Modeling $p$-adic Whittaker Functions
Whittaker functions

- $F$ – a locally compact field
- $G$ – a split reductive group over $F$
- $B$ – positive Borel subgroup
- $T$ – maximal torus
- $U$ – unipotent radical of $B = TU$
- $\psi$ – nondegenerate character of $U$

Example: $G = \text{GL}_n$

\[
U = \left\{ \begin{pmatrix} 1 & u_{12} & \cdots & u_{n-1,n} \\ & 1 & \cdots & u_{n-2,n} \\ & & \ddots & \ddots \\ & & & 1 \end{pmatrix} \right\}
\]

$\psi(u) = \psi_0(u_{12} + u_{23} + \cdots)$

$\psi_0: F \rightarrow \mathbb{C}$ a nontriv char.

Theorem. (Gelfand-Graev, Shalika, Piatetski-Shapiro) The representation $\text{Ind}_U^G(\psi)$ is multiplicity-free.

A Whittaker model of an irreducible representation $(\pi, V)$ is a space of functions $W_\pi$ on $G$ that satisfy

\[ W(u g) = \psi(u) W(g), \quad u \in U, \]

that is closed under right translation:

\[ W \in W_\pi \quad \Rightarrow \quad \rho(g) W \in W_\pi, \quad \rho(g) W(x) = W(x g) \]

and such that $W_\pi \cong V$ as $G$-modules. The content of the theorem is that the Whittaker model (if it exists) is unique.
Principal Series representations

Let $\chi$ be a character of $T_F$. Extend $\chi$ to $B_F$ (Borel subgroup) and induce:

$$V(\chi) = \{ f: G_F \rightarrow \mathbb{C} \mid f(bg) = (\delta^{1/2} \chi)(b)f(g) \}$$

($\delta$ = modular character of $B_F$)

$G_F$ acts by right translation:

$$\pi(g)f(x) = f(xg)$$

- $V(\chi)$ is usually irreducible.
- If $w \in W$ (Weyl group) $V(\chi)$ and $V(w\chi)$ are isomorphic (if irreducible).

Suppose $F$ is nonarchimedean, $\mathfrak{o} =$ integers in $F$. Let $K = G(\mathfrak{o})$, max’l compact.

- Given any representation $(\pi, V)$, let $V^K =$ space of $K$-fixed vectors.

**Proposition 1.** If $(\pi, V)$ is irreducible $\dim(V^K) \leq 1$.

The irreducible representation is spherical if $\dim(V^K) = 1$.

If $\chi$ is a character of $T_F$, $\chi$ is unramified if $\chi(T_\mathfrak{o}) = 1$.

**Proposition.** If $\chi$ is unramified $V(\chi)$ is spherical.

(If $V(\chi)$ is reducible, it has a unique spherical quotient.)
The L-group

Given a group $G$ there is a group $\hat{G}$ whose root data are dual to $G$.

- $G$ – a split reductive group
- $T$ – maximal split torus in $G$
- $\Phi$ – root system of $G$
- $P$ – weight lattice in $G$
- $\hat{T}$ – maximal split torus in $\hat{G}$
- $\hat{G}$ – the L-group
- $\hat{\Phi}$ – root system of $\hat{G}$
- $P^\vee$ – the weight lattice in $\hat{G}$

| Example: | $G = \text{GL}_n$ | $P = \mathbb{Z}^n$ |
| Example: | $G = \text{SO}_{2r+1}$ | $P^\vee = \mathbb{Z}^r$ |

Assume that the ground field $F$ is nonarchimedean local.

