Mixed Volume Computation in Solving Polynomial Systems

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How to solve:

$$P(x) = x^5 + 2x^4 - 4x^3 + x - 5 = 0$$

Pick random complex numbers c_1, c_2, c_3, c_4, c_5 and consider

$$Q(x) = x^5 + 2x^4 - 4x^3 + x - 5 = 0$$

Form the homotopy

$$H(x,t) = (1-t)\gamma Q(x) + tP(x) = 0$$

$$H(x,1) \equiv P(x) \qquad H(x,0) \equiv Q(x)$$

Theorem

For almost all choice of $c_1, \ldots, c_5$, this homotopy "works".

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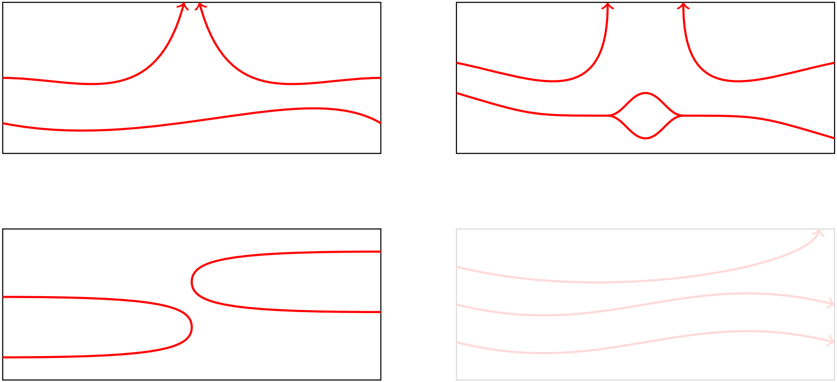

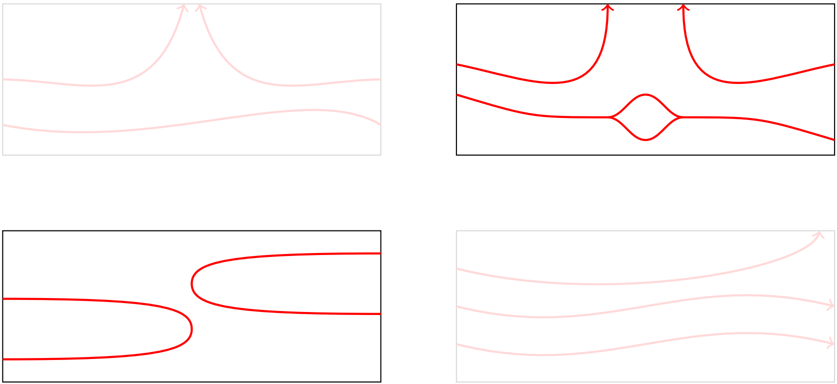

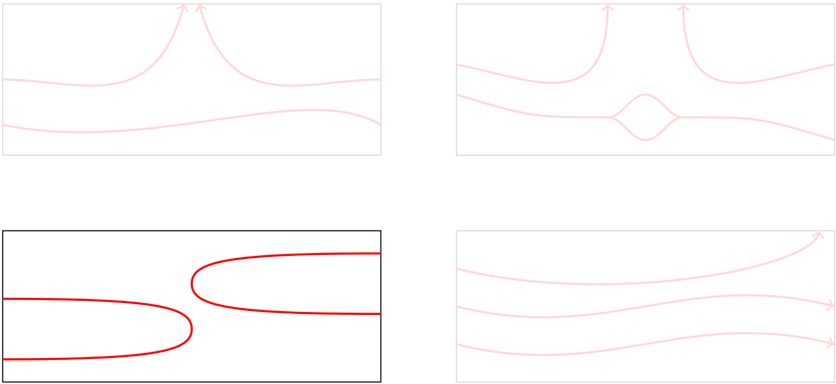

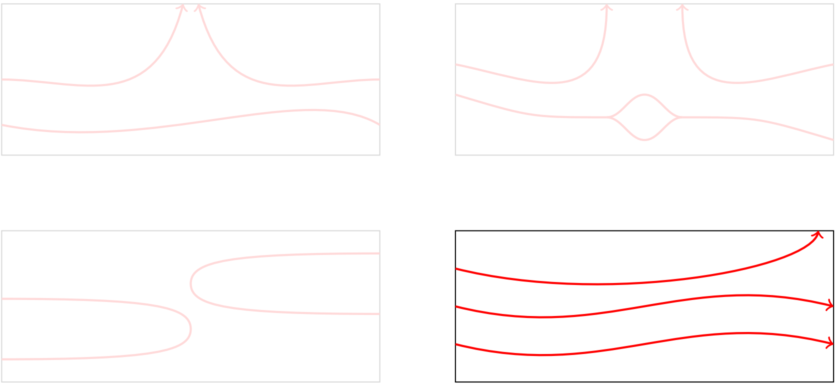

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Equation of 2 terms: can be solved easily, no matter the degree.

