Free biholomorphic classification of noncommutative domains

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Plan

- Free holomorphic functions on noncommutative domains
- Free biholomorphic functions and noncommutative Cartan type results
- Free biholomorphic classification of noncommutative domains
- Isomorphisms of noncommutative Hardy algebras

- Reference : Free biholomorphic classification of noncommutative domains, *Int. Math. Res. Not.*, in press.

Noncommutative Reinhardt domains

- $\mathbb{F}_n^+$ is the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0.
- $|\alpha|$ stands for the length of the word $\alpha \in \mathbb{F}_n^+$.
- If $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$, we set $X_\alpha := X_{i_1} \cdots X_{i_k}$ if $\alpha := g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$, and $X_{g_0} := I$.
- $f := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha$ is a *free holomorphic function* on a ball $[B(\mathcal{H})^n]_\gamma$, $\gamma > 0$, if $\limsup_{k\to\infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2\right)^{1/2k} < \infty$.
- f is called *positive regular free holomorphic function* if $a_\alpha \geq 0$, $a_{g_i} \neq 0$, $i = 1, \ldots, n$.

Noncommutative Reinhardt domains

- Given $m, n \in \mathbb{N} := \{1, 2, \ldots\}$ and a positive regular free holomorphic function f, define the noncommutative domain

$$\mathbf{D}_f^m(\mathcal{H}) := \left\{ X \in B(\mathcal{H})^n : \ (id - \Phi_{f,X})^k(I) \geq 0, \ 1 \leq k \leq m \right\},$$

where $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by

$$\Phi_{f,X}(Y) := \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha X_\alpha Y X_\alpha^*, \qquad Y \in B(\mathcal{H}),$$

and the convergence is in the week operator topology.

- $\mathbf{D}_f^m(\mathcal{H})$ can be seen as a *noncommutative Reinhardt domain*, i.e.,

$$(e^{i\theta_1} X_1, \ldots, e^{i\theta_n} X_n) \in \mathbf{D}_f^m(\mathcal{H}),$$

for $X \in \mathbf{D}_f^m(\mathcal{H})$ and $\theta_i \in \mathbb{R}$.

Noncommutative Reinhardt domains

- If $m = 1$, $p = X_1 + \cdots + X_n$, then $\mathbf{D}_p^1(\mathcal{H})$ coincides with

$$[B(\mathcal{H})^n]_1 := \{(X_1, \ldots, X_n) : \ \|X_1X_1^* + \cdots + X_nX_n^*\| \leq 1\}.$$

- The study of $[B(\mathcal{H})^n]_1$ has generated a *free analogue of Sz.-Nagy–Foiaş theory*.
- Frazho, Bunce, Popescu, Arias-Popescu, Davidson-Pitts-Katsoulis, Ball-Vinnikov, and others.
- The domain $\mathbf{D}_f^1(\mathcal{H})$ was studied in

  G. Popescu, Operator theory on noncommutative domains, *Mem. Amer. Math. Soc.* **205** (2010), No.964, vi+124 pp.

Noncommutative Reinhardt domains

- The domain $\mathbf{D}_f^m(\mathcal{H})$, $m \geq 2$, was considered in

  G. POPESCU, Noncommutative Berezin transforms and multivariable operator model theory, *J. Funct. Anal.*, **254** (2008), 1003–1057.

- If $q = X_1 + \cdots + X_n$ and $m \geq 1$, then $\mathbf{D}_q^m(\mathcal{H})$ coincides with the set of all row contractions $(X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1$ satisfying the positivity condition

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} X_\alpha X_\alpha^* \geq 0.$$

  The elements of $\mathbf{D}_q^m(\mathcal{H})$ can be seen as multivariable noncommutative analogues of Agler's *m*-hypercontractions (when $n = 1, m \geq 2, q = X$)

Universal model for $\mathbf{D}_f^m$

- Let H_n be an n-dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. The *full Fock space* of H_n is defined by

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{k\geq 1} H_n^{\otimes k}.$$

- The *weighted left creation operators* associated with $\mathbf{D}_f^m(\mathcal{H})$ are defined by setting $W_i : F^2(H_n) \to F^2(H_n)$,

$$W_i e_\alpha = \frac{\sqrt{b_\alpha^{(m)}}}{\sqrt{b_{g_i\alpha}^{(m)}}} e_{g_i\alpha}, \quad \alpha \in \mathbb{F}_n^+,$$

where $b_{g_0}^{(m)} = 1$ and

Universal model for $\mathbf{D}_f^m$

$$b_\alpha^{(m)} = \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \ldots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \binom{j+m-1}{m-1} \quad \text{if } |\alpha| \geq 1,$$

where a_α are the coefficients of f.

- $(W_1, \ldots, W_n)$ is the universal model for $\mathbf{D}_f^m$.
- The domain algebra $\mathcal{A}_n(\mathbf{D}_f^m)$ associated with the noncommutative domain $\mathbf{D}_f^m$ is the norm closure of all polynomials in $W_1, \ldots, W_n$, and the identity.
- The Hardy algebra $F_n^\infty(\mathbf{D}_f^m)$ is the SOT-(resp. WOT-, w^*-) closure of all polynomials in $W_1, \ldots, W_n$, and the identity.