- $\hat{T}(\mathbb{C}) \cong $ group of characters of $T_F/T_0$  \[ z \in \hat{T}(\mathbb{C}) \iff \chi_z \in X(T_F/T_0) \]
- $T_F/T_0 \cong $ coweight lattice $P^\vee$  \[ \lambda \in P^\vee \iff t_{\lambda^\vee} \in T_F \]
  (if $G$ is of adjoint type, otherwise $\subseteq P^\vee$.)
- Dominant $\lambda \in P^\vee$ parametrize irreducible characters of $\hat{G}(\mathbb{C})$  \[ \lambda \in P^\vee \iff \xi_\lambda, \text{ irreducible} \]

If $z \in \hat{T}(\mathbb{C})$, we may consider the induced representation $V(\chi_z)$. 
Duality

Recap: (Semisimple) (spherical)
Conjugacy classes of $\hat{G}(\mathbb{C})$ correspond to irreps of $G(F)$
(Semisimple) (finite-dimensional)
Conjugacy classes of $G(F)$ correspond to irreps of $\hat{G}(\mathbb{C})$
(Not bijectively: $t_\lambda$ is only determined up to multiplication by a unit.)

- $z \in \hat{T}(\mathbb{C})$
  - L-group torus element
  - $\chi z \in X(T_F/T_0)$

- $z \in \hat{T}(\mathbb{C})$
  - $V(\chi z) = \text{Ind}(\delta^{1/2} \chi z)$
  - (If irreducible – usually)

- $z' = wz$ ($w \in W$)
  - $V(\chi z) \cong V(\chi z')$

- $\lambda \in P^\vee$
  - (Coweight)
  - $t_\lambda \in T_F/T_0$

- $\lambda \in P^\vee$
  - (Dominant weight)
  - $\xi_\lambda$ irr char of $\hat{G}(\mathbb{C})$
  - L-group elements index unramified chars of $T_F$
  - and by induction, irreps of $G_F$. Conjugate $z$
  - index isomorphic $V(\chi)$. Elements of $T_F/T_0$ are
  - indexed by coweights; dominant coweights
  - index irreps of $\hat{G}(\mathbb{C})$. Each conjugacy class
  - contains a unique coset $t_\lambda \mod T_0$ with
  - $\lambda$ dominant.
Casselman-Shalika Formula

Let \( z \in \hat{T}(\mathbb{C}) \). Let \( W_{\hat{z}} \) be the spherical vector in the Whittaker model of \( V(\chi_{\hat{z}}) \). Langlands conjectured that the values of \( W_{\hat{z}} \) are the values of irreducible characters of \( \hat{G} \). This was proved by Shintani, S. Kato, Casselman and Shalika and is referred to as the \textbf{Casselman-Shalika formula}.

**Theorem.** We have

\[
W_{\hat{z}}(t_{\lambda^\vee}) = \begin{cases} \text{const} \times \delta^{1/2}(t_{\lambda^\vee}) \chi_{\lambda}(z) & \text{if } \lambda^\vee \text{ is dominant}, \\ 0 & \text{otherwise} \end{cases}
\]

In a natural normalization the constant is \( \prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha) \). More precisely, we may define \( W_{\hat{z}} \) as an integral, thus:

\[
W_{\hat{z}}(g) = \int_{U_F} f^\circ(w_0ug)\psi(u)^{-1}du, \quad w_0 = \text{long } W \text{ element},
\]

where \( f^\circ(bk) = \delta^{1/2}\chi(b), \) \( b \in B_F, k \in K \). Then \( \text{const} = \prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha) \).
Why Seek Other Models?

The Casselman-Shalika formula is the complete story for the spherical Whittaker function. **Why look any further?**

- The constant $\prod_{\alpha \in \Phi^+} (1 - q^{-1}z^\alpha)$ is a deformation of Weyl’s denominator. So we seek a deformation of the Weyl character formula.

- The study of such deformations leads us to crystal bases and statistical (ice-type) models.

- Furthermore such models work for *metaplectic Whittaker functions* where the Casselman-Shalika formula does not apply.

Suppose that $F \supset \mu_n$ (the $n$-th roots of unity). Weil, Kubota and Matsumoto defined a *metaplectic cover* which is a central extension

$$1 \to \mu_n \to \tilde{G}(F) \to G(F) \to 1.$$  

The cover splits over $U(F)$ so one may still consider Whittaker models.

