Descartes' rule of signs.

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Descartes' rule of signs is easy.

Let $f = \sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ be a non-zero polynomial of degree d.

- $R(f)$ is the number of positive roots of f counted with multiplicities.
- $S(f)$ is the number of changes of signs in the sequence of coefficients of f, ignoring the zeros.

Theorem (Descartes (1637) - Gauss (1828))

$R(f) \leq S(f)$ and $S(f) - R(f)$ is even.

Descartes' rule of signs is correct.

Proven by Gauss (1828), Albert (1943), Wang (2004), ...
The proofs are based on:

Lemma

$S((x-1)f(x)) \geq S(f) + 1.$

Lemma

- $a_0 a_d > 0 \implies S(f)$ *and* $R(f)$ *are both even.*
- $a_0 a_d < 0 \implies S(f)$ *and* $R(f)$ *are both odd.*

Descartes' rule of signs is sharp.

- If $f = (x - r_1) \cdots (x - r_n) \in \mathbb{R}[x]$ where $r_i > 0 \, \forall \, i$, then $S(f) = R(f) = n$.
- [Grabiner (1999)] For any sequence of signs (no zeros), there exists a non-zero $f \in \mathbb{R}[x]$ with coefficients of the given signs and $S(f) = R(f)$.

Descartes' rule of signs is inexact.

- If $f = x^2 + bx + c \in \mathbb{R}[x]$ where $b < 0$ and $c > b^2/4$, then $S(f) = 2$ and $R(f) = 0$.
- [Anderson, Jackson, Sitharam (1998)] For any sequence of signs or zeros with n changes of signs and an even integer k such that $0 \leq k \leq n$, there exists a non-zero $f \in \mathbb{R}[x]$ with coefficients of the given signs and $R(f) = n - k$.

Descartes' rule of signs is almost exact.

- [Poincare (1888)] There exists $g \in \mathbb{R}[x]$, that depends on f, such that $R(f) = S(fg)$.
- [Polya (1928)] If f has no positive roots, then there exists $n \in \mathbb{N}_0$ such that $S((x+1)^n f(x)) = 0$.
- [Powers, Reznick (2007)] If f has no positive roots and

$$n > \binom{d}{2} \frac{\max_{0 \le i \le d}\{a_i / \binom{d}{i}\}}{\min_{\lambda \in [0,1]}\{(1-\lambda)^d f\left(\frac{\lambda}{1-\lambda}\right)\}} - d$$

then $S((x+1)^n f(x)) = 0$.

Descartes' rule of signs is exact!

Theorem (Avendano (2009))

For any non-zero $f \in \mathbb{R}[x]$*, the sequence* $S((x+1)^n f(x))$ *is monotone decreasing and it stabilizes at* $R(f)$*.*

Intuitive proof I

Recall that $f = a_d x^d + \cdots + a_1 x + a_0$.

- Then $(x+1)^n f(x) = c_n^{n+d} x^{n+d} + \cdots + c_n^1 x + c_n^0$ where

$$c_n^k = \sum_{i=0}^{d} a_i \binom{n}{k-i}.$$

- Encode the (signs of the) coefficients c_n^k in the piecewise constant functions $g_n : [0,1) \to \mathbb{R}$ given by

$$g_n(\lambda) = \binom{n+d}{[\lambda(n+d+1)]}^{-1} c_n^{[\lambda(n+d+1)]}.$$

- $\operatorname{sgn}(c_n^k) = \operatorname{sgn}(g_n(k/(n+d+1)))$.

Example I

Consider the polynomial

$$f = (x-2)(x-7)(9x^6 - x^5 + 2x^4 - 4x^3 + 2x^2 + 4x + 1)$$
$$= 9x^8 - 82x^7 + 137x^6 - 36x^5 + 66x^4 - 70x^3 - 7x^2 + 47x + 14.$$

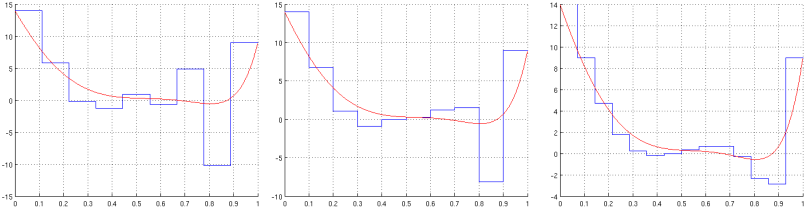

Figure: Functions $g_0(\lambda)$, $g_1(\lambda)$ and $g_5(\lambda)$ compared with $g(\lambda)$.