Universal model for $\mathbf{D}_f^m$

$$b_\alpha^{(m)} = \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \ldots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \binom{j+m-1}{m-1} \quad \text{if } \ |\alpha| \geq 1,$$

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- $(W_1, \ldots, W_n)$ is the universal model for $\mathbf{D}_f^m$.
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Universal model for $\mathbf{D}_f^m$

- Assumptions :
  (i) $\mathcal{H}$ is a separable infinite dimensional Hilbert space ;
  (ii) $\mathbf{D}_f^m(\mathcal{H})$ is closed in the operator norm topology ;
  (iii) $\mathbf{D}_f^m(\mathcal{H})$ is starlike domain, i.e.

$$r\mathbf{D}_f^m(\mathcal{H}) \subset \mathbf{D}_f^m(\mathcal{H}), \qquad r \in [0,1).$$

- Examples of closed starlike domains :
  (i) $\mathbf{D}_f^1(\mathcal{H})$;
  (ii) $\mathbf{D}_p^m(\mathcal{H})$ if $p = a_1 X_1 + \ldots + a_n X_n$, $a_i > 0$.

Free holomorphic functions

- The radial part of $\mathbf{D}_f^m(\mathcal{H})$ is defined by

$$\mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H}) := \bigcup_{0\leq r<1} r\mathbf{D}_f^m(\mathcal{H}).$$

- if q is any positive regular noncommutative polynomial, then
$Int(\mathbf{D}_q^1(\mathcal{H})) = \mathbf{D}_{q,\mathrm{rad}}^1(\mathcal{H})$ and $\overline{Int(\mathbf{D}_q^1(\mathcal{H}))} = \mathbf{D}_q^1(\mathcal{H})$.
- A formal power series $G := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha$, $c_\alpha \in \mathbb{C}$, is called *free holomorphic function* on $\mathbf{D}_{f,\mathrm{rad}}^m$ if its representation on any Hilbert space $\mathcal{H}$, i.e., $G : \mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H}) \to B(\mathcal{H})$ given by

$$G(X) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha X_\alpha, \qquad X \in \mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H}),$$

is well-defined in the operator norm topology.

Free holomorphic functions

- The map G is called free holomorphic function on $\mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})$.
- $Hol(\mathbf{D}^m_{f,\mathrm{rad}})$ denotes the algebra of all free holomorphic functions on $\mathbf{D}^m_{f,\mathrm{rad}}$.

Free holomorphic functions

Theorem

Let $G := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha$ be a formal power series and let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then the following statements are equivalent :

(i) *G is a free holomorphic function on $\mathbf{D}_{f,\mathrm{rad}}^m$.*

(ii) *For any $r\in[0,1)$, the series*

$$G(rW_1,\ldots,rW_n) := \sum_{k=0}^{\infty}\sum_{|\alpha|=k} r^{|\alpha|} c_\alpha W_\alpha$$

is convergent in the operator norm topology, where $(W_1,\ldots,W_n)$ is the universal model associated with $\mathbf{D}_f^m$.

Free holomorphic functions

(iii) The inequality

$$\limsup_{k\to\infty} \left\| \sum_{|\alpha|=k} \frac{1}{b_\alpha^{(m)}} |c_\alpha|^2 \right\|^{\frac{1}{2k}} \leq 1,$$

holds.

(iv) For any $r \in [0, 1)$, the series $\sum_{k=0}^\infty \left\| \sum_{|\alpha|=k} r^{|\alpha|} c_\alpha W_\alpha \right\|$ is convergent.

(v) For any $X \in \mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})$, the series $\sum_{k=0}^\infty \left\| \sum_{|\alpha|=k} c_\alpha X_\alpha \right\|$ is convergent.

Free holomorphic functions

- Connection between the theory of free holomorphic functions on noncommutative domains $\mathbf{D}^m_{f,\mathrm{rad}}$ and the theory of holomorphic functions on domains in $\mathbb{C}^d$.

Remark

If $p \in \mathbb{N}$ and $F(X) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha X_\alpha$ is a free holomorphic function on $\mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})$, then its representation on $\mathbb{C}^p$, i.e., the map

$$\mathbb{C}^{np^2} \supset \mathbf{D}^m_{f,\mathrm{rad}}(\mathbb{C}^p) \ni \Lambda \mapsto F(\Lambda) \in M_{p\times p} \subset \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $\mathbf{D}^m_f(\mathbb{C}^p)$.

Free holomorphic functions

- Connection between the theory of free holomorphic functions on noncommutative domains $\mathbf{D}_{f,\mathrm{rad}}^m$ and the theory of holomorphic functions on domains in $\mathbb{C}^d$.

Remark

If $p \in \mathbb{N}$ and $F(X) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha X_\alpha$ is a free holomorphic function on $\mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H})$, then its representation on $\mathbb{C}^p$, i.e., the map

$$\mathbb{C}^{np^2} \supset \mathbf{D}_{f,\mathrm{rad}}^m(\mathbb{C}^p) \ni \Lambda \mapsto F(\Lambda) \in M_{p\times p} \subset \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $\mathbf{D}_f^m(\mathbb{C}^p)$.

Free holomorphic functions

- When $p = 1$, the interior $Int(\mathbf{D}_f^m(\mathbb{C}))$ is a Reinhardt domain in $\mathbb{C}^n$.
- When $p \geq 2$, $Int(\mathbf{D}_f^m(\mathbb{C}^p))$ are circular domains in $\mathbb{C}^{np^2}$.

- $A(\mathbf{D}_{f,\mathrm{rad}}^m)$ denotes the set of all elements G in $Hol(\mathbf{D}_{f,\mathrm{rad}}^m)$ such that the mapping
$$\mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H}) \ni X \mapsto G(X) \in B(\mathcal{H})$$
has a continuous extension to $[\mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H})]^- = \mathbf{D}_f^m(\mathcal{H})$.
- $A(\mathbf{D}_{f,\mathrm{rad}}^m)$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$ and has a unital operator algebra structure.