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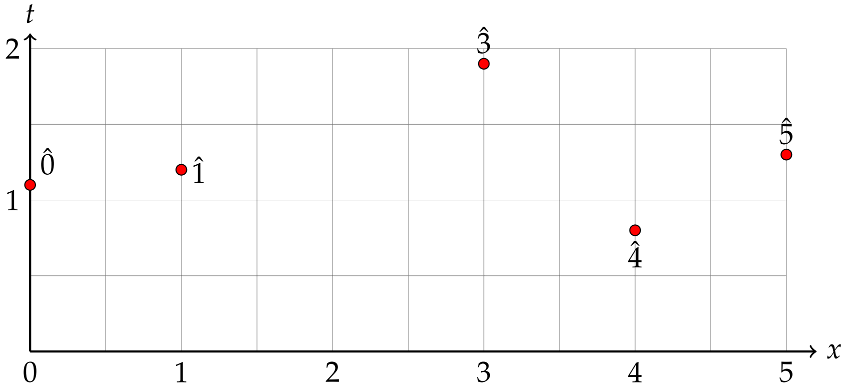

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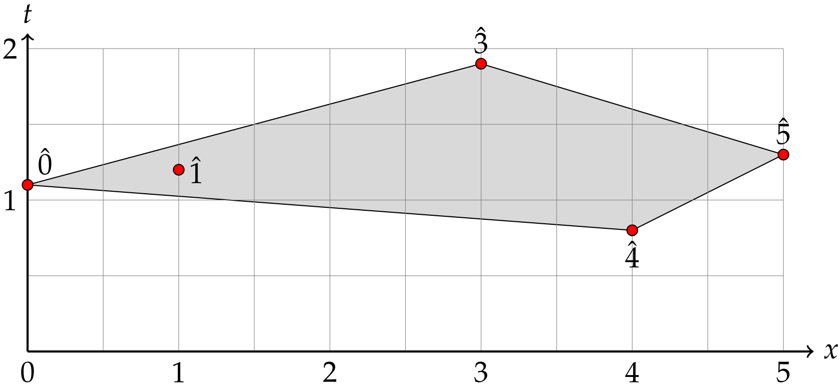

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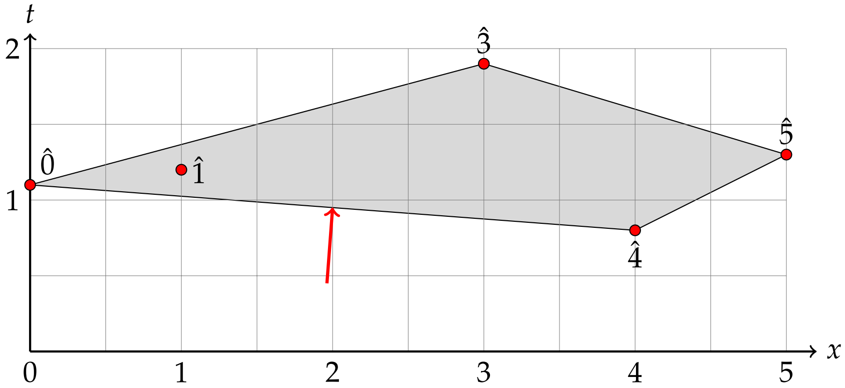

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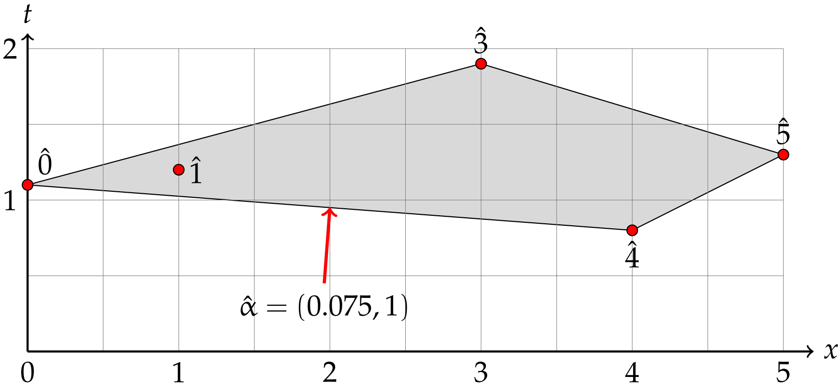