- Uniqueness of Whittaker models fails. **Still spherical Whittaker functions have expressions in terms of crystal or ice models.**
Deformations of the Weyl Character formula

A deformations of the Weyl character formula was found by Tokuyama (1988). Others considered deformations of the Weyl denominator.

- Kuperberg, Okada, Simpson, Hamel and King.
- Beineke, Brubaker, Bump, Chinta, Friedberg, Frechette, Gunnells, Ivanov, Tabony.

There are different ways of writing Tokuyama’s formula.

- Sum over strict Gelfand-Tsetlin patterns (original paper).
- Sum over crystal $B_{\lambda+\rho}$. 

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha
\]

- Six-vertex model.

The last two approaches are subtly different suggesting different tools.
Weyl Characters

Let $\mathcal{G}$ be a complex Lie group. **Note:** Eventually $\mathcal{G}$ will be $\hat{\mathcal{G}}(\mathbb{C})$ so $\Phi$ will become $\Phi^\vee$ (coroots) and $P$ will become $P^\vee$ coweights.

- Let $\lambda \in P$ be dominant. Let $\xi_\lambda$ be the irr character of highest weight $\lambda$.
- Decompose $\xi_\lambda$ into a sum of weights with multiplicities.

Example: $G = \text{GL}_3(\mathbb{C})$, $P = \mathbb{Z}^3$. $\lambda = (3, 1, 0)$

<table>
<thead>
<tr>
<th>Elements of $P$</th>
<th>Positive Weyl Chamber (dominant weights)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Weights with multiplicity 1</td>
</tr>
<tr>
<td></td>
<td>Weights with multiplicity 2</td>
</tr>
</tbody>
</table>

Observe that the “weight diagram” is invariant under $W$ (which is the group generated by the reflections in the two hyperplanes bounding the positive Weyl chamber).
**Root operators**

Let $G$ be a complex Lie group. Let $P$ be the weights (char’s of max’l torus $T$).

**Note:** Eventually $G$ will be $\hat{G}(\mathbb{C})$ so $\Phi$ will become $\Phi^\vee$ (coroots) and $P$ will become $P^\vee$ coweights.

- $\Phi$ – The root system
- $\Phi^+$ – The positive roots
- $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$ – The simple roots
- $V$ – A $G$-module
- $\mu \in P$ – a weight of $G$.
- $V(\mu)$ – The weight space

A positive root is called **simple** if it cannot be decomposed as a sum of other positive roots.

We have $V = \bigoplus_{\mu \in P} V(\mu)$.

If $X \in \text{Lie}(G)$ then $X$ acts on $V$. Let $\alpha \in \Phi$ and $X_\alpha \in \text{Lie}(G)$ be in the one-dimensional root eigenspace. Then

$$X_\alpha: V(\mu) \longrightarrow V(\mu + \alpha).$$

We choose $E_i = X_{\alpha_i}$ and $F_i = X_{-\alpha_i}$ to be the Chevalley generators. Then

$$E_i: V(\mu) \longrightarrow V(\mu + \alpha_i), \quad F_i: V(\mu) \longrightarrow V(\mu - \alpha_i).$$
Crystals

A (Kashiwara) crystal is a combinatorial substitute for $V(\mu)$. The crystal $\mathcal{B}_\lambda$ of highest weight $\lambda$ is a set with cardinality $\dim(V(\mu))$.

- It is equipped with a weight map $\text{wt}: \mathcal{B}_\lambda \to P$.
- The number of $\mathcal{B}_\lambda$ with weight $\mu$ is $m(\mu) = \dim V(\mu)$.
- Root operators $E_i, F_i: \mathcal{B}_\lambda \to \mathcal{B}_\lambda \cup \{0\}$ are defined.
- If $E_i(v) = w \neq 0$ then $F_i(w) = v$ and $\text{wt}(v) = \text{wt}(w) + \alpha_i$.