Free holomorphic functions

Theorem

The map $\Phi : A(\mathbf{D}^m_{f,\mathrm{rad}}) \to \mathcal{A}_n(\mathbf{D}^m_f)$ *defined by*

$$\Phi\left(\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha\right) := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha W_\alpha$$

is a completely isometric isomorphism of operator algebras. Moreover, if $G := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha$ *is a free holomorphic function on the domain* $\mathbf{D}^m_{f,\mathrm{rad}}$*, then the following statements are equivalent :*

(i) $G \in A(\mathbf{D}^m_{f,\mathrm{rad}})$;

(iii) $G(rW_1, \ldots, rW_n) := \sum_{k=0}^\infty \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} W_\alpha$ *is convergent in the operator norm topology as* $r \to 1$.

Free holomorphic functions

(ii) there exists $\widetilde{G} \in \mathcal{A}_n(\mathbf{D}_f^m)$ with $G = \mathbf{B}[\widetilde{G}]$.

In this case, $\Phi(G) = \widetilde{G} = \lim_{r\to 1} G(rW_1, \ldots, rW_n)$ and

$$\Phi^{-1}(\widetilde{G}) = G = \mathbf{B}[\widetilde{G}], \quad \widetilde{G} \in \mathcal{A}_n(\mathbf{D}_f^m),$$

where $\mathbf{B}$ is the noncommutative Berezin transform associated with $\mathbf{D}_f^m$.

Free holomorphic functions

Corollary

If $p \in \mathbb{N}$ *and* $F(X) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha X_\alpha$ *is in* $A(\mathbf{D}^m_{f,\mathrm{rad}})$*, then its representation on* $\mathbb{C}^p$*, i.e., the map*

$$\mathbb{C}^{np^2} \supset \mathbf{D}^m_f(\mathbb{C}^p) \ni \Lambda \mapsto F(\Lambda) \in M_{p\times p} \subset \mathbb{C}^{p^2}$$

is a continuous map on $\mathbf{D}^m_f(\mathbb{C}^p)$ *and holomorphic on the interior of* $\mathbf{D}^m_f(\mathbb{C}^p)$*.*

Composition of free holomorphic functions

Theorem

Let f and g be positive regular free holomorphic functions with n and p indeterminates, respectively, and let $m, l \geq 1$.
If $F : \mathbf{D}^l_{g,\mathrm{rad}}(\mathcal{H}) \to B(\mathcal{H})$ and $\Phi : \mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H}) \to \mathbf{D}^l_{g,\mathrm{rad}}(\mathcal{H})$ are free holomorphic functions, then $F \circ \Phi$ is a free holomorphic function on $\mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})$.
If, in addition, F is bounded, then $F \circ \Phi$ is bounded and $\|F \circ \Phi\|_\infty \leq \|F\|_\infty$.

Composition of free holomorphic functions

Theorem

Let f and g be positive regular free holomorphic functions with n and p indeterminates, respectively, and let $m, l \geq 1$. If $F : \mathbf{D}^l_{g,\mathrm{rad}}(\mathcal{H}) \to B(\mathcal{H})$ and $\Phi : \mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H}) \to \mathbf{D}^l_g(\mathcal{H})$ are bounded free holomorphic functions which have continuous extensions to the noncommutative domains $\mathbf{D}^l_g(\mathcal{H})$ and $\mathbf{D}^m_f(\mathcal{H})$, respectively, then $F \circ \Phi$ is a bounded free holomorphic function on $\mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})$ which has continuous extension to $\mathbf{D}^m_f(\mathcal{H})$.

Composition of free holomorphic functions

Moreover,

(a) $(F \circ \Phi)(X) = \mathcal{B}_X \left\{ \mathcal{B}_{\tilde{\Phi}}[\tilde{F}] \right\}$, $X \in \mathbf{D}_f^m(\mathcal{H})$, where $\mathcal{B}_X$, $\mathcal{B}_{\tilde{\Phi}}$ are the noncommutative Berezin transforms ;

(b) the model boundary function of $F \circ \Phi$ satisfies

$$\widetilde{F \circ \Phi} = \lim_{r \to 1} F(r\widetilde{\Phi}_1, \ldots, r\widetilde{\Phi}_p) = \mathcal{B}_{\tilde{\Phi}}[\widetilde{F}],$$

where the convergence is in the operator norm.

Free biholomorphic functions

- Let f and g be positive regular free holomorphic functions with n and q indeterminates, respectively, and let $m, l \geq 1$.
- A map $F : \mathbf{D}_f^m(\mathcal{H}) \to \mathbf{D}_g^l(\mathcal{H})$ is called free biholomorphic function if F is a homeomorphism in the operator norm topology and $F|_{\mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H})}$ and $F^{-1}|_{\mathbf{D}_{g,\mathrm{rad}}^l(\mathcal{H})}$ are free holomorphic functions on $\mathbf{D}_{f,\mathrm{rad}}^m(\mathcal{H})$ and $\mathbf{D}_{g,\mathrm{rad}}^l(\mathcal{H})$, respectively.
- $\mathbf{D}_f^m(\mathcal{H})$ and $\mathbf{D}_g^l(\mathcal{H})$ are called free biholomorphic equivalent and denote $\mathbf{D}_f^m \simeq \mathbf{D}_g^l$.
- $Bih(\mathbf{D}_f^m, \mathbf{D}_g^l)$ denotes the set of all the free biholomorphic functions $F : \mathbf{D}_f^m(\mathcal{H}) \to \mathbf{D}_g^l(\mathcal{H})$.

Free biholomorphic functions

- Two domains Ω_1, Ω_2 in $\mathbb{C}^d$ are called biholomorphic equivalent if there are holomorphic maps $\varphi : \Omega_1 \to \Omega_2$ and $\psi : \Omega_2 \to \Omega_1$ be such that $\varphi \circ \psi = id_{\Omega_2}$ and $\psi \circ \varphi = id_{\Omega_1}$.

