D}(E^\sigma)^* = \{(Z_1, Z_2, \cdots, Z_d) \in M_n(\mathbb{C})^d \mid \|\sum_{i=1}^d Z_i Z_i^*\| < 1\}$, i.e., $\mathbb{D}(E^\sigma)^*$ is the set of all strict row contractions of length d consisting of $n \times n$ matrices. If $F \in \mathscr{T}_+(\mathbb{C}^d)$, then $F = \sum_{w \in \mathbb{F}_d^+} a_w S_w$, where $w \in \mathbb{F}_d^+$, and S_w is the corresponding word in the creation operators, S_i, $S_i\xi := e_i \otimes \xi$, $\xi \in \mathscr{F}(\mathbb{C}^d)$, - $\{e_i\}_{i=1}^d$ an o.n. basis for $\mathbb{C}^d$. The function $\widehat{F}$ on $\mathbb{D}(E^\sigma)^*$ defined using equation (3) is given by the formula $\widehat{F}(Z_1, Z_2, \cdots, Z_d) = \sum_{w \in \mathbb{F}^+} a_w Z_w$, where the Z_w are the appropriate words in the Z_i and the series converges uniformly and absolutely on compact subsets of $\mathbb{D}(E^\sigma)^*$; i.e., $\widehat{F}$ lies in a certain completion of the *algebra of d generic $n \times n$ matrices*.

Fully Matricial Function Theory

Definition

(J. Taylor, Voiculescu) Let G be a Banach space and for each n let $\mathfrak{M}_n(G)$ denote the $n \times n$ matrices over G. A *fully matricial G-set* is a sequence $\{\Omega_n\}_{n \geq 1}$ such that

1. Ω_n is a subset of $\mathfrak{M}_n(G)$ for each n.
2. $\Omega_{n+m} \cap (\mathfrak{M}_m(G) \oplus \mathfrak{M}_n(G)) = \Omega_m \oplus \Omega_n$, $m, n \geq 1$.
3. If $X \in \Omega_m$, $Y \in \Omega_n$ and if $S \in GL(m+n, \mathbb{C})$ is such that $Ad(S)(X \oplus Y) \in \Omega_{m+n}$, then there is an $S' \in GL(m, \mathbb{C})$ and an $S'' \in GL(n, \mathbb{C})$ so that $Ad(S')(X) \in \Omega_m$ and $Ad(S'')(Y) \in \Omega_n$.

Fully Matricial Function Theory

Definition

If H is another Banach space, then a sequence $\mathbf{R} = \{R_n\}_{n\geq 1}$ of functions, with R_n defined on Ω_n , is called a *fully matricial H-valued function defined on* $\{\Omega_n\}_{n\geq 1}$ in case

1. R_n maps Ω_n into $\mathfrak{M}_n(H)$;
2. If $X \oplus Y \in \Omega_m \oplus \Omega_n$, then $R_{m+n}(X \oplus Y) = R_m(X) \oplus R_n(Y)$; and
3. if $X \in \Omega_n$ and if $S \in GL(n, \mathbb{C})$ is such that $Ad(S) \otimes I_G(X)$ lies in Ω_n, then $R_n(Ad(S) \otimes I_G(X)) = Ad(S) \otimes I_H(R_n(X))$.

Fully Matricial Function Theory

Example

Fix a representation $\sigma : M \to B(H_\sigma)$ and let $n\sigma$ be the n-fold multiple of σ acting on $H_\sigma \oplus H_\sigma \oplus \cdots \oplus H_\sigma$. Then, in a natural way, $E^{n\sigma}$ may be identified with $\mathfrak{M}_n(E^\sigma)$ and when this is done, $\{\mathbb{D}(E^{n\sigma})^*\}_{n\geq 1}$ becomes a fully matricial $E^{\sigma *}$-set. Further, if $F \in \mathscr{T}_+(E)$, and if $\widehat{F}_n$ is defined on $\mathbb{D}(E^{n\sigma})^*$ as before, then $\{\widehat{F}_n\}_{n\geq 1}$ is a fully matricial $B(H_\sigma)$-valued function on $\{\mathbb{D}(E^{n\sigma})^*\}_{n\geq 1}$.

Observation

In our setting, we should take into consideration the relations among all the (normal) representations of M.

