Operator algebraic geometry: Classifying universal operator algebras with algebraic varieties

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Contents of this talk

I will define a class of commutative operator algebras, and I will describe the classification of these algebras up to **isomorphism** and up to **(completely) isometric isomorphism**.

Missing from this talk

- Why these algebras?
- Context and history, contributions by others.
- Other aspects of our recent work, especially the **noncommutative** case.
- The connection to **subproduct systems**.

The setting 1

Let $p_1, \dots, p_k$ be homogeneous polynomials in d variables. Let $\mathcal{I}$ be the (homogeneous) ideal in $\mathbb{C}[z] := \mathbb{C}[z_1, \dots, z_d]$ that they generate. We will now describe the **universal unital operator algebra** $\mathcal{A}_\mathcal{I}$ that is generated by a commuting row contraction $(T_1, \dots, T_d)$ satisfying system of equations

$$p_1(T_1, \dots, T_d) = 0,$$

$$\vdots$$

$$p_k(T_1, \dots, T_d) = 0.$$

The setting 2

Let H^2_d be Drury-Arveson space.
Reminder: H^2_d is the RKHS on $\mathbb{B}_d$ with kernel

$$K(\lambda, \nu) = \frac{1}{1 - \langle \lambda, \mu \rangle}.$$

$\mathbb{C}[z] \subseteq H^2_d$ as a dense subset, monomials are orthogonal and

$$\|z_1^{\alpha_1} \cdots z_d^{\alpha_d}\|^2 = \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \ldots + \alpha_d)!} .$$

We define the d-**shift** on H^2_d to be the tuple $(S_1, \ldots, S_d)$ given by

$$(S_j f)(z) = z_j f(z).$$

The setting 3 - the algebra $\mathcal{A}_\mathcal{I}$

Define

$$\mathcal{F}_\mathcal{I} = H^2_d \ominus \mathcal{I}.$$

Let $S_j^\mathcal{I}$ be the compression of S_j to $\mathcal{F}_\mathcal{I}$. $(S_1^\mathcal{I}, \ldots, S_d^\mathcal{I})$ is a commuting row contraction, and it is the universal commuting row contraction satisfying

$$p(S_1^\mathcal{I}, \ldots, S_d^\mathcal{I}) = 0 \quad , \quad p \in \mathcal{I}. \tag{1}$$

$\mathcal{A}_\mathcal{I}$ is defined to be the norm closed unital algebra generated by $(S_1^\mathcal{I}, \ldots, S_d^\mathcal{I})$.

Fact (Popescu): $\mathcal{A}_\mathcal{I}$ is the universal operator algebra generated by a commuting row contraction satisfying (1).

All ideals below are **homogeneous**.

Basic problem number 1:

How does the ideal $\mathcal{I}$ determine the structure of $\mathcal{A}_{\mathcal{I}}$?

Example

Let $\mathcal{I} = \langle p \rangle$ and $\mathcal{J} = \langle q \rangle$, where

$$p(z_1, z_2) = z_1^2 - z_1 z_2 \quad , \quad q(z_1, z_2) = z_2^2 - z_1 z_2 .$$

It is clear that $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are completely isometrically isomorphic.

Fact: if $\mathcal{I}$ and $\mathcal{J}$ are related by some unitary change of variables, then $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are isometrically isomorphic.
What about the other direction?

The connection between algebra and geometry

Given an ideal $\mathcal{I} \subseteq \mathbb{C}[z]$, we define

$$V(\mathcal{I}) = \{z \in \mathbb{C}^d : p(z) = 0 \text{ for all } p \in \mathcal{I}\}.$$

1. When restricting to radical ideals, $\mathcal{I}$ and $V(\mathcal{I})$ completely determine each other.
2. $\mathbb{C}[z]/\mathcal{I} \cong \mathbb{C}[V(\mathcal{I})]$ is the universal unital algebra generated by d commuting elements satisfying the relations in $\mathcal{I}$.
3. $\mathbb{C}[V(\mathcal{I})]$ is isomorphic to $\mathbb{C}[V(\mathcal{J})]$ if and only if $V(\mathcal{I})$ and $V(\mathcal{J})$ are isomorphic (in homogeneous case: related by a linear map).

Basic problem number 2:

How does the geometry of $V(\mathcal{I})$ determine the algebraic and isometric structure of $\mathcal{A}_{\mathcal{I}}$?

Example

Let $d = 1$, $\mathcal{I} = \langle p \rangle$ and $\mathcal{J} = \langle q \rangle$, where

$$p(z) = z \quad , \quad q(z) = z^2.$$

$V(\mathcal{I}) = V(\mathcal{J}) = \{0\}$.
On the other hand $\mathcal{A}_{\mathcal{I}} = \mathbb{C}$, $\mathcal{A}_{\mathcal{J}}$ is two dimensional.

Just as in classical algebraic geometry, we will need to assume that the ideals are **radical** to obtain a strong connection between algebra and geometry. But "geometry" will have a different meaning.