Following Kashiwara and Nakashima, if $\Phi = A_r$ the elements of $\mathcal{B}_\lambda$ are semistandard Young tableaux of shape $\lambda$ in the alphabet $\{1, 2, 3, \ldots, r\}$. These are fillings of the Young diagram with shape $\lambda$ by elements of the alphabet with weakly increasing rows and strictly increasing columns, like this:

```
 1 1 2 3 4
 2 3
 3
```
**Example: GL₃**

Here is the crystal with highest weight $\lambda = (3, 1, 0)$. Compare it with the weight diagram (above) for $V(\lambda)$.

Kashiwara: elements of $\mathcal{B}_\lambda$ are labeled by tableaux of shape $\lambda$ in \{1, 2, 3\}.

If $F_i(v) = w$ and $E_i(w) = v$ we draw an arrow $v \xrightarrow{i} w$.

We have drawn the crystal so that the elements of equal weight overlap.

The crystal is mapped to the weight diagram (left) by $\text{wt}: \mathcal{B}_\lambda \rightarrow P$. 
Tokuyama functions

By a **Tokuyama function** on the crystal $\mathcal{B}_{\lambda+\rho}$ we mean a function

$$G: \mathcal{B}_{\lambda+\rho} \times \mathbb{C} \to \mathbb{C}$$

such that

$$\sum_{v \in \mathcal{B}_{\lambda+\rho}} G(v, t) z^{wt(v)} = \left[ \prod_{\alpha \in \Phi^+} (1 + tz^\alpha) \right] \xi_\lambda(z).$$

- If $t = -1$ the formula should reduce to the Weyl character formula.
- If $t = 0$, then $G(v, t)$ should vanish unless $v$ is in the image of a map $\mathcal{B}_\lambda \to \mathcal{B}_{\lambda+\rho}$, and the formula should reduce to $\xi_\lambda(z) = \sum_{v \in \mathcal{B}_\lambda} z^{wt(v)}$.
- If $t = -q^{-1}$ the formula should give the Casselman-Shalika formula (with deformed Weyl denominator.)
- There are also **metaplectic Tokuyama functions**. These produce not characters but **metaplectic Whittaker functions**.
- “Natural” Tokuyama functions can be given in many cases, beginning with Tokuyama (1988). The Tokuyama function is not unique. Using results of McNamara one gets **one Tokuyama function for each reduced word** decomposing the long Weyl group element into simple reflections.
Statistical Models

Solvable lattice models in statistical mechanics are 2-dimensional systems in which the partition function can be evaluated explicitly. The first example was the Ising model, solved by Onsager (1944). The six-vertex model is an important example.

- Solved by Lieb and Sutherland in the 1960’s.
- Baxter developed the **star-triangle relation** or **Yang-Baxter equation** as a powerful tool.
- Hamel and King showed how characters (together with deformed Weyl denominators) are **partition functions** of systems of this type.
- Brubaker, Bump and Friedberg showed how to use the Yang-Baxter equation to investigate these models.
- Metaplectic Whittaker functions can also be represented as such partition functions.
Six-Vertex Model

We describe a statistico-physical system $\mathcal{S}$. Take a square lattice of finite size.

For Example:

To specify the system, we require some further data.

- Signs or spins $\pm$ on the boundary edges are fixed.
- At each vertex $v$ there are assigned six values $a_1(v), a_2(v), b_1(v), b_2(v), c_1(v), c_2(v)$ which are also part of the data defining the system.
States

- A state $s$ of the system $\mathcal{G}$ consists of an assignment of signs $\pm$ to the interior edges.
- Recall that the signs of the boundary edges are fixed.

For example, here is a state of the system shown earlier.

We will also consider more general planar graphs in which some of the edges are rotated.
The Partition Function

Given a state of the system, every vertex \( v \) is assigned a value \( \beta_s(v) \), its **Boltzmann weight**. This is either zero or one of the six values \( a_1(v), a_2(v), b_1(v), b_2(v), c_1(v), c_2(v) \).