Fully Matricial Function Theory

Example

Fix a representation $\sigma : M \to B(H_\sigma)$ and let $n\sigma$ be the n-fold multiple of σ acting on $H_\sigma \oplus H_\sigma \oplus \cdots \oplus H_\sigma$. Then, in a natural way, $E^{n\sigma}$ may be identified with $\mathfrak{M}_n(E^\sigma)$ and when this is done, $\{\mathbb{D}(E^{n\sigma})^*\}_{n\geq 1}$ becomes a fully matricial $E^{\sigma*}$-set. Further, if $F \in \mathscr{T}_+(E)$, and if $\widehat{F}_n$ is defined on $\mathbb{D}(E^{n\sigma})^*$ as before, then $\{\widehat{F}_n\}_{n\geq 1}$ is a fully matricial $B(H_\sigma)$-valued function on $\{\mathbb{D}(E^{n\sigma})^*\}_{n\geq 1}$.

Observation

In our setting, we should take into consideration the relations among all the (normal) representations of M.

Fully Matricial E-sets

Notation

We write Σ for the set of all normal representations σ of M on a Hilbert space H_σ.

Remarks

- *Really only want to consider separable Hilbert spaces, so may assume $H_\sigma \subseteq \ell^2(\mathbb{N})$, but we don't always want to pass to unitary equivalence classes of σ's. So keep the H_σ dependent on σ.*
- *It is frequently convenient to let Σ be the faithful representations.*

Fully Matricial E-sets

Notation

We write Σ for the set of all normal representations σ of M on a Hilbert space H_σ.

Remarks

- *Really only want to consider separable Hilbert spaces, so may assume $H_\sigma \subseteq \ell^2(\mathbb{N})$, but we don't always want to pass to unitary equivalence classes of σ's. So keep the H_σ dependent on σ.*
- *It is frequently convenient to let Σ be the faithful representations.*

Fully Matricial E-sets

Notation

We write Σ for the set of all normal representations σ of M on a Hilbert space H_σ.

Remarks

- *Really only want to consider separable Hilbert spaces, so may assume $H_\sigma \subseteq \ell^2(\mathbb{N})$, but we don't always want to pass to unitary equivalence classes of σ's. So keep the H_σ dependent on σ.*

- *It is frequently convenient to let Σ be the faithful representations.*

Fully Matricial E-sets

Definition

Let $\mathbf{G} = \{G(\sigma)\}_{\sigma\in\Sigma}$ *be a family of groups, with* $G(\sigma) \subseteq GL(\sigma(M)')$, *and let* $\mathbf{S} = \{S(\sigma)\}_{\sigma\in\Sigma}$ *be a family of sets, with* $S(\sigma) \subseteq E^{\sigma}$. *The* $\mathbf{S}$ *is called a* $\mathbf{G}$*-invariant, fully matricial* E*-set in case:*

1. $S(\sigma) \oplus S(\tau) \subseteq S(\sigma \oplus \tau)$, $\sigma, \tau \in \Sigma$.
2. *For each* $\mathfrak{g} \in G(\sigma)$ *and for each* $\mathfrak{s} \in S(\sigma)$, $\mathfrak{g} \cdot \mathfrak{s} \cdot \mathfrak{g}^{-1} \in S(\sigma)$.

Fully Matricial E-sets

Remarks

- $\mathfrak{g} \cdot \mathfrak{s} \cdot \mathfrak{g}^{-1}$ *is the bimodule product:* $(I_E \otimes \mathfrak{g})\mathfrak{s}(\mathfrak{g}^{-1})$.
- *Most of the time,* $G(\sigma)$ *will be the unitary subgroup of* $GL(\sigma(M)')$, $\mathscr{U}(\sigma(M)')$, *or* $GL(\sigma(M)')$, *itself.*

Fully Matricial E-sets

Remarks

- $\mathfrak{g} \cdot \mathfrak{s} \cdot \mathfrak{g}^{-1}$ *is the bimodule product:* $(I_E \otimes \mathfrak{g})\mathfrak{s}(\mathfrak{g}^{-1})$.
- *Most of the time,* $G(\sigma)$ *will be the unitary subgroup of* $GL(\sigma(M)')$, $\mathscr{U}(\sigma(M)')$, *or* $GL(\sigma(M)')$, *itself.*

Fully Matricial E-sets

Examples

- If $\mathscr{U} = \{\mathscr{U}(\sigma(M)')\}_{\sigma\in\Sigma}$, then $\{\mathbb{D}(E^\sigma)\}_{\sigma\in\Sigma}$ and $\{\overline{\mathbb{D}(E^\sigma)}\}_{\sigma\in\Sigma}$ are $\mathscr{U}$-invariant fully matricial E-sets.
- Let $\mathbf{G} = \{GL(\sigma(M)')\}_{\sigma\in\Sigma}$, and let $C(\sigma) = \bigcup_{\mathfrak{g}\in GL(\sigma(M)')} \mathfrak{g}\cdot\overline{\mathbb{D}(E^\sigma)}\cdot\mathfrak{g}^{-1}$. Then $\mathbf{C} := \{C(\sigma)\}_{\sigma\in\Sigma}$ is the $\mathbf{G}$-invariant fully matricial E-set which parametrizes all the *completely bounded* representations of $\mathscr{T}_+(E)$, ρ, with the property that $\rho\circ\varphi_\infty = \sigma$, $\sigma\in\Sigma$.