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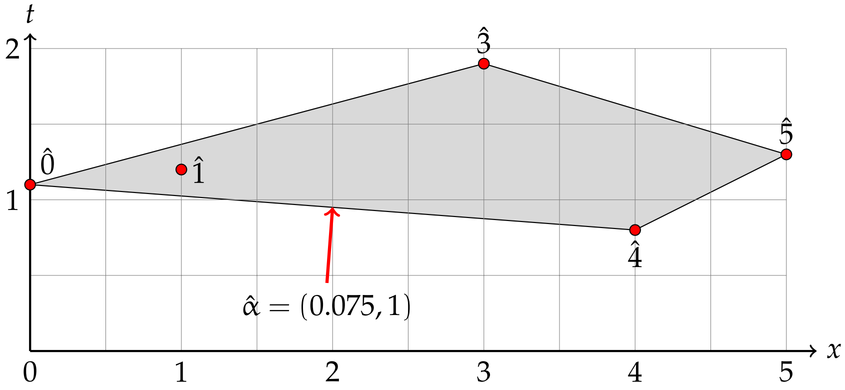
$\langle \hat{1}, \hat{\alpha} \rangle = 1.275$ $\langle \hat{0}, \hat{\alpha} \rangle = 1.1$



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Solution: Use change of variables with $\alpha = 0.075$

$$x = yt^{\alpha}$$

Note that

$$\text{at} \quad t = 1 \qquad\qquad x = y$$

Then

$$\begin{aligned}
H(x,t) &= c_1x^5t^{1.3} + c_2x^4t^{0.8} + c_3x^3t^{1.9} + c_4xt^{1.2} + c_5t^{1.1} \\
&= c_1y^5t^{5\alpha+1.3} + \cdots \\
&= c_1y^5t^{1.675} + c_2y^4t^{1.1} + c_3y^3t^{2.125} + c_4y^1t^{1.275} + c_5t^{1.1} \\
&= t^{1.1}(c_1y^5t^{1.675-1.1} + c_2y^4 + c_3y^3t^{2.125-1.1} + c_4y^1t^{1.275-1.1} + c_5) \\
&= t^{1.1}[c_2y^4 + c_5 + (\text{terms with positive powers of } t)] \\
H^{\alpha}(y,t) &= t^{-1.1}H(yt^{\alpha},t) \\
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Now the new starting system

$$H^{\alpha}(y,0) = 0$$
$$c_2 y^4 + c_5 = 0$$
$$y^4 = -c_5/c_2$$

can be solved and it generally has 4 solutions. Hope:

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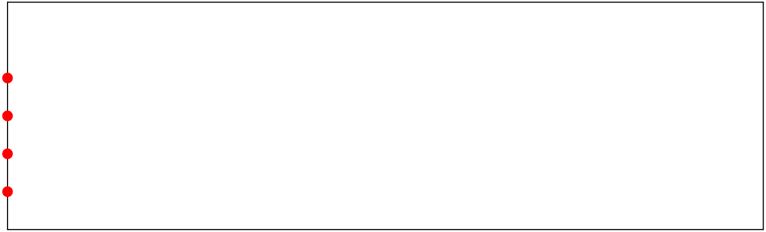

$t = 0$ $t = 1$

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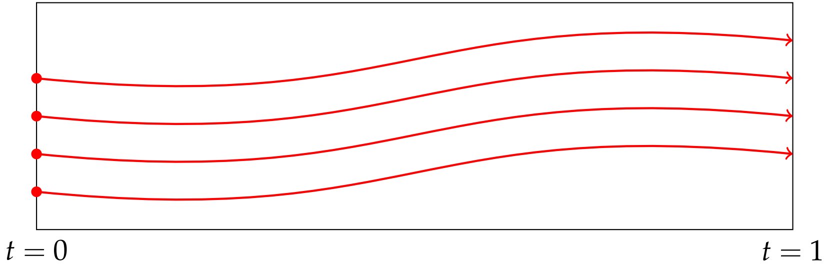

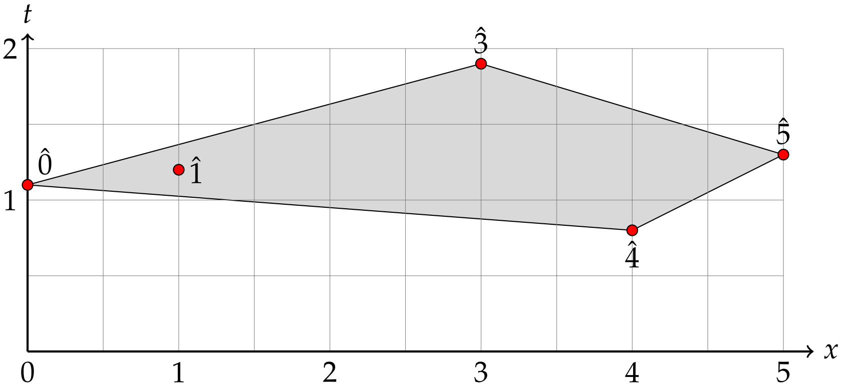

$\langle \hat{5}, \hat{\alpha} \rangle = -1.2$ $\langle \hat{4}, \hat{\alpha} \rangle = -1.2$ $\langle \hat{3}, \hat{\alpha} \rangle = 0.4$