Theorem

Let f and g be positive regular free holomorphic functions with n and q indeterminates, respectively, and let $m, l, p \geq 1$. If $F : \mathbf{D}_f^m(\mathcal{H}) \to \mathbf{D}_g^l(\mathcal{H})$ is a free biholomorphic function, then $n = q$ and its representation on $\mathbb{C}^p$, i.e., the map

$$\mathbb{C}^{np^2} \supset \mathbf{D}_f^m(\mathbb{C}^p) \ni \Lambda \mapsto F(\Lambda) \in \mathbf{D}_g^l(\mathbb{C}^p) \subset \mathbb{C}^{qp^2}$$

is a homeomorphism from $\mathbf{D}_f^m(\mathbb{C}^p)$ onto $\mathbf{D}_g^l(\mathbb{C}^p)$ and a biholomorphic function from $Int(\mathbf{D}_f^m(\mathbb{C}^p))$ onto $Int(\mathbf{D}_g^l(\mathbb{C}^p))$.

Free biholomorphic functions

- The theory of functions in several complex variables $\Longrightarrow$ results on the classification of the noncommutative domains $\mathbf{D}_f^m(\mathcal{H})$.

Corollary

Let f and g be positive regular free holomorphic functions with n and q indeterminates, respectively, and let $m, l \geq 1$. If $n \neq q$ or there is $p \in \{1, 2, \ldots\}$ such that $Int(\mathbf{D}_f^m(\mathbb{C}^p))$ is not biholomorphic equivalent to $Int(\mathbf{D}_g^l(\mathbb{C}^p))$, then the noncommutative domains $\mathbf{D}_f^m(\mathcal{H})$ and $\mathbf{D}_g^l(\mathcal{H})$ are not free biholomorphic equivalent.

Free biholomorphic functions

- Since $Int(\mathbf{D}_f^m(\mathbb{C})) \subset \mathbb{C}^n$ and $Int(\mathbf{D}_g^l(\mathbb{C})) \subset \mathbb{C}^q$ are Reinhardt domains which contain 0, Sunada's result implies that there exists a permutation σ of the set $\{1, \ldots, n\}$ and scalars $\mu_1, \ldots, \mu_n > 0$ such that the map

$$(z_1, \ldots, z_n) \mapsto (\mu_1 z_{\sigma(1)}, \ldots, \mu_n z_{\sigma(n)})$$

  is a biholomorphic map from $Int(\mathbf{D}_f^m(\mathbb{C}))$ onto $Int(\mathbf{D}_g^l(\mathbb{C}))$.

- Open question : Is there an analogue of Sunada's result for the noncommutative domains $\mathbf{D}_f^m$.

Cartan type results

- $N := (N_1, \ldots, N_n) \in B(\mathcal{H})^n$ is called *nilpotent* if there is $p \in \mathbb{N} := \{1, 2, \ldots\}$ such that $N_\alpha = 0$ for any $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = p$.
- The nilpotent part of the noncommutative domain $\mathbf{D}_f^m(\mathcal{H})$ is defined by

$$\mathbf{D}_{f,\mathrm{nil}}^m(\mathcal{H}) := \{N \in \mathbf{D}_f^m(\mathcal{H}) : \ N \text{ is nilpotent}\}.$$

Cartan type results

Theorem

Let f be a positive regular free holomorphic function with n indeterminates and let $m \geq 1$. Let $H_1, \ldots, H_n$ be formal power series in n noncommuting indeterminates $Z = (Z_1, \ldots, Z_n)$ of the form

$$H_i(Z) := \sum_{k=2} \sum_{|\alpha|=k} a_\alpha^{(i)} Z_\alpha, \qquad i = 1, \ldots, n.$$

If $F(Z) := (Z_1 + H_1(Z), \ldots, Z_n + H_n(Z))$ has the property that

$$F(\mathbf{D}^m_{f,\mathrm{nil}}(\mathcal{H})) \subseteq \mathbf{D}^m_{f,\mathrm{nil}}(\mathcal{H})$$

for any Hilbert space $\mathcal{H}$, then $F(Z) = Z$.

Cartan type results

Theorem

Let f and g be positive regular free holomorphic functions with n indeterminates and let $m, l \geq 1$. Let $F = (F_1, \ldots, F_n)$ and $G = (G_1, \ldots, G_n)$ be n-tuples of formal power series in n noncommuting indeterminates such that

$$F(0) = G(0) = 0 \quad \textit{and} \quad F \circ G = G \circ F = \mathrm{id}.$$

If $F(\mathbf{D}^m_{f,\mathrm{nil}}(\mathcal{H})) = \mathbf{D}^l_{g,\mathrm{nil}}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$, then F has the form

$$F(Z_1, \ldots, Z_n) = [Z_1, \ldots, Z_n]U,$$

where U is an invertible bounded linear operator on $\mathbb{C}^n$.

- $(W_1^{(f)}, \ldots, W_n^{(f)})$ is the universal model associated with $\mathbf{D}^m_f$.

Cartan type results

Theorem

Let f and g be positive regular free holomorphic functions with n indeterminates and let $m, l \geq 1$. Let $F = (F_1, \ldots, F_n)$ and $G = (G_1, \ldots, G_n)$ be n-tuples of formal power series in n noncommuting indeterminates such that

$$F(0) = G(0) = 0 \quad \text{and} \quad F \circ G = G \circ F = \mathrm{id}.$$

If $F(\mathbf{D}^m_{f,\mathrm{nil}}(\mathcal{H})) = \mathbf{D}^l_{g,\mathrm{nil}}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$, then F has the form

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where U is an invertible bounded linear operator on $\mathbb{C}^n$.