Fully Matricial E-sets

Examples

- If $\mathscr{U} = \{\mathscr{U}(\sigma(M)')\}_{\sigma\in\Sigma}$, then $\{\mathbb{D}(E^\sigma)\}_{\sigma\in\Sigma}$ and $\{\overline{\mathbb{D}(E^\sigma)}\}_{\sigma\in\Sigma}$ are $\mathscr{U}$-invariant fully matricial E-sets.
- Let $\mathbf{G} = \{GL(\sigma(M)')\}_{\sigma\in\Sigma}$, and let $C(\sigma) = \bigcup_{\mathfrak{g}\in GL(\sigma(M)')} \mathfrak{g}\cdot\overline{\mathbb{D}(E^\sigma)}\cdot\mathfrak{g}^{-1}$. Then $\mathbf{C} := \{C(\sigma)\}_{\sigma\in\Sigma}$ is the $\mathbf{G}$-invariant fully matricial E-set which parametrizes all the *completely bounded* representations of $\mathscr{T}_+(E)$, ρ, with the property that $\rho\circ\varphi_\infty = \sigma$, $\sigma\in\Sigma$.

Fully Matricial E-sets

Examples

- For $\sigma \in \Sigma$, let $\mathscr{AC}(\sigma) := \{\mathfrak{z} \in \overline{\mathbb{D}(E^\sigma)} \mid \mathfrak{z}^* \times \sigma$ is ultra-weakly continuous$\}$ and let $\mathscr{AC}(E) := \bigcup_{\sigma\in\Sigma} \mathscr{AC}(\sigma)$. Then $\mathscr{AC}(E)$ is a $\mathscr{U}$-invariant fully matricial E-set.

When $\sigma(M)'$ is finite dimensional, $\mathscr{AC}(\sigma) = \{\mathfrak{z} \in \overline{\mathbb{D}(E^\sigma)} \mid \mathfrak{z}^*$ is completely non-coisometric$\}$. In general, $\mathscr{AC}(\sigma)$ is a (mysterious) subset of $\overline{\mathbb{D}(E^\sigma)}$ that contains $\mathbb{D}(E^\sigma)$.

Fully Matricial E-sets

Examples

- For $\sigma \in \Sigma$, let $\mathscr{AC}(\sigma) := \{\mathfrak{z} \in \overline{\mathbb{D}(E^\sigma)} \mid \mathfrak{z}^* \times \sigma \text{ is ultra-weakly continuous}\}$ and let $\mathscr{AC}(E) := \bigcup_{\sigma \in \Sigma} \mathscr{AC}(\sigma)$. Then $\mathscr{AC}(E)$ is a $\mathscr{U}$-invariant fully matricial E-set.

When $\sigma(M)'$ is finite dimensional, $\mathscr{AC}(\sigma) = \{\mathfrak{z} \in \overline{\mathbb{D}(E^\sigma)} \mid \mathfrak{z}^* \text{ is completely non-coisometric}\}$.
In general, $\mathscr{AC}(\sigma)$ is a (mysterious) subset of $\overline{\mathbb{D}(E^\sigma)}$ that contains $\mathbb{D}(E^\sigma)$.

Fully Matricial E-sets

Examples

- For $\sigma \in \Sigma$, let $\mathscr{AC}(\sigma) := \{\mathfrak{z} \in \overline{\mathbb{D}(E^\sigma)} \mid \mathfrak{z}^* \times \sigma \text{ is ultra-weakly continuous}\}$ and let $\mathscr{AC}(E) := \bigcup_{\sigma \in \Sigma} \mathscr{AC}(\sigma)$. Then $\mathscr{AC}(E)$ is a $\mathscr{U}$-invariant fully matricial E-set.

When $\sigma(M)'$ is finite dimensional, $\mathscr{AC}(\sigma) = \{\mathfrak{z} \in \overline{\mathbb{D}(E^\sigma)} \mid \mathfrak{z}^* \text{ is completely non-coisometric}\}$. In general, $\mathscr{AC}(\sigma)$ is a (mysterious) subset of $\overline{\mathbb{D}(E^\sigma)}$ that contains $\mathbb{D}(E^\sigma)$.