- If the weight does not appear in the table it is **zero**.
- Given the state \( s \), the **Boltzmann weight** \( \beta(s) = \prod_v \beta_s(v) \).
- The **Partition function** \( Z(\mathcal{G}) = \sum_{s \text{ states}} \beta(s) \).
Transfer Matrices

Let \( v \) be a vertex type with Boltzmann weights \( a_i(v), b_i(v), c_i(v) \). Define

\[
V_v(\alpha, \gamma) = Z \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\gamma_1 & \gamma_2 & \cdots & \gamma_n
\end{pmatrix}
\]

where \( \alpha_i, \gamma_i \in \{ \pm \} \). There are \( 2^n \) possibilities for \( \alpha = (\alpha_1, \ldots, \alpha_n) \), so we think of \( V_v \) as being a \( 2^n \times 2^n \) matrix, the row transfer matrix for \( v \). Clearly

\[
Z \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\beta_1 & \beta_2 & \cdots & \beta_n
\end{pmatrix} = \sum_{\gamma} V_v(\alpha, \gamma)V_w(\gamma, \beta).
\]

We may compute the partition function by multiplying transfer matrices!
Baxter

- Baxter: organize transfer matrices into commuting families.
- A maximal commuting family of operators is like a maximal torus.
- This leads to evaluation of the partition function.

Example: the Field-Free Case
Suppose $a_1(v) = a_2(v) = a(v)$, $b_1(v) = b_2(v) = b(v)$, $c_1(v) = c_2(v) = c(v)$. Let

$$\Delta(v) = \frac{a(v)^2 + b(v)^2 - c(v)^2}{2a(v)b(v)}.$$ 

**Theorem. (Baxter)** If $\Delta(v) = \Delta(w)$ then $V_v$ and $V_w$ commute.

**Proof.** Use the Yang-Baxter equation. □
The Yang-Baxter Equation

Let $v, w, r$ be three types of vertices, with Boltzmann weights $a_i(x), b_i(x), c_i(x)$ for $x \in \{v, w, r\}$. Then we write $[r, v, w] = 0$ if for all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6 \in \{\pm\}$:

$$Z \begin{pmatrix} 
\varepsilon_2 \\
\varepsilon_1 \\
\varepsilon_6 \\
\varepsilon_5 \\
\varepsilon_4 \\
\varepsilon_3 \\
\end{pmatrix}
\begin{pmatrix} r \\
v \\
w \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\end{pmatrix}
= Z \begin{pmatrix} 
\varepsilon_2 \\
\varepsilon_1 \\
\varepsilon_6 \\
\varepsilon_5 \\
\varepsilon_4 \\
\varepsilon_3 \\
\end{pmatrix}
\begin{pmatrix} r \\
w \\
v \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\end{pmatrix}.$$ 

This means that summing over the three unlabeled edges gives the same result on both sides.

**Lemma. (Baxter)** If $\Delta(v) = \Delta(w) = \Delta$ there exists a third field-free vertex $r$ with $\Delta(r) = \Delta$ such that $[r, v, w] = 0$. 


The R-matrix in action ...

To prove Baxter’s commutativity in the field-free case, that $Z(\mathcal{G})$,

\[
\mathcal{G} = \begin{pmatrix}
\alpha_1 & v & \alpha_2 & v & \ldots & v \\
+ & + & v & + & + & + \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_n \\
\end{pmatrix}
\]

is unchanged if $v$ and $w$ are interchanged, attach the \textbf{R-matrix} vertex $r$:

\[
Z \begin{pmatrix}
\alpha_1 & v & \alpha_2 & v & \ldots & v \\
+ & + & v & + & + & + \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_n \\
\end{pmatrix} = a_1(r)Z(\mathcal{G}),
\]

because $a_1(r)$ is the value of $\oplus_r$, the only legal configuration at $r$. 


... Yang-Baxter equation $n$ times ...

$$
Z \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
v & v & \cdots & v \\
w & w & \cdots & w \\
\beta_1 & \beta_2 & \cdots & \beta_n
\end{pmatrix}
= a_2(r) Z(\mathcal{G}'),
$$

where $\mathcal{G}'$ is the system $\mathcal{G}$ with $v$ and $w$ interchanged. Since $a_1(r) = a_2(r)$ we may cancel them and get $Z(\mathcal{G}) = Z(\mathcal{G}')$.