- $(W_1^{(f)}, \ldots, W_n^{(f)})$ is the universal model associated with $\mathbf{D}^m_f$.

Cartan type results

Theorem

Let f and g be positive regular free holomorphic functions with n and q indeterminates, respectively, and let $m, l \geq 1$. A map $F : \mathbf{D}_f^m(\mathcal{H}) \to \mathbf{D}_g^l(\mathcal{H})$ is a free biholomorphic function with $F(0) = 0$ if and only if $n = q$ and F has the form

$$F(X) = [X_1, \ldots, X_n]U, \qquad X = (X_1, \ldots, X_n) \in \mathbf{D}_f^m(\mathcal{H}),$$

where U is an invertible bounded linear operator on $\mathbb{C}^n$ such that

$$[W_1^{(f)}, \ldots, W_n^{(f)}]U \in \mathbf{D}_g^l(F^2(H_n))$$

and

$$[W_1^{(g)}, \ldots, W_n^{(g)}]U^{-1} \in \mathbf{D}_f^m(F^2(H_n)).$$

Cartan type results

- Characterization the unit ball of $B(\mathcal{H})^n$ among the noncommutative domains $\mathbf{D}_f^m(\mathcal{H})$, up to free biholomorphisms.

Corollary

Let g be a positive regular free holomorphic function with q indeterminates and let $l \geq 1$. Then the noncommutative domain $\mathbf{D}_g^l(\mathcal{H})$ is biholomorphic equivalent to the unit ball $[B(\mathcal{H})^n]_1$ if and only if $q = n$ and there is an invertible operator $U \in B(\mathbb{C}^n)$ such that

$$[S_1, \ldots, S_n]U \in \mathbf{D}_g^l(F^2(H_n))$$

and

$$[W_1^{(g)}, \ldots, W_n^{(g)}]U^{-1} \in [B(\mathcal{H})^n]_1.$$

Cartan type results

- $Aut_0(\mathbf{D}_f^m)$ denotes the subgroup of all free holomorphic automorphisms of $\mathbf{D}_f^m(\mathcal{H})$ that fix the origin.

Corollary

A map $\Psi : \mathbf{D}_f^m(\mathcal{H}) \to \mathbf{D}_f^m(\mathcal{H})$ is in the subgroup $Aut_0(\mathbf{D}_f^m)$ if and only if

$$\Psi(X) = [X_1, \ldots, X_n]U, \qquad X = (X_1, \ldots, X_n) \in \mathbf{D}_f^m(\mathcal{H}),$$

for some invertible operator U on $\mathbb{C}^n$ such that

$$[W_1, \ldots, W_n]U \quad \text{and} \quad [W_1, \ldots, W_n]U^{-1}$$

are in $\mathbf{D}_f^m(F^2(H_n))$.

Cartan type results

- The theory of functions in several complex variables $\Longrightarrow$ results on the classification of the domains $\mathbf{D}_f^m(\mathcal{H})$.

Theorem

Let f and g be positive regular free holomorphic functions with n indeterminates and let $m, l \geq 1$. Assume that there is $p' \in \{1, 2, \ldots\}$ such that the domains $Int(\mathbf{D}_f^m(\mathbb{C}^{p'}))$ and $Int(\mathbf{D}_g^l(\mathbb{C}^{p'}))$ are linearly equivalent and all the automorphisms of $Int(\mathbf{D}_f^m(\mathbb{C}^{p'}))$ fix the origin.
Then $\mathbf{D}_f^m(\mathcal{H})$ and $\mathbf{D}_g^l(\mathcal{H})$ are free biholomorphic equivalent if and only if there is an invertible operator $U \in B(\mathbb{C}^n)$ such that

$$[W_1^{(f)}, \ldots, W_n^{(f)}]U \in \mathbf{D}_g^l(F^2(H_n))$$

$$[W_1^{(g)}, \ldots, W_n^{(g)}]U^{-1} \in \mathbf{D}_f^m(F^2(H_n)).$$

Cartan type results

- **Thullen's theorem.** *If a bounded Reinhardt domain in $\mathbb{C}^2$ has a biholomorphic map that does not fix the origin, then the domain is linearly equivalent to one of the following : polydisc, unit ball, or the so-called Thullen domain.*

Cartan type results

Corollary

Let f and g be positive regular free holomorphic functions with 2 indeterminates and let $m, l \geq 1$. Assume that the Reinhardt domains $Int(\mathbf{D}_f^m(\mathbb{C}))$ and $Int(\mathbf{D}_g^l(\mathbb{C}))$ are linearly equivalent but they are not linearly equivalent to either the polydisc, the unit ball, or any Thullen domain in $\mathbb{C}^2$.
Then the noncommutative domains $\mathbf{D}_f^m(\mathcal{H})$ and $\mathbf{D}_g^l(\mathcal{H})$ are free biholomorphic equivalent if and only if there is an invertible bounded linear operator $U \in B(\mathbb{C}^2)$ such that

$$[W_1^{(f)}, W_2^{(f)}]U \in \mathbf{D}_g^l(F^2(H_2)), \quad [W_1^{(g)}, W_2^{(g)}]U^{-1} \in \mathbf{D}_f^m(F^2(H_2)).$$