Intertwiners

Definition

Let $\sigma, \tau \in \Sigma$, let $\mathfrak{y} \in E^{\sigma}$, and let $\mathfrak{z} \in E^{\tau}$. We say a map $C : H_\sigma \to H_\tau$ intertwines $\mathfrak{y}^*$ and $\mathfrak{z}^*$, and write $C \in \mathscr{I}(\mathfrak{y}^*, \mathfrak{z}^*)$, in case

1. $C\sigma(\cdot) = \tau(\cdot)C$, *and*
2. $C\mathfrak{y}^* = \mathfrak{z}^*(I_E \otimes C)$ *as maps* $E \otimes_\sigma H_\sigma$ *to* H_τ.

Observation

If $\|\mathfrak{y}\|, \|\mathfrak{z}\| \leq 1$, then C intertwines $\mathfrak{y}^*$ and $\mathfrak{z}^*$ if and only if C intertwines the representations of $\mathscr{T}_+(E)$, $\mathfrak{y}^* \times \sigma$ and $\mathfrak{z}^* \times \tau$, i.e., $\mathscr{I}(\mathfrak{y}^*, \mathfrak{z}^*) = Hom_{\mathscr{T}_+(E)}(H_\sigma, H_\tau)$.

Intertwiners

Definition

Let $\sigma, \tau \in \Sigma$, let $\mathfrak{y} \in E^\sigma$, and let $\mathfrak{z} \in E^\tau$. We say a map $C : H_\sigma \to H_\tau$ intertwines $\mathfrak{y}^*$ and $\mathfrak{z}^*$, and write $C \in \mathscr{I}(\mathfrak{y}^*, \mathfrak{z}^*)$, in case

1. $C\sigma(\cdot) = \tau(\cdot)C$, *and*
2. $C\mathfrak{y}^* = \mathfrak{z}^*(I_E \otimes C)$ *as maps* $E \otimes_\sigma H_\sigma$ *to* H_τ.

Observation

If $\|\mathfrak{y}\|, \|\mathfrak{z}\| \leq 1$, then C intertwines $\mathfrak{y}^*$ and $\mathfrak{z}^*$ if and only if C intertwines the representations of $\mathscr{T}_+(E)$, $\mathfrak{y}^* \times \sigma$ and $\mathfrak{z}^* \times \tau$, i.e., $\mathscr{I}(\mathfrak{y}^*, \mathfrak{z}^*) = Hom_{\mathscr{T}_+(E)}(H_\sigma, H_\tau)$.

Intertwiners

Definition

Let $\mathbf{S} = \{S(\sigma)\}_{\sigma\in\Sigma}$ be a fully matricial E-set. (The group is irrelevant.) A family $\mathbf{f} = \{f_\sigma\}_{\sigma\in\Sigma}$ of functions, with $f_\sigma : S(\sigma) \to B(H_\sigma)$, is said to *preserve intertwiners* $\mathscr{I}(\mathfrak{y}^*,\mathfrak{z}^*) \subseteq \mathscr{I}(f_\sigma(\mathfrak{y}^*), f_\tau(\mathfrak{z}^*))$ for all $\mathfrak{y} \in S(\sigma)$, $\mathfrak{z} \in S(\tau)$, $\sigma,\tau \in \Sigma$, i.e., if $C \in \mathscr{I}(\mathfrak{y}^*,\mathfrak{z}^*)$, then $Cf_\sigma(\mathfrak{y}^*) = f_\tau(\mathfrak{z}^*)C$.

Intertwiners

Definition

Let E and F be W^*-correspondences over M, let $\mathbf{S}$ be a fully matricial E-set and let $\mathbf{S}'$ be a fully matricial F-set. A family $\mathbf{f} = \{f_\sigma\}_{\sigma\in\Sigma}$ of maps, with $f_\sigma : S(\sigma) \to S^{'}(\sigma)$, is said to *preserve intertwiners* in case $\mathscr{I}(\mathfrak{y}^*, \mathfrak{z}^*) \subseteq \mathscr{I}(f_\sigma(\mathfrak{y}^*), f_\tau(\mathfrak{z}^*))$ for all $\mathfrak{y} \in S(\sigma)$, $\mathfrak{z} \in S(\tau)$, $\sigma, \tau \in \Sigma$.

Intertwining vs. Holomorphic, I.

A non-linear double commutant theorem

Theorem

(After K-V & V) Let $\mathfrak{f} = \{f_\sigma\}_{\sigma\in\Sigma}$ *be a family with* $f_\sigma : \mathscr{AC}(\sigma) \to B(H_\sigma)$, $\sigma \in \Sigma$. *Then there is an element* $F \in H^\infty(E)$ *such that*

$$f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$$

for all $\sigma \in \Sigma$ *and all* $\mathfrak{z} \in \mathscr{AC}(\sigma)$ *if and only if* $\mathfrak{f}$ *preserves intertwiners.*

Remark

There is an analogous theorem for intertwiners and free holomorphic functions on $\{\mathbb{D}(E^\sigma)\}_{\sigma\in\Sigma}$, *i.e., functions that have a "tensor power series" expansion.*

Intertwining vs. Holomorphic, I.