That is, the transfer matrices $V_v$ and $V_w$ commute, as promised.
Parametrized Yang-Baxter Equation

Let $\Delta \in \mathbb{C}$ be fixed and let $R_\Delta$ be the set of field-free Boltzmann weights

$$a_1 = a_2 = a, \quad b_1 = b_2 = b, \quad c_1 = c_2 = c, \quad \frac{a^2 + b^2 - c^2}{2ab} = \Delta.$$  

Recall:

**Lemma.** (Baxter) If $\Delta(v) = \Delta(w) = \Delta$ there exists a third field-free vertex $r$ with $\Delta(r) = \Delta$ such that $[r, v, w] = 0$.

We have actually a **parametrized Yang-Baxter equation**.

**Theorem.** (Baxter) There is a map $R: \mathbb{C}^\times \rightarrow R_\Delta$ such that

$$[R(t), R(tu), R(u)] = 0. \quad \text{So } r = R(t), v = R(tu), w = R(u).$$

Discard the field free assumption and impose **free Fermionic condition**. Let:

$$R_{ff} = \{v | a_1(v)a_2(v) + b_1(v)b_2(v) - c_1(v)c_2(v) = 0\}.$$

**Lemma.** (BBF) There is a map $R: \text{GL}_1(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \rightarrow R_{ff}$ such that

$$[R(t), R(tu), R(u)] = 0.$$
The Yang-Baxter Commutator

Let $V = \mathbb{C}^2$ with basis $+$ and $−$. Given $T \in \text{End}(V \otimes V)$ let

$$T(\varepsilon_i \otimes \varepsilon_j) = \sum_{k, l} T_{ij}^{kl} \varepsilon_k \otimes \varepsilon_l \quad (\varepsilon_i \in \{ \pm \}).$$

We interpret the coefficients $T_{ij}^{kl}$ as a Boltzmann weight of $\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4$. With respect to basis $+ \otimes +$, $+ \otimes −$, $− \otimes +$, $− \otimes −$ of $V \otimes V$ the vertex $v$ is

$$v$$

the linear transformation with matrix

$$
\begin{pmatrix}
    a_1(v) & b_1(v) & c_1(v) \\
    c_2(v) & b_2(v) & a_2(v)
\end{pmatrix}.
$$

If $T \in \text{End}(V \otimes V)$ let $T_{ij} \in \text{End}(V \otimes V \otimes V)$ be $T$ acting on the $i$-th and $j$-th components and $I_V$ acts on the $k$-th component ($k \neq i, j$). The Yang-Baxter commutator is

$$[A, B, C] = A_{12} B_{13} C_{23} - C_{23} B_{13} A_{12}.$$
Quantum Groups

With this framework many (Faddeev, Kulish, Sklyanin, Kirillov, Reshetikhin, Takhtadjan, Jimbo, Miwa, Drinfeld, ...) sought an explanation for the Yang-Baxter equation. This led to the invention of Quantum groups.

The explanation for Baxter’s parametrized YBE

\[ [R(t), R(tu), R(u)] = 0 \]

is that \( V(t) \) is a module for the Hopf algebra \( H = U_q(\hat{\mathfrak{sl}}_2) \) (completed) and there is an element \( R \in H \otimes H \) that induces an endomorphism \( V(t) \otimes V(u) \) for every pair of modules. The quantum group \( H \) is a quasitriangular Hopf algebra which means that \( R \) satisfies conditions implying the Yang-Baxter equation.

**Question 1:** Give a similar treatment of the Free fermionic case.

**Question 2:** Extend the free Fermionic story to the eight vertex model.
Schur Polynomials

- Hamel and King extended Tokuyama’s deformation for Cartan Type $A_r$ by giving a generalized deformation. They also treated Cartan Type $C_r$.
- Brubaker, Bump and Friedberg found two families of deformations, one of which is Hamel and King’s. These are called Gamma ice and Delta ice.
- They gave proofs based on the Yang-Baxter equation.
- Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$.
- Let $z_1, \ldots, z_n \in \mathbb{C}^\times$ be spectral parameters.
- Let $t_1, \ldots, t_n \in \mathbb{C}$ be deformation parameters.