The connection between operator-algebra and geometry

Define

$$Z(\mathcal{I}) = V(\mathcal{I}) \cap \overline{\mathbb{B}}_d \ .$$

By universality of $\mathcal{A}_{\mathcal{I}}$, $Z(\mathcal{I})$ can be identified with the space of **characters** - complex multiplicative linear functionals on $\mathcal{A}_{\mathcal{I}}$:

$$\rho \longleftrightarrow \left(\rho(S_1^{\mathcal{I}}), \ldots, \rho(S_d^{\mathcal{I}})\right) \in Z(\mathcal{I}).$$

An isomorphism $\varphi : \mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$ induces a homeomorphism $\varphi^* : Z(\mathcal{J}) \to Z(\mathcal{I})$:

$$\varphi^*(\rho) = \rho \circ \varphi.$$

Lemma

φ^* preserves the **analytic** *structure of $Z(\mathcal{J})$.*

The vacuum state

We define the **vacuum state** ρ_0 on $\mathcal{A}_{\mathcal{I}}$ to be the functional corresponding to $0 \in Z(\mathcal{I})$, that is

$$\rho_0(I) = 1,$$

$$\rho_0(S_i^{\mathcal{I}}) = 0 \quad , \quad i = 1, \ldots, d.$$

Theorem (SS09)

There exists a unitary change of variables sending $\mathcal{I}$ to $\mathcal{J}$ if and only there exists a **vacuum preserving** *isometric isomorphism $\varphi : \mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$.*

There do exist isomorphisms that are not vacuum preserving, so this theorem does not solve the isometric isomorphism problem for these algebras.

Classification up to (complete) isometric isomorphism

Theorem

If $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are (isometrically) isomorphic, then there exists a **vacuum preserving** *(isometric) isomorphism $\varphi : \mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$.*

Combining this result and the theorem on the previous slide:

Corollary

$\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are isometrically isomorphic if and only if $\mathcal{I}$ and $\mathcal{J}$ are related by a unitary change of variables. In this case the algebras are unitarily equivalent.

Outline of proof of the theorem

Let $\varphi : \mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$ be an (isometric) isomorphism.

1. There is a ball B in $Z(\mathcal{J})$, called **the singular nucleus** of $Z(\mathcal{J})$ that is mapped onto a ball, the singular nucleus of $Z(\mathcal{I})$.
2. Since $\varphi^* : Z(\mathcal{J}) \to Z(\mathcal{I})$ preserves analytic structure, $\varphi^*\big|_B$ is a (multivariate) Mobius map. Thus there is a disc $0 \in D_1 \subseteq B$ that is mapped onto a disc D_2.
3. $\mathcal{O}(\mathcal{I}, \mathcal{J}) := \{\rho \in D_2 : \rho = \psi^*(\rho_0) \text{ for some } \psi : \mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}\}$ $\mathcal{O}(\mathcal{I}, \mathcal{J})$ is rotation invariant, so it contains $C := \{z \in D_2 : |z| = \varphi^*(\rho_0)\}$. So $(\varphi^*)^{-1}(C) \subseteq \mathcal{O}(\mathcal{J}, \mathcal{J})$ is a circle through 0 in D_1. It is rotation invariant, so its interior is in $\mathcal{O}(\mathcal{J}, \mathcal{J})$. Thus $0 \in \mathcal{O}(\mathcal{I}, \mathcal{J})$.

That concludes the classification of the algebras $\mathcal{A}_{\mathcal{I}}$ up to (complete) isometric isomorphism.
We now concentrate on radical ideals $\cong$ varieties. We will be able to classify up to isomorphism, the classifying objects being geometric.

Theorem

Let $\mathcal{I}$ and $\mathcal{J}$ be radical homogeneous ideals. $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are isometrically isomorphic if and only if there is a unitary on $\mathbb{C}^d$ sending $V(\mathcal{I})$ onto $V(\mathcal{J})$.

Caveat

We will speak about (radical) ideals/varieties in general, but for the results stated below we currently need to impose some technical restrictions on the ideals/varieties. Varieties satisfying these technical restrictions include:

- Irreducible varieties (prime ideals).
- Varieties with two irreducible components.
- Varieties with irreducible components lying in disjoint subspaces.

Mappings between the varieties

If $\varphi : \mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$ is an isomorphism then we have $\varphi^* : Z(\mathcal{J}) \to Z(\mathcal{I})$. When $\mathcal{I}$ and $\mathcal{J}$ are radical we have more:

Lemma

There exists a holomorphic $F : \overline{\mathbb{B}_d} \to \mathbb{C}^d$ such that

$$\varphi^* = F\big|_{Z(\mathcal{J})}.$$

Lemma

If $F : \overline{\mathbb{B}_d} \to \mathbb{C}^d$ is holomorphic fixing 0 which maps $Z(\mathcal{J})$ bijectively onto $Z(\mathcal{I})$, then $F\big|_{Z(\mathcal{J})}$ is the restriction of a linear map.