A non-linear double commutant theorem

Theorem

(After K-V & V) Let $\mathbf{f} = \{f_\sigma\}_{\sigma \in \Sigma}$ *be a family with* $f_\sigma : \mathscr{A}\mathscr{C}(\sigma) \to B(H_\sigma)$, $\sigma \in \Sigma$. *Then there is an element* $F \in H^\infty(E)$ *such that*

$$f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$$

for all $\sigma \in \Sigma$ *and all* $\mathfrak{z} \in \mathscr{A}\mathscr{C}(\sigma)$ *if and only if* $\mathbf{f}$ *preserves intertwiners.*

Remark

There is an analogous theorem for intertwiners and free holomorphic functions on $\{\mathbb{D}(E^\sigma)\}_{\sigma \in \Sigma}$, *i.e., functions that have a "tensor power series" expansion.*

Intertwining vs. Holomorphic, II.

Theorem

(After H,K & McC) Suppose E and F are W^*-correspondences over M and suppose $\mathbf{f} = \{f_\sigma\}_{\sigma\in\Sigma}$ is a family of maps from $\mathscr{AC}(E)$ to $\mathscr{AC}(F)$. Then $\mathbf{f}$ preserves intertwiners if and only if there is an ultra-weakly continuous homomorphism $\alpha : H^\infty(F) \to H^\infty(E)$ such that for all $F \in H^\infty(F)$ and all $\mathfrak{z} \in \mathscr{AC}(E)(\sigma)$,

$$\widehat{\alpha(F)}(\mathfrak{z}^*) = \widehat{F}(f_\sigma(\mathfrak{z})^*),$$

i.e. iff the f_σ are ball maps that "preserve absolute continuity".

Kernels and r.k.h.m.'s

Definition

Let Ω be a set and let A and B be C^*-algebras. A kernel K on Ω with values in the bounded linear maps from A to B, $\mathscr{B}(A, B)$, is called *completely positive definite* in case K is a function from $\Omega \times \Omega$ to $\mathscr{B}(A, B)$ such that for each finite set of points $\{\omega_1, \omega_2, \ldots, \omega_n\}$ in Ω, for each choice of n elements in A, $a_1, a_2, \ldots, a_n$, and for each choice of n elements in B, $b_1, b_2, \ldots, b_n$, the inequality

$$\sum_{i,j=1}^{n} b_i^* K(\omega_i, \omega_j)[a_i^* a_j] b_j \geq 0$$

holds in B.

Kernels and r.k.h.m.'s

Barreta, Bhat, Liebscher, and Skeide show how to build a "reproducing kernel A, B-correspondence" from a completely positive definite kernel and they prove a kind of Kolmogorov decomposition of the kernel. Based on their analysis we can prove the following generalization of the Nevanlinna-Pick interpolation theorem.

The Szegö Kernel

Definition

If σ is a normal representation of M on the Hilbert space H_σ, then the kernel $K_S : \mathbb{D}(E^\sigma)^* \times \mathbb{D}(E^\sigma)^* \to \mathscr{B}(\sigma(M)', \sigma(M)')$, defined by the formula

$$K_S(\mathfrak{z}^*, \mathfrak{w}^*) = (\iota - \theta_{\mathfrak{z},\mathfrak{w}})^{-1},$$

where ι denotes the identity map on $\sigma(M)'$ and where $\theta_{\mathfrak{z},\mathfrak{w}}(a) = \langle \mathfrak{z}, a \cdot \mathfrak{w} \rangle = \mathfrak{z}^*(I_E \otimes a)\mathfrak{w}$, $a \in \sigma(M)'$, is called the *Szegö kernel* for $\mathbb{D}(E^\sigma)^*$.

Schur Class Functions

Theorem

Let $\sigma : M \to B(H_\sigma)$ be a normal representation of M. Then the Szegö kernel for $\mathbb{D}(E^\sigma)^*$ is completely positive definite, and a function $\Phi : \mathbb{D}(E^\sigma)^* \to B(H_\sigma)$ is $\widehat{F}$ for an $F \in H^\infty(E)$, with $\|F\| \leq 1$, if and only if the kernel

$$K_\Phi(\mathfrak{z}^*, \mathfrak{w}^*) := (\iota - Ad(\Phi(\mathfrak{z}^*), \Phi(\mathfrak{w}^*)) \circ K_S(\mathfrak{z}^*, \mathfrak{w}^*)$$

is a completely positive definite kernel from $\sigma(M)'$ to $B(H_\sigma)$, where, in general, for any Hilbert space H and any $a, b \in B(H)$, $Ad(a, b) : B(H) \to B(H)$ is defined by $Ad(a, b)(X) := aXb^*$.