The character $\xi_\lambda(z)$ is the Schur polynomial $s_\lambda(z)$.

There are two statistical systems $\mathcal{G}_\lambda^\Gamma$ and $\mathcal{G}_\lambda^\Delta$ with

$$Z(\mathcal{G}_\lambda^\Gamma) = \prod_{i<j} (t_i z_j + z_i) s_\lambda(z_1, \ldots, z_n), \quad Z(\mathcal{G}_\lambda^\Delta) = \prod_{i<j} (t_j z_j + z_i) s_\lambda(z_1, \ldots, z_n).$$
Gamma Ice

Label columns 0, 1, 2, \ldots from right to left, rows 1, 2, 3, \ldots, \( n \) top to bottom.

Use these weights in the \( i \)-th row:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a_1(i) )</td>
<td>( a_2(i) )</td>
<td>( b_1(i) )</td>
<td>( b_2(i) )</td>
<td>( c_1(i) )</td>
<td>( c_2(i) )</td>
</tr>
<tr>
<td>1</td>
<td>( z_i )</td>
<td>( t_i )</td>
<td>( z_i )</td>
<td>( z_i(t_i + 1) )</td>
<td>1</td>
</tr>
</tbody>
</table>

+ on left and bottom boundary edges,
− on right boundary edges.

On top edges in \( \rho + \lambda \) put −,
On remaining top edges put +.

**Theorem:** 
\[
Z(\mathcal{G}_\lambda^\Gamma) = \prod_{i < j} (t_i z_j + z_i)s_\lambda(z_1, \ldots, z_n).
\]

**Example:** 
\( \lambda = (2, 1, 0, 0) \)
\( \lambda + \rho = (5, 3, 1, 0) \)
Tokuyama Function

Recall that a **Tokuyama function** is a map $G: \mathcal{B}_{\lambda + \rho} \times \mathbb{C} \rightarrow \mathbb{C}$ such that

$$
\left[ \prod_{\alpha \in \Phi^+} (1 + t z^\alpha) \right] \xi_\lambda(z) = \sum_{v \in \mathcal{B}_{\lambda + \rho}} G(v, t) z^{wt(v)}.
$$

- There are different Tokuyama functions (one for every reduced decomposition of the long Weyl group element) but we discuss a particular one.
- This Tokuyama function has another description using an embedding of Berenstein, Zelevinsky, Lusztig, Littelmann (BZL) of $\mathcal{B}_{\lambda + \rho}$ into cones.

We will describe an injection

$$
c: \{\text{states of } \mathfrak{S}_\lambda^\Gamma\} \rightarrow \mathcal{B}_{\lambda + \rho}.
$$

When all $t_i = t$

$$
\beta(s) = G(v, t) z^{wt(v)}, \quad v = c(s),
$$
and $G(v, t) = 0$ if $v$ is not in the image of $c$. Thus the **nonzero** terms in the crystal description coincide with the states of the ice.
A subtle shift of viewpoint

The nonzero terms in the crystal description coincide with the states of the statistical model. So we might think that the two descriptions are identical. However there is a subtle shift of viewpoint between the two pictures.

- The image of $c$ is (in some sense) most but not all of $\mathcal{B}_{\lambda+\rho}$.
- The tool sets are different in the two pictures since $c$ is not bijective.
- The image of $c$ is not stable under the Schützenberger involution of $\mathcal{B}_{\lambda+\rho}$, so that involution has no significance in the statistical picture.
- But the Yang-Baxter equation is not available in the crystal picture.
- An aggravating fact about the crystal picture is that the terms in the sum are usually invariant under the Schützenberger involution, yet there are some on the boundary of the BZL polytopes that are not invariant under the involution. Using the involution to understand the sums leads one to group these exceptional terms together in packets resulting in Combinatorial Difficulties.
- These are surmountable but the Yang-Baxter equation is a welcome alternative.
Associate a Gelfand-Tsetlin pattern with a State

- Identify states with strict Gelfand-Tsetlin Patterns.

