Quantized Function Theory

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(Joint work with Meghna Mittal)

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Outline

- **Quantized Function Theory**
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- **Rough Statement of Theorem**
- **Examples**
- **Operator Algebras of Functions, General Theory**
- **Quantized Function Theory, Redux**

Overview: Quantized Function Theory

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Given $G \subseteq \mathbb{C}^N$ open and $\mathcal{R} = \{F_k : G \to M_{m_k,n_k} : k \in I\}$ a set of analytic matrix-valued functions such that $\|F_k(z)\| < 1 \ \forall \ z \in G, k \in I$. We call this an **"analytic presentation of G",** provided certain hypotheses are met.

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The "quantized" version of G.

Set

$$\mathcal{Q}(G) = \{T : \sigma(T) \subseteq G, \|F_k(T)\| \leq 1 \ \forall \ k \in I\}$$

where $T = (T_1, T_2, ..., T_N)$ is a commuting N-tuple of operators on some Hilbert space.

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Similarly, define norms on $M_n(H^\infty_{\mathcal{R}}(G))$.
These are examples of "Abstract Operator Algebras of Functions".
We prove theorems about $H^\infty_{\mathcal{R}}(G)$ by applying some new general theorems about such algebras.

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4. $H^\infty_{\mathcal{R}}(G)$ is wk*-RFD, and consequently, the $\mathcal{R}$-norm is generally the sup over matrices in $\mathcal{Q}(G)$,
5. when I is a finite set, $P = (p_{i,j}) \in M_{m,n}(Hol(G))$, we have that $\|P\|_{\mathcal{R}} \le 1$ if and only if there exist analytic functions $H_k : G \to B(\mathbb{C}^m, \mathcal{H}_k)$, such that $I - P(z)P(w)^* = \sum_{k=1}^{K} H_k(z)[(I_m - F_k(z)F_k(w)^*) \otimes I]H_k(w)^*$.
"Agler-Ball-Bolotnikov factorization"

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Proposition

Let $\mathcal{A}$ be an operator algebra of functions on X, then $\mathcal{A} \subseteq \ell^\infty(X)$, and for every n and every $(f_{i,j}) \in M_n(\mathcal{A})$, we have $\|(f_{ij})\|_\infty \leq \|(f_{ij})\|_{M_n(\mathcal{A})}$ and $\|\pi_x\|_{cb} = 1$

Note: Given a finite subset $Y \subseteq X$, $I_Y = \{f \in \mathcal{A} : f|_Y = 0\}$ - ideal $\mathcal{A}/I_Y$- quotient op. alg. ($\cong \mathbb{C}^{|Y|}$), $\pi_Y : \mathcal{A} \to \mathcal{A}/I_Y$- quotient map.

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1. **local** *if* $\forall$ n *and* $\forall$ $(f_{ij}) \in M_n(\mathcal{A})$, $\|(f_{ij})\| = \sup_Y \|(\pi_Y(f_{ij}))\|$.

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1. **local** *if* $\forall$ *n and* $\forall$ $(f_{ij}) \in M_n(\mathcal{A})$, $\|(f_{ij})\| = \sup_Y \|(\pi_Y(f_{ij}))\|$.
2. **Residually finite dimensional(RFD)** *if* $\forall$ *n and* $\forall$ $(f_{ij}) \in M_n(\mathcal{A})$, $\|(f_{ij})\| = \sup\{\|(\pi(f_{ij}))\|\}$ *where supremum is taken over all cc homo.* $\pi : \mathcal{A} \to M_m$ *and for all integers m.*

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Remark: Every finite dimensional C*-algebra is RFD, but there are finite dimensional operator algebras that are not RFD.

Proposition

Let $\mathcal{A}$ be an operator algebra of functions on X. Then $\mathcal{A}_L = \mathcal{A}$ equipped with the matrix norms, $\|(f_{i,j})\|_L = \sup_Y \|(\pi_Y(f_{i,j}))\|$ is a local operator algebra of functions on X.