Induced Representations

Definition

Let π be a normal representation of M on H_π. Let $K = \mathscr{F}(E) \otimes_\pi H_\pi$. Set $\sigma = \pi^{\mathscr{F}(E)} \circ \varphi_\infty$ and let $S(\xi) := T_\xi \otimes I_{H_\pi} = \pi^{\mathscr{F}(E)}(T_\xi)$. Then (S, σ) is called the *induced (covariant) representation* determined by π.

An induced representation is a generalization of a unilateral shift.

Definition

A c^3 representation (V, σ) is called *isometric* (resp. (fully) *coisometric*) in case $\widetilde{V}$ is an isometry (resp. a coisometry).

Induced Representations

Definition

Let π be a normal representation of M on H_π. Let $K = \mathscr{F}(E) \otimes_\pi H_\pi$. Set $\sigma = \pi^{\mathscr{F}(E)} \circ \varphi_\infty$ and let $S(\xi) := T_\xi \otimes I_{H_\pi} = \pi^{\mathscr{F}(E)}(T_\xi)$. Then (S, σ) is called the *induced (covariant) representation* determined by π.

An induced representation is a generalization of a unilateral shift.

Definition

A c^3 representation (V, σ) is called *isometric* (resp. (fully) *coisometric*) in case $\widetilde{V}$ is an isometry (resp. a coisometry).

Induced Representations

Definition

Let π be a normal representation of M on H_π. Let $K = \mathscr{F}(E) \otimes_\pi H_\pi$. Set $\sigma = \pi^{\mathscr{F}(E)} \circ \varphi_\infty$ and let $S(\xi) := T_\xi \otimes I_{H_\pi} = \pi^{\mathscr{F}(E)}(T_\xi)$. Then (S, σ) is called the *induced (covariant) representation* determined by π.

An induced representation is a generalization of a unilateral shift.

Definition

A c^3 representation (V, σ) is called *isometric* (resp. (fully) *coisometric*) in case $\widetilde{V}$ is an isometry (resp. a coisometry).

Wold Decomposition

Theorem

Every isometric covariant representation (V, σ) *decomposes uniquely as* $(V, \sigma) = (V_1, \sigma_1) \oplus (V_2, \sigma_2)$ *where* (V_1, σ_1) *is induced and where* (V_2, σ_2) *is (fully) coisometric.*

(V_1, σ_1) is induced by the restriction of σ to $(\widetilde{V}(E \otimes_\sigma H_\sigma))^\perp$.
The Hilbert space of (V_2, σ_2) is

$$\bigcap_{n \geq 1} \widetilde{V}(I_E \otimes \widetilde{V})(I_{E^{\otimes 2}} \otimes \widetilde{V}) \cdots (I_{E^{\otimes n}} \otimes \widetilde{V})(E^{\otimes(n+1)} \otimes_\sigma H_\sigma).$$

Wold Decomposition

Theorem

Every isometric covariant representation (V, σ) *decomposes uniquely as* $(V, \sigma) = (V_1, \sigma_1) \oplus (V_2, \sigma_2)$ *where* (V_1, σ_1) *is induced and where* (V_2, σ_2) *is (fully) coisometric.*

(V_1, σ_1) is induced by the restriction of σ to $(\widetilde{V}(E \otimes_\sigma H_\sigma))^\perp$.

The Hilbert space of (V_2, σ_2) is

$$\bigcap_{n \geq 1} \widetilde{V}(I_E \otimes \widetilde{V})(I_{E^{\otimes 2}} \otimes \widetilde{V}) \cdots (I_{E^{\otimes n}} \otimes \widetilde{V})(E^{\otimes(n+1)} \otimes_\sigma H_\sigma).$$

Wold Decomposition

Theorem

Every isometric covariant representation (V,σ) decomposes uniquely as $(V,\sigma) = (V_1,\sigma_1) \oplus (V_2,\sigma_2)$ where (V_1,σ_1) is induced and where (V_2,σ_2) is (fully) coisometric.