Noncommutative domain algebras

- If $\Phi : A(\mathbf{D}^m_{f,\mathrm{rad}}) \to A(\mathbf{D}^l_{g,\mathrm{rad}})$ is a unital algebra homomorphism, it induces a unique unital homomorphism $\widehat{\Phi} : \mathcal{A}_n(\mathbf{D}^m_f) \to \mathcal{A}_q(\mathbf{D}^l_g)$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}_n(\mathbf{D}^m_f) & \xrightarrow{\ \widehat{\Phi}\ } & \mathcal{A}_q(\mathbf{D}^l_g) \\ \Big\downarrow \mathbf{B} & & \Big\downarrow \mathbf{B} \\ A(\mathbf{D}^m_{f,\mathrm{rad}}) & \xrightarrow{\ \Phi\ } & A(\mathbf{D}^l_{g,\mathrm{rad}}) \end{array}$$

is commutative, i.e., $\Phi\mathbf{B} = \mathbf{B}\widehat{\Phi}$. The homomorphisms Φ and $\widehat{\Phi}$ uniquely determine each other by the formulas :

$$[\Phi(\chi)](X) = \mathbf{B}_X[\widehat{\Phi}(\widetilde{\chi})], \qquad X \in \mathbf{D}^l_{g,\mathrm{rad}}(\mathcal{H}),$$

and

$$\widehat{\Phi}(\widetilde{\chi}) = \widetilde{\Phi(\chi)}, \qquad \widetilde{\chi} \in \mathcal{A}_n(\mathbf{D}^m_f).$$

Noncommutative domain algebras

- Consider the closed operator systems

$$\mathcal{S}_f := \overline{\text{span}}\{W_\alpha^{(f)} W_\beta^{(f)^*};\ \alpha, \beta \in \mathbb{F}_n^+\}$$

and

$$\mathcal{S}_g := \overline{\text{span}}\{W_\alpha^{(g)} W_\beta^{(g)^*};\ \alpha, \beta \in \mathbb{F}_q^+\},$$

where $(W_1^{(f)}, \ldots, W_n^{(f)})$ and $(W_1^{(g)}, \ldots, W_q^{(g)})$ are the universal models of $\mathbf{D}_f^m$ and $\mathbf{D}_g^l$, respectively.

Noncommutative domain algebras

- Let $\Phi : A(\mathbf{D}^m_{f,\mathrm{rad}}) \to A(\mathbf{D}^l_{g,\mathrm{rad}})$ be a unital completely isometric isomorphism. We say that Φ has *completely contractive hereditary extension* if the linear maps $\widehat{\Phi}_* : \mathcal{S}_f \to \mathcal{S}_g$ defined by

$$\widehat{\Phi}_*\left(W_\alpha^{(f)} W_\beta^{(f)*}\right) := \widehat{\Phi}\left(W_\alpha^{(f)}\right)\widehat{\Phi}\left(W_\beta^{(f)}\right)^*$$

and $(\widehat{\Phi}^{-1})_* : \mathcal{S}_g \to \mathcal{S}_f$ defined by

$$(\widehat{\Phi}^{-1})_*\left(W_\alpha^{(g)} W_\beta^{(g)*}\right) := \widehat{\Phi}^{-1}\left(W_\alpha^{(g)}\right)\widehat{\Phi}^{-1}\left(W_\beta^{(g)}\right)^*$$

are completely contractive.

- Consequently, the map $\widehat{\Phi}_* : \mathcal{S}_f \to \mathcal{S}_g$ is a unital completely isometric linear isomorphism which extends $\widehat{\Phi}$.

Classification of noncommutative domains

Theorem

Let f and g be positive regular free holomorphic functions with n and q indeterminates, respectively, and let $m, l \geq 1$. Then the following statements are equivalent :

(i) $\Psi : A(\mathbf{D}^m_{f,\mathrm{rad}}) \to A(\mathbf{D}^l_{g,\mathrm{rad}})$ *is a unital completely isometric isomorphism with completely contractive hereditary extension ;*

(ii) *there is* $\varphi \in Bih(\mathbf{D}^l_g, \mathbf{D}^m_f)$ *such that*

$$\Psi(\chi) = \chi \circ \varphi, \qquad \chi \in A(\mathbf{D}^m_{f,\mathrm{rad}}).$$

In this case, $\widehat{\Psi}(\tilde{\chi}) = \mathcal{B}_{\tilde{\varphi}}[\tilde{\chi}]$, $\tilde{\chi} \in \mathcal{A}_n(\mathbf{D}^m_f)$, *where* $\mathcal{B}_{\tilde{\varphi}}$ *is the noncommutative Berezin transform at* $\tilde{\varphi}$.

Classification of noncommutative domains

In the particular case when $m = l = 1$, any unital completely isometric isomorphism has c.c. hereditary extension.

Remark

Let $\Psi : A(\mathbf{D}^m_{f,\mathrm{rad}}) \to A(\mathbf{D}^l_{g,\mathrm{rad}})$ be a unital algebra homomorphism. Then Ψ is a unital completely isometric isomorphism having completely contractive hereditary extension if and only if Ψ is a continuous homeomorphism such that

$$(\hat{\Psi}(W_1^{(f)}), \ldots, \hat{\Psi}(W_n^{(f)})) \in \mathbf{D}_f^m(F^2(H_n))$$

and

$$(\hat{\Psi}^{-1}(W_1^{(g)}), \ldots, \hat{\Psi}^{-1}(W_q^{(g)})) \in \mathbf{D}_g^l(F^2(H_q)).$$

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and

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Classification of noncommutative domains

Corollary

Let f be a positive regular free holomorphic function with n indeterminates, and let $m \geq 1$. Then the following statements are equivalent :

(i) $\Psi \in Aut_{ci}^*(A(\mathbf{D}_{f,\mathrm{rad}}^m))$;

(ii) *there is* $\varphi \in Aut(\mathbf{D}_f^m)$ *such that*

$$\Psi(\chi) = \chi \circ \varphi, \quad \chi \in A(\mathbf{D}_{f,\mathrm{rad}}^m).$$

Consequently, $Aut_{ci}^*(A(\mathbf{D}_{f,\mathrm{rad}}^m)) \simeq Aut(\mathbf{D}_f^m)$.