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Theorem

If $\mathcal{A}$ is a local operator algebra of functions then $\mathcal{A}$ is RFD.

Definition

A function $f : X \to \mathbb{C}$ *is called a* **bounded pointwise(BPW)** *limit of* $\mathcal{A}$*, if there exists a net* $f_\lambda \in \mathcal{A}, f_\lambda \to f$ *ptw and* $\|f_\lambda\| \le C$*. We let* $\tilde{\mathcal{A}}$ *denote the set of functions that are BPW limits from* $\mathcal{A}$*.*
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If we equip $\tilde{\mathcal{A}}$ with the family of norms given by $\|(f_{ij})\| = inf\{C : \|(f_{ij}^\lambda)\|_{\mathcal{A}} \leq C,\ f_{ij}^\lambda \to f_{ij} \text{ ptw } \}$, then $\tilde{\mathcal{A}}$ is a BPW complete local operator algebra of functions and $\mathcal{A}_L \hookrightarrow \tilde{\mathcal{A}}$ completely isometrically.

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We call $\tilde{\mathcal{A}}$ the **BPW completion** *of $\mathcal{A}$.*

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Given a set X and a Hilbert space $\mathcal{H}$, then we call a vector space $\mathcal{L}$ of $\mathcal{H}$-valued functions, an $\mathcal{H}$**-valued RKHS** *if it is equipped with an inner product that makes it a Hilbert space and it has the property that for every $x \in X$, the evaluation map $E_x : \mathcal{L} \to \mathcal{H}$, is a bounded, linear map.*

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2. *$\exists$ $\mathcal{H}$-valued RKHS, $\mathcal{L}$ such that $\mathcal{A} = \mathcal{M}(\mathcal{L})$ complete isometric, wk*-isomorphism.*

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then we call $\mathcal{R} = \{F_k : k \in I\}$ an **analytic presentation of G** *and we call $\mathcal{A}$ the* **algebra of the presentation.**

Definition

Given an analytic presentation of G and $\pi : \mathcal{A} \to B(\mathcal{H})$ a homomorphism of the algebra of the presentation, then we call π an **admissible representation** *provided that $\|(\pi(f_{k,i,j}))\| \leq 1$ for all $k \in I$ and an* **admissible strict representation** *when these inequalities are all strictly less than 1.*

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Let $(f_{ij}) \in M_n(\mathcal{A})$, set $\|(f_{ij})\|_u = \sup\{\|(\pi(f_{i,j}))\| : \pi \text{ admissible }\}$ and set $\|(f_{i,j})\|_{u_0} = \sup\{\|(\pi(f_{i,j}))\| : \pi \text{ strictly admissible }\}$.

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$$I - P(z)P(w)^* = \sum_{k=1}^{K} L_k(z)[(I - F_k(z)F_k(w)^*) \otimes I_{\mathcal{H}_k}]L_k(w)^*.$$

What we don't know

Part 1, above tells us that for $f \in \mathcal{A}$, we have that $\|f\|_{\mathcal{R}}$ is the local norm derived from $\|f\|_u$, but we do not know in general if $\|f\|_u = \|f\|_{u_0}$, if $\|f\|_{\mathcal{R}} = \|f\|_u$ or if $\|f\|_{\mathcal{R}} = \|f\|_{u_0}$.

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But we can't prove, in general, that $\|f\|_{\mathcal{R}} = \sup \|f(T)\|$ over N-tuples $T = (T_1, ..., T_N)$ of commuting matrices in $\mathcal{Q}(G)$.