(V_1,σ_1) is induced by the restriction of σ to $(\widetilde{V}(E \otimes_\sigma H_\sigma))^\perp$.
The Hilbert space of (V_2,σ_2) is

$$\bigcap_{n\geq 1} \widetilde{V}(I_E \otimes \widetilde{V})(I_{E^{\otimes 2}} \otimes \widetilde{V}) \cdots (I_{E^{\otimes n}} \otimes \widetilde{V})(E^{\otimes(n+1)} \otimes_\sigma H_\sigma).$$

Double Commutant Theorem

Theorem

If (V, σ) *is an induced covariant representation, then*

$$V \times \sigma(H^\infty(E))'' = V \times \sigma(H^\infty(E)).$$

Deus Ex Machina

Definition

Let π be a faithful normal representation of M of infinite multiplicity and let (S_0, σ_0) be the isometric covariant representation induced by π. We call (S_0, σ_0) *the universal induced representation*.

Proposition

(S_0, σ_0) is unique up to unitary equivalence and every induced representation is a direct summand of (S_0, σ_0).

Deus Ex Machina

Definition

Let π be a faithful normal representation of M of infinite multiplicity and let (S_0, σ_0) be the isometric covariant representation induced by π. We call (S_0, σ_0) *the universal induced representation*.

Proposition

(S_0, σ_0) is unique up to unitary equivalence and every induced representation is a direct summand of (S_0, σ_0).

Scattering

Theorem

Let $\sigma : M \to B(H_\sigma)$ and let $\mathfrak{z} \in \mathbb{D}(E^\sigma)$. Then $\mathfrak{z} \in \mathscr{AC}(\sigma)$ if and only if

$$H_\sigma = \bigcup \{Ran(C) \mid C \in \mathscr{I}(S_0, \mathfrak{z}^*)\}.$$

Intertwining vs. Holomorphic, I

Proof outline

Theorem

(After K-V & V) Let $\mathbf{f} = \{f_\sigma\}_{\sigma \in \Sigma}$ *be a family with* $f_\sigma : \mathscr{AC}(\sigma) \to B(H_\sigma)$, $\sigma \in \Sigma$. *Then there is an element* $F \in H^\infty(E)$ *such that*

$$f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$$

for all $\sigma \in \Sigma$ *and all* $\mathfrak{z} \in \mathscr{AC}(\sigma)$ *if and only if* $\mathbf{f}$ *preserves intertwiners.*

Lemma

If $F \in H^\infty(E)$ *and* $\mathfrak{z} \in \mathscr{AC}(\sigma)$, *then for all* $C \in \mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$,

$$\widehat{F}(\mathfrak{z}^*)C = C\widehat{F}(\widetilde{S}_0) = C\pi^{\mathscr{F}(E)}(F).$$

Intertwining vs. Holomorphic, I

Proof outline

Theorem

(After K-V & V) Let $\mathbf{f} = \{f_\sigma\}_{\sigma\in\Sigma}$ *be a family with* $f_\sigma : \mathscr{AC}(\sigma) \to B(H_\sigma)$, $\sigma \in \Sigma$. *Then there is an element* $F \in H^\infty(E)$ *such that*

$$f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$$

for all $\sigma \in \Sigma$ *and all* $\mathfrak{z} \in \mathscr{AC}(\sigma)$ *if and only if* $\mathbf{f}$ *preserves intertwiners.*

Lemma

If $F \in H^\infty(E)$ *and* $\mathfrak{z} \in \mathscr{AC}(\sigma)$, *then for all* $C \in \mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$,

$$\widehat{F}(\mathfrak{z}^*)C = C\widehat{F}(\widetilde{S}_0) = C\pi^{\mathscr{F}(E)}(F).$$

Intertwining vs. Holomorphic, I

Proof outline

Proof.

If $\mathbf{f}$ *preserves intertwiners and if* $R \in B(\mathscr{F}(E) \otimes_\pi H_\pi)$ *commutes with* $\pi^{\mathscr{F}(E)}(H^\infty(E))$, *then* R *commutes with* $f_{\sigma_0}(\widetilde{S}_0)$, *by what it means for* $\mathbf{f}$ *to preserve Intertwiners, i.e.,* $f_{\sigma_0}(\widetilde{S}_0)$ *is in the double commutant of* $\pi^{\mathscr{F}(E)}(H^\infty(E))$. *So* $f_{\sigma_0}(\widetilde{S}_0) = \widehat{F}(\widetilde{S}_0)$ *for some* $F \in H^\infty(E)$.

If $\mathfrak{z} \in \mathscr{AC}(\sigma)$, *then for every* $C \in \mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$,

$$f_\sigma(\mathfrak{z}^*)C = Cf_{\sigma_0}(\widetilde{S}_0) = C\widehat{F}(\widetilde{S}_0) = \widehat{F}(\mathfrak{z}^*)C,$$

where the last equality is by the lemma.