Example II

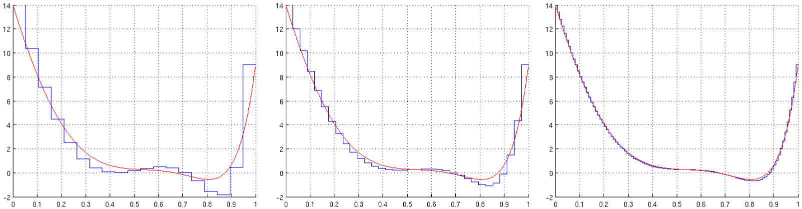

Figure: Functions $g_{10}(\lambda)$, $g_{25}(\lambda)$ and $g_{100}(\lambda)$ compared with $g(\lambda)$.

Intuitive proof II

- Show that the sequence of functions $\{g_n\}_{n\geq 0}$ converge uniformly to
$$g(\lambda) = (1-\lambda)^d f\left(\frac{\lambda}{1-\lambda}\right)$$
in the interval $[0, 1)$.
- Note that the homography $\lambda \mapsto \frac{\lambda}{1-\lambda}$ is a bijection from $[0, 1)$ to $[0, \infty)$. Its inverse is given by $x \mapsto \frac{x}{x+1}$.
- For large enough n, the number of sign alternations in c_n^k is equal to the number of changes of signs of $g(\lambda)$, i.e. the number of positive roots of f.

What else?

- The n required to get $S((x+1)^n f) = R(f)$ is usually (very) large. An analysis of the optimal n is in progress.
- Can we change $x+1$ by some other polynomial?
- For large enough n, the coefficients of $(x+1)^n f(x)$ and the values of f, after some normalization, almost coincide. Can we use this for finding roots?
- The proof uses that a Binomial probability distribution can be approximated well by a Poisson distribution. Also, we are multiplying by powers of $(x+1)$. Is this technique related with random walks?

What is a Descartes' rule of signs?

Let $\mathfrak{M}$ be the set of sequences of real numbers indexed by the non-negative integers, with finite support. We use this sequences to encode the coefficients of polynomials in $\mathbb{R}[x]$.

Consider a function $\hat{S} : \mathfrak{M} \to \mathbb{N}_0$ such that:

1. $\hat{S}(\square\square * a) \leq \hat{S}(a)$
2. $\hat{S}(a) \geq$ "positive regions in a" $+$ "negative regions in a" $- 1$

for all $a \in \mathfrak{M}$. Then $\hat{S}$ is a DRS, i.e. $R(f) \leq \hat{S}(f)$ for all $f \in \mathbb{R}[x]$.

Here $*$ denotes convolution of sequences (or multiplication of polynomials) and $\square\square$ corresponds to the binomial $1 + x$.

Is there any other Descartes' rule of signs?

Yes, sure!

Define $\hat{S}(a)$ as the number of times the sequence changes from $+$ to $-$ plus twice the number of changes from $-$ to $+$. This gives a DRS.

Want more?

For any sequence $a \in \mathfrak{M}$ define $\hat{a} \in \mathfrak{M}$ by

$$\hat{a}_n = \sum_{i=n}^{\infty} a_i \binom{i}{n} (-1)^{i-n}.$$

Then the function $\hat{S} : \mathfrak{M} \to \mathbb{N}_0$ given by $\hat{S}(a) = S(\hat{a})$ is a DRS.

How to go to several variables?

Let $\mathfrak{M}_2$ denote the set of two-dimensional sequences (indexed by $\mathbb{N}_0 \times \mathbb{N}_0$) of real numbers with finite support. Consider a function $\hat{S} : \mathfrak{M}_2 \to \mathbb{N}_0$ such that

1. $\hat{S}(\square\square * a) \leq \hat{S}(a)$
2. $\hat{S}(\substack{\square \\ \square} * a) \leq \hat{S}(a)$
3. $\hat{S}(a) \geq$ "positive regions in a" + "negative regions in a"

for all $a \in \mathfrak{M}_2$. Then $\hat{S}$ gives a DRS in two variables, i.e. for any non-zero $f \in \mathbb{R}[x, y]$, it gives an upper bound for the number of connected components of the complement of the zero set of f.

Is there any DRS in two variables?

Yes, sure!

For any $a \in \mathfrak{M}_2$ define $Q(a) =$ "positive regions in a" + "negative regions in a" and

$$\hat{S}(a) = \max_{n,m \geq 0} Q(\square\square^n * {\substack{\square \\ \square}}^m * a).$$

The function $\hat{S}$ is a DRS in two variables.

Is there any DRS in two variables with a simple formula?