What we don't know

Part 1, above tells us that for $f \in \mathcal{A}$, we have that $\|f\|_{\mathcal{R}}$ is the local norm derived from $\|f\|_u$, but we do not know in general if $\|f\|_u = \|f\|_{u_0}$, if $\|f\|_{\mathcal{R}} = \|f\|_u$ or if $\|f\|_{\mathcal{R}} = \|f\|_{u_0}$.
When the domain has a Fejer-like kernel, we can prove equality of these norms.
Part 3, tells us that for $f \in H^\infty_{\mathcal{R}}(G)$, we have $\|f\|_{\mathcal{R}} = \sup\{\|\pi(f)\|\}$, where the supremum is over all m and all $\pi : H^\infty_{\mathcal{R}}(G) \to M_m$ weak*-continuous.
But we can't prove, in general, that $\|f\|_{\mathcal{R}} = \sup \|f(T)\|$ over N-tuples $T = (T_1, ..., T_N)$ of commuting matrices in $\mathcal{Q}(G)$.
We don't have useful characterizations of the pre-duals of $H^\infty_{\mathcal{R}}(G)$.

Proof of the factorization result

Our proof of the factorization result(part 4) relies on first proving a factorization result for $\mathcal{A}$ via abstract operator algebra methods. We wish to mention that result, since it is another place where the abstract theory of operator algebras plays a key role.

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Definition

Let F_0 denote the constant function. Then a block diagonal matrix-valued function of the form $D(z) = diag(F_{k_1}, ..., F_{k_m})$ where $k_i \in I$ or $k_i = 0$, for $1 \leq i \leq m$ is called **admissible block diagonal matrix over G.**

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Theorem

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(iii) *(Agler-type factorization) there exists a positive, invertible matrix $R \in M_m$ and matrices $P_k \in M_{m,r_k}(\mathcal{A}), 0 \le k \le K$, such that $I_m - P(z)P(w)^* =$*
$R + P_0(z)P_0(w)^* + \sum_{k=1}^{K} P_k(z)(I - F_k(z)F_k(w)^*)^{(q_k)} P_k(w)^*$
where $r_k = q_k m_k$ and $z = (z_1, \dots, z_N),\ w = (w_1, \dots, w_N) \in G$.

Applications of Theorem

1. **Agler:** $G = \mathbb{D}^N, K = N$. For $1 \leq k \leq N$, define $F_k : G \to \mathbb{C}$ via $F_k(z) = z_k$, where $z = (z_1, \cdots, z_N)$,

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Moreover, $\mathcal{R}$-norm obtained by supping over commuting N-tuples of matrices.

2. **Drury, Arveson, et al:** $G = \mathbb{B}^N \subseteq \mathbb{C}^N$, $K = 1$. Define $F : G \to \mathbb{M}_{1,N}$ via $F(z) = (z_1, \cdots, z_n)$

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Each of these factorization theorems also covered by Ball-Bolotnikov theory, with slightly different hypotheses for $\mathcal{Q}(G)$.
NEW: i) $H^\infty_{\mathcal{R}}(G)$ is a dual operator algebra,
ii) $H^\infty_{\mathcal{R}}(G)$ is the multiplier algebra of a vector-valued RKHS on G,
iii) these norms could all be acheived by taking the suprema over commuting tuples of matrices satisfying the defining inequalities.

4. **Agler-McCarthy, Mittal:** $G = \mathbb{A}_r = \{z : r < |z| < 1\}$, $K = 2$. Define $F_1(z) = z, \ F_2(z) = rz^{-1}$

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Mittal proves that this norm is also attained over matrices, finds the extremals of the rational family $\mathcal{Q}(\mathbb{A}_r)$ and computes the C*-envelope, $C_e^*(\mathcal{A})$.

5. **Kalyuzhnyi-Verbovetzkii:** $G = \{z \in \mathbb{C}^N : Re(z_i) > 0\}$, and $F_i : G \to \mathbb{C}, F_i(z) = \frac{z_i - 1}{z_i + 1}$,

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The difference between Example 6 and Examples 1–5, is that in 1–5 the coordinate functions belong to $H^\infty_{\mathcal{R}}(G)$.

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   $\|f\|_{\mathcal{R}} \leq 1$ iff $(1 - f(z)\overline{f(w)})$ is a pointwise limit of sums of terms of the form $(1 - F_\theta(z)\overline{F_\theta(w)})K(z,w)$
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