Since $\mathfrak{z} \in \mathscr{AC}(\sigma)$, *the ranges of the* C *in* $\mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$ *cover* H_σ. *Therefore,* $f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$. $\square$

Intertwining vs. Holomorphic, I

Proof outline

Proof.

If $\mathfrak{f}$ *preserves intertwiners and if* $R \in B(\mathscr{F}(E) \otimes_\pi H_\pi)$ *commutes with* $\pi^{\mathscr{F}(E)}(H^\infty(E))$*, then* R *commutes with* $f_{\sigma_0}(\widetilde{S}_0)$*, by what it means for* $\mathfrak{f}$ *to preserve Intertwiners, i.e.,* $f_{\sigma_0}(\widetilde{S}_0)$ *is in the double commutant of* $\pi^{\mathscr{F}(E)}(H^\infty(E))$*. So* $f_{\sigma_0}(\widetilde{S}_0) = \widehat{F}(\widetilde{S}_0)$ *for some* $F \in H^\infty(E)$*.*

If $\mathfrak{z} \in \mathscr{AC}(\sigma)$*, then for every* $C \in \mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$*,*

$$f_\sigma(\mathfrak{z}^*)C = Cf_{\sigma_0}(\widetilde{S}_0) = C\widehat{F}(\widetilde{S}_0) = \widehat{F}(\mathfrak{z}^*)C,$$

where the last equality is by the lemma.

Since $\mathfrak{z} \in \mathscr{AC}(\sigma)$*, the ranges of the* C *in* $\mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$ *cover* H_σ*. Therefore,* $f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$*.* $\square$

Intertwining vs. Holomorphic, I

Proof outline

Proof.

If $\mathfrak{f}$ *preserves intertwiners and if* $R \in B(\mathscr{F}(E) \otimes_\pi H_\pi)$ *commutes with* $\pi^{\mathscr{F}(E)}(H^\infty(E))$*, then* R *commutes with* $f_{\sigma_0}(\widetilde{S}_0)$*, by what it means for* $\mathfrak{f}$ *to preserve Intertwiners, i.e.,* $f_{\sigma_0}(\widetilde{S}_0)$ *is in the double commutant of* $\pi^{\mathscr{F}(E)}(H^\infty(E))$*. So* $f_{\sigma_0}(\widetilde{S}_0) = \widehat{F}(\widetilde{S}_0)$ *for some* $F \in H^\infty(E)$*.*

If $\mathfrak{z} \in \mathscr{AC}(\sigma)$*, then for every* $C \in \mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$*,*

$$f_\sigma(\mathfrak{z}^*)C = Cf_{\sigma_0}(\widetilde{S}_0) = C\widehat{F}(\widetilde{S}_0) = \widehat{F}(\mathfrak{z}^*)C,$$

where the last equality is by the lemma.
Since $\mathfrak{z} \in \mathscr{AC}(\sigma)$*, the ranges of the* C *in* $\mathscr{I}(\widetilde{S}_0, \mathfrak{z}^*)$ *cover* H_σ*.*
Therefore, $f_\sigma(\mathfrak{z}^*) = \widehat{F}(\mathfrak{z}^*)$*.* □

Another Tool

Extended Nevanlinna-Pick

Observation

If Φ, Ψ are positive maps such that $id - \Phi$ is invertible then the map $(id - \Psi) \circ (id - \Phi)^{-1}$ is positive if and only if :

$$\{a \geq 0 : \Phi(a) \leq a\} \subseteq \{a \geq 0 : \Psi(a) \leq a\}.$$

The last statement makes sense even if $id - \Phi$ is not invertible. It is related to the Lyapunov preorder studied in matrix theory. This was pointed out to us by Nir Cohen.

Extended Nevanlinna-Pick

Theorem

Let σ be a faithful normal representation of M on H. If $\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_k \in AC(\sigma)$ are distinct and if $D_1, D_2, \ldots, D_k \in B(H)$, then there is an $F \in H^\infty(E)$, with $\|F\| \leq 1$, such that

$$\widehat{F}(\mathfrak{z}_i^*) = D_i \ ,$$

$i = 1, 2, \ldots, k$, if and only if for every $m \geq 1$, for every $i : \{1, \ldots, m\} \to \{1, \ldots, k\}$ and for every m-tuple, $C_1, C_2, \ldots, C_m$, with $C_j \in \mathscr{I}(S_0, \mathfrak{z}_{i(j)}^*)$, the inequality,

$$(D_{i(l)} C_l C_j^* D_{i(j)}^*)_{l,j} \leq (C_l C_j^*)_{l,j},$$

is satisfied.