Classification of noncommutative domains

Corollary

The noncommutative domains $\mathbf{D}_f^1(\mathcal{H})$ *and* $\mathbf{D}_g^1(\mathcal{H})$ *are free biholomorphic equivalent if and only if the domain algebras* $\mathcal{A}_n(\mathbf{D}_f^1)$ *and* $\mathcal{A}_q(\mathbf{D}_g^1)$ *are completely isometrically isomorphic. Moreover,*

$$Aut_{ci}^*(A(\mathbf{D}_{f,\mathrm{rad}}^1)) = Aut_{ci}(A(\mathbf{D}_{f,\mathrm{rad}}^1)) \simeq Aut(\mathbf{D}_f^1).$$

- **Remarks.** The case $f = g = X_1 + \cdots + X_n$.
- $Aut_{ci}(\mathcal{A}_n) \simeq Aut(B(\mathcal{H})_1^n)$.
- (Davidson-Pitts '98) $Aut_{ci}(\mathcal{A}_n) \simeq Aut(\mathbb{B}_n)$.
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Classification of noncommutative domains

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Classification of noncommutative domains

Corollary

Let g be a positive regular free holomorphic function with q indeterminates. Then $\mathbf{D}_g^1(\mathcal{H})$ is biholomorphic equivalent to the unit ball $[B(\mathcal{H})^n]_1$ if and only if $q = n$ and $g = c_1 X_1 + \cdots + c_n X_n$ for some $c_i > 0$.

Classification of noncommutative domains

Theorem

A map $\Psi : A(\mathbf{D}^m_{f,\mathrm{rad}}) \to A(\mathbf{D}^l_{g,\mathrm{rad}})$ is a unital completely isometric isomorphism having completely contractive hereditary extension and such that its symbol φ fixes the origin if and only if $n = q$ and Ψ is given by

$$\Psi(\chi) = \chi \circ \varphi, \qquad \chi \in A(\mathbf{D}^m_{f,\mathrm{rad}}),$$

for some $\varphi \in Bih(\mathbf{D}^l_g, \mathbf{D}^m_f)$ of the form $\varphi(X) = [X_1, \ldots, X_n]U$, $X \in \mathbf{D}^l_{g,\mathrm{rad}}(\mathcal{H})$, where U is an invertible operator on $\mathbb{C}^n$ such that

$$[W_1^{(g)}, \ldots, W_n^{(g)}]U \in \mathbf{D}^m_f(F^2(H_n)),$$

and

Classification of noncommutative domains

$$[W_1^{(f)},\ldots,W_n^{(f)}]U^{-1} \in \mathbf{D}_g^l(F^2(H_n)).$$

In this case, we have

$$[\widehat{\Psi}(W_1^{(f)}),\ldots,\widehat{\Psi}(W_n^{(f)})] = \widetilde{\varphi} = [W_1^{(g)},\ldots,W_n^{(g)}]U.$$

- When $m = l = 1$, Arias and Latrémolière proved that if there is a completely isometric isomorphism between $\mathcal{A}_n(\mathbf{D}_f^1)$ and $\mathcal{A}_n(\mathbf{D}_g^1)$, whose dual map fixes the origin, then the algebras are related by a linear relation of their generators. Our Theorem implies and strengthens their result and also provides a converse.

Classification of noncommutative domains

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Noncommutative Hardy algebras

- $H^\infty(\mathbf{D}^m_{f,\mathrm{rad}})$ denotes the set of all elements φ in $Hol(\mathbf{D}^m_{f,\mathrm{rad}})$ such that
$$\|\varphi\|_\infty := \sup \|\varphi(X)\| < \infty,$$
where the sup is taken over all n-tuples $X \in \mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})$.

- $H^\infty(\mathbf{D}^m_{f,\mathrm{rad}})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$, and has a unital operator algebra structure.

Noncommutative Hardy algebras

Theorem

The map $\Phi : H^\infty(\mathbf{D}^m_{f,\mathrm{rad}}) \to F^\infty_n(\mathbf{D}^m_f)$ *defined by*

$\Phi\left(\sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha\right) := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha W_\alpha$ *is a completely isometric isomorphism of operator algebras. Moreover, if* $G := \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha Z_\alpha$ *is a free holomorphic function on* $\mathbf{D}^m_{f,\mathrm{rad}}$*, then the following statements are equivalent :*

(i) $G \in H^\infty(\mathbf{D}^m_{f,\mathrm{rad}})$;

(ii) $\sup_{0\le r<1} \|G(rW_1,\ldots,rW_n)\| < \infty$, *where*

$$G(rW_1,\ldots,rW_n) := \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha r^{|\alpha|} W_\alpha;$$

Noncommutative Hardy algebras

(iii) there exists $\widetilde{G} \in F_n^\infty(\mathbf{D}_f^m)$ with $G = \mathbf{B}[\widetilde{G}]$.

In this case, $\Phi(G) = \widetilde{G} = \text{SOT-}\lim_{r\to 1} G(rW_1, \ldots, rW_n)$ *and*

$$\Phi^{-1}(\varphi) = G = \mathbf{B}[\widetilde{G}], \qquad \widetilde{G} \in F_n^\infty(\mathbf{D}_f^m),$$

where $\mathbf{B}$ *is the noncommutative Berezin transform.*

- $T := (T_1, \ldots, T_n) \in \mathbf{D}_f^m(\mathcal{H})$ is *pure* if

$$\text{SOT-}\lim_{k\to\infty} \Phi_{f,T}^k(I) = 0,$$

where $\Phi_{f,T}(X) = \sum_{k=1}^\infty \sum_{|\alpha|=k} a_\alpha T_\alpha X T_\alpha^*$.