Conclusion:

If there is a vacuum preserving isomorphism $\mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$, then there exists a linear map A on $\mathbb{C}^d$ such that $AZ(\mathcal{J}) = Z(\mathcal{I})$.

Compare:

There is a grading preserving ($=$ vacuum preserving) isomorphism $\mathbb{C}[V(\mathcal{I})] \to \mathbb{C}[V(\mathcal{J})]$ if and only if there exists a linear map A on $\mathbb{C}^d$ such that $AV(\mathcal{J}) = V(\mathcal{I})$.

1. Note the difference in "geometry".
2. Isomorphic algebras with non-isomorphic closures.
3. We seek the other direction: does an invertible linear map between $Z(\mathcal{J})$ and $Z(\mathcal{I})$ induce an (algebraic) isomorphism $\mathcal{A}_{\mathcal{I}} \to \mathcal{A}_{\mathcal{J}}$?

Connection to reproducing kernel Hilbert spaces

Recall: H^2_d is the RKHS on $\mathbb{B}_d$ with Kernel functions

$$\nu_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle} \ , \ \lambda \in \mathbb{B}_d.$$

Recall: $\mathcal{F}_\mathcal{I} = H^2_d \ominus \mathcal{I}$. When $\mathcal{I}$ is radical,

$$\mathcal{F}_\mathcal{I} = \overline{span}\{\nu_\lambda : \lambda \in Z^o(\mathcal{I})\},$$

where $Z^o(\mathcal{I}) = Z(\mathcal{I}) \cap \mathbb{B}_d$. $\mathcal{F}_\mathcal{I}$ is then a RKHS on $Z^o(\mathcal{I})$ and $\mathcal{A}_\mathcal{I}$ is an algebra of **continuous multipliers** on $\mathcal{F}_\mathcal{I}$, in particular, it is a function algebra.

A key lemma

Lemma

Let A be a linear map such that $AZ(\mathcal{J}) = Z(\mathcal{I})$. Then there exists a bounded linear map $\tilde{A} : \mathcal{F}_{\mathcal{J}} \to \mathcal{F}_{\mathcal{I}}$ such that

$$\tilde{A}\nu_\lambda = \nu_{A\lambda}.$$

Lemma

Let A be a linear map such that $AZ(\mathcal{J}) = Z(\mathcal{I})$. Then there exists a linear map $\tilde{A} : \mathcal{F}_{\mathcal{J}} \to \mathcal{F}_{\mathcal{I}}$ such that

$$\tilde{A}\nu_\lambda = \nu_{A\lambda}.$$

Corollary

Let A be an invertible linear map such that $AZ(\mathcal{J}) = Z(\mathcal{I})$. Then the map

$$\varphi : f \to f \circ A$$

is a completely bounded isomorphism from $\mathcal{A}_{\mathcal{I}}$ onto $\mathcal{A}_{\mathcal{J}}$, and it is given by conjugation with $\tilde{A}^*$:

$$\varphi(f) = \tilde{A}^* f (\tilde{A}^*)^{-1}.$$

Algebraic-geometric classification of the operator algebras

Theorem

Let $\mathcal{I}$ and $\mathcal{J}$ be two radical homogeneous ideals (satisfying an additional condition). $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are isomorphic if and only if there is an invertible linear map A such that $AZ(\mathcal{J}) = Z(\mathcal{I})$. In this case, $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are similar.

Examples

Consider $\mathcal{A}_\mathcal{I}$ and $\mathcal{A}_\mathcal{J}$, where $V(\mathcal{I})$ and $V(\mathcal{J})$ is a pair of homogeneous varieties in $\mathbb{C}^2$, in one of the following categories

- One line - always isometrically isomorphic.
- Two lines - always isomorphic. Isometrically isomorphic iff angle is the same.
- Three lines - Even the non-closed algebras might be not isomorphic. When they are, sometimes the closed algebras are isomorphic and sometimes not.

However, the C^*-algebra generated by $\mathcal{A}_\mathcal{I}$ depends only on the **number** of lines. I have a feeling that the C^*-algebra $C^*(\mathcal{A}_\mathcal{I})$ is completely determined by the **topology** of $V(\mathcal{I})$ (to be continued...)

Rigidity of varieties and operator algebras

Lemma

Let V be an irreducible variety, and let A be a linear map such that

$$\|Ax\| = \|x\| \ , \ x \in V.$$

Then A is isometric on the span of V.

Theorem

Let $\mathcal{I}$ be a homogeneous prime ideal, and $\mathcal{J}$ a homogenous radical ideal. If $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are isomorphic then they are unitarily equivalent. Every vacuum preserving isomorphism is a complete isometry and is unitarily implemented.

A similar statement can be made for ideals corresponding to nonlinear hypersurfaces.