A Gelfand-Tsetlin Pattern is a triangular array of partitions of descending length whose rows interleave.

For each row, write down the column numbers of vertices in the above 3 configurations (having a $-$ above the vertex). **Example:**

The pattern is strict meaning each row is strictly decreasing.
Associate a tableaux with that Gelfand-Tsetlin P.

Striking all \( n \)’s, then all \( n - 1 \)’s, etc. from a tableau gives a sequence of shapes.

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 \\
3 \\
\end{array} \Rightarrow \begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 \\
3 \\
\end{array} \Rightarrow \begin{array}{ccc}
1 & 1 & 2 \\
2 \\
\end{array} \Rightarrow \begin{array}{c}
1 & 1 \\
\end{array}
\]

\{5, 2, 1, 0\} \rightarrow \{4, 2, 1\} \rightarrow \{3, 2\} \rightarrow \{2\}

Taking those shapes and arranging them gives a Gelfand-Tsetlin pattern:

\[
\left\{ \begin{array}{cccc}
5 & 2 & 1 & 0 \\
4 & 2 & 1 \\
3 & 1 \\
2 \\
\end{array} \right\}
\]

Thus

- States correspond to strict Gelfand-Tsetlin patterns with top row \( \lambda + \rho \).
- Gelfand-Tsetlin patterns biject with tableaux with shape \( \lambda + \rho \).
- Not all patterns are strict so the map \( c \) is an injection but not a bijection.
Metaplectic Ice

- For Type A and arbitrary metaplectic covers, there are ice models.
- Key facts amount to commutativity of transfer matrices.
- Still the Yang-Baxter equation remains elusive for Type A.

But there is a model for the Whittaker function on the metaplectic double cover of $Sp_4(F)$ where the Yang-Baxter equation plays a significant role.

Use Delta ice on Blue rows
Use Gamma ice on Red rows
For the “cap vertices” use the metaplectic weights:

(for the $i$-th pair of rows)
Related Nonmetaplectic Work

- Related to **U-Turn Ice** and **Alternating sign matrices** of Kuperberg, Okada and Hamel and King.

- Those models are related to work of Beineke, Brubaker and Frechette on crystal models for Type C (nonmetaplectic).

- Thesis of Dmitriy Ivanov introduces **Yang-Baxter equation** in such models introducing a novel **caduceus relation** which we will discuss.
The Caduceus

The so-called Caduceus braid

bears a noted resemblance to the fabled “staff of Hermes”

We have attached a caduceus braid preparing to prove a functional equation with respect to the first simple reflection in the Weyl group.

- The caduceus braid first appeared in the thesis of D. Ivanov.
- This multiplies $Z(\mathcal{G})$ by $(t z_j + z_i^{-1})(z_i + t z_j)(t z_i^{-1} + z_j^{-1})(t z_i + z_j^{-1})$.
- Using the Yang-Baxter equation, the caduceus moves to the right.
The Caduceus Identity

**Lemma.** For any \(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\) we have

\[
\begin{pmatrix}
\varepsilon_4 \\
\varepsilon_3 \\
\varepsilon_2 \\
\varepsilon_1
\end{pmatrix}
= \text{const} \times
\begin{pmatrix}
\varepsilon_4 \\
\varepsilon_3 \\
\varepsilon_2 \\
\varepsilon_1
\end{pmatrix}
\]

where the constant is \((tz_i + z_j^{-1})(tz_i + z_j)(tz_i^{-1} + z_j^{-1})(tz_j + z_i^{-1})\), independent of the \(\varepsilon_i\).

- Discarding the caduceus this way shows how the partition function changes under the interchange of spectral parameters.

- We are aware of caduceus identities for three different sets of cap weights. (The archetype is in Ivanov’s thesis.)