- Define

$$\mathbf{D}_{f,\text{pure}}^m(\mathcal{H}) := \{X \in \mathbf{D}_f^m(\mathcal{H}) : X \text{ is pure}\}.$$

Noncommutative Hardy algebras

(iii) there exists $\widetilde{G} \in F_n^\infty(\mathbf{D}_f^m)$ with $G = \mathbf{B}[\widetilde{G}]$.

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Noncommutative Hardy algebras

- Note that : $\mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H}) \subset \mathbf{D}^m_{f,\mathrm{pure}}(\mathcal{H}) \subset \mathbf{D}^m_f(\mathcal{H})$.
- $Bih(\mathbf{D}^l_{g,\mathrm{pure}}, \mathbf{D}^m_{f,\mathrm{pure}})$ is the set of all bijections

$$\varphi : \mathbf{D}^l_{g,\mathrm{pure}}(\mathcal{H}) \to \mathbf{D}^m_{f,\mathrm{pure}}(\mathcal{H})$$

such that $\varphi|_{\mathbf{D}^l_{g,\mathrm{rad}}(\mathcal{H})}$ and $\varphi^{-1}|_{\mathbf{D}^m_{f,\mathrm{rad}}(\mathcal{H})}$ are free holomorphic functions with their model boundary functions pure, and φ and φ^{-1} are their radial extensions in the strong operator topology, respectively, i.e.,

$$\varphi(X) = \text{SOT-}\lim_{r\to 1} \varphi(rX), \qquad X \in \mathbf{D}^l_{g,\mathrm{pure}}(\mathcal{H}),$$

and

$$\varphi^{-1}(X) = \text{SOT-}\lim_{r\to 1} \varphi^{-1}(rX), \qquad X \in \mathbf{D}^m_{f,\mathrm{pure}}(\mathcal{H}).$$

Unitarily implemented isomorphisms

Theorem

A map $\Psi : H^\infty(\mathbf{D}^m_{f,\mathrm{rad}}) \to H^\infty(\mathbf{D}^l_{g,\mathrm{rad}})$ is a unitarily implemented isomorphism if and only if it has the form

$$\Psi(\chi) := \chi \circ \varphi, \qquad \chi \in H^\infty(\mathbf{D}^m_{f,\mathrm{rad}}),$$

for some $\varphi \in Bih(\mathbf{D}^l_{g,\mathrm{pure}}, \mathbf{D}^m_{f,\mathrm{pure}})$ such that $\tilde{\varphi}$ is unitarily equivalent to the universal model $(W_1^{(f)}, \ldots, W_n^{(f)})$ associated with $\mathbf{D}^m_f$. In this case,

$$\widehat{\Psi}(\tilde{\chi}) = \mathcal{B}_{\tilde{\varphi}}[\tilde{\chi}] := K^{(m)^*}_{f,\tilde{\varphi}}(\tilde{\chi} \otimes I_{\mathcal{D}_{f,m,\tilde{\varphi}}})K^{(m)}_{f,\tilde{\varphi}}, \qquad \tilde{\chi} \in F_n^\infty(\mathbf{D}^m_f),$$

where the noncommutative Berezin kernel $K^{(m)}_{f,\tilde{\varphi}}$ is a unitary operator and $\dim \mathcal{D}_{f,m,\tilde{\varphi}} = 1$.

Unitarily implemented isomorphisms

Remark

If $m = 1$, then $\tilde{\varphi}$ is unitarily equivalent to the universal model $(W_1^{(f)}, \ldots, W_n^{(f)})$ if and only if

i) $\text{SOT-}\lim_{k\to\infty} \Phi_{f,\tilde{\varphi}}^k(I) = 0$

(ii) $\mathit{rank}\,[I - \Phi_{f,\tilde{\varphi}}(I)] = 1$

(iii) *the characteristic function* $\Theta_{\tilde{\varphi}} = 0$.

- $Aut_w(\mathbf{D}_{f,\mathrm{pure}}^m)$ is the group of all $\varphi \in Bih(\mathbf{D}_{f,\mathrm{pure}}^m, \mathbf{D}_{f,\mathrm{pure}}^m)$ such that $\tilde{\varphi}$ is unitarily equivalent to $(W_1^{(f)}, \ldots, W_n^{(f)})$.

Corollary

$$Aut_u(F_n^\infty(\mathbf{D}_f)) \simeq Aut_w(\mathbf{D}_{f,\mathrm{pure}}^m).$$

Unitarily implemented isomorphisms

- The case $m = 1$ and $f = X_1 + \cdots + X_n$.
- (Davidson-Pitts '98) $Aut_u(F_n^\infty) \simeq Aut(\mathbb{B}_n)$.
- (P.'10) $Aut(B(\mathcal{H})_1^n) \simeq Aut(\mathbb{B}_n) \simeq Aut_u(F_n^\infty)$.

Theorem

Let $Aut_w(\mathbf{D}_f^m))$ be the group of all $\psi \in Bih(\mathbf{D}_g^l, \mathbf{D}_f^m)$ such that $\widetilde{\psi}$ is unitarily equivalent to the universal model $(W_1^{(f)}, \ldots, W_n^{(f)})$. Then

$$Aut_u(\mathcal{A}_n(\mathbf{D}_f^m)) \simeq Aut_w(\mathbf{D}_f^m).$$

- THANK YOU