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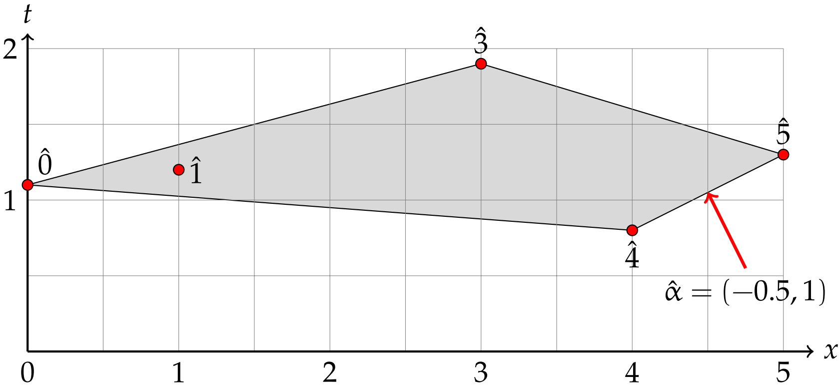

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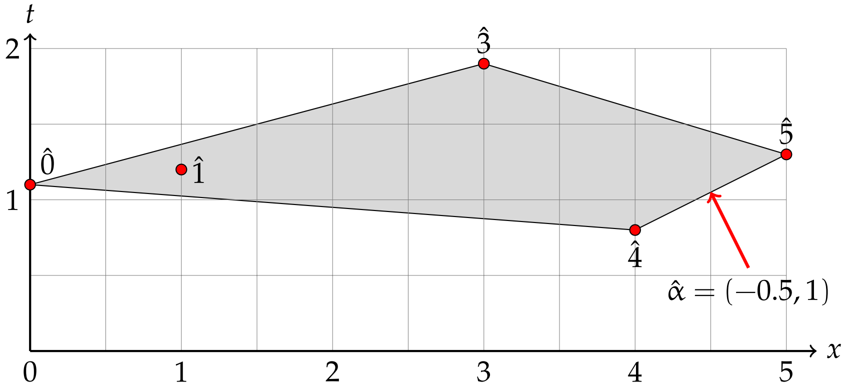

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Solution: Use change of variables with $\alpha = -0.5$

$$x = yt^{\alpha}$$

Note that

at $t = 1$ $x = y$

$$H(x,t) = c_1 x^5 t^{1.3} + c_2 x^4 t^{0.8} + c_3 x^t 1.9 + c_4 x t^{1.2} + c_5 t 1.1$$

$$H(yt^{\alpha},t) = c_1 y^5 t^{-1.2} + c_2 y^4 t^{-1.2} + c_3 y^3 t^{0.4} + c_4 y^1 t^{0.7} + c_5 t^{1.1}$$

$$= t^{-1.2}[c_1 y^5 + c_2 y^4 + (\text{terms with positive powers of } t)]$$

$$H^{\alpha}(y,t) = t^{-(-1.2)} H(yt^{\alpha},t)$$

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Solution: Use change of variables with $\alpha = -0.5$

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Note that

at $t = 1$ $\qquad$ $x = y$

$$\begin{aligned} H(x,t) &= c_1 x^5 t^{1.3} + c_2 x^4 t^{0.8} + c_3 x^t 1.9 + c_4 x t^{1.2} + c_5 t 1.1 \\ H(yt^{\alpha},t) &= c_1 y^5 t^{-1.2} + c_2 y^4 t^{-1.2} + c_3 y^3 t^{0.4} + c_4 y^1 t^{0.7} + c_5 t^{1.1} \\ &= t^{-1.2}[c_1 y^5 + c_2 y^4 + (\text{terms with positive powers of } t)] \end{aligned}$$

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$$H^{\alpha}(y,0) = c_1y^5 + c_2y^4$$

$$H^{\alpha}(y,0) = 0$$
$$c_1 y^5 + c_2 y^4 = 0$$
$$c_1 y^5 = -c_2 y^4$$
$$y = -c_2/c_1$$

Similarly, for almost all choices of $c_1 \ldots, c_5$, the homotopy works

$t = 0$ $t = 1$

$$
\begin{aligned}
H^{\alpha}(y,0) &= 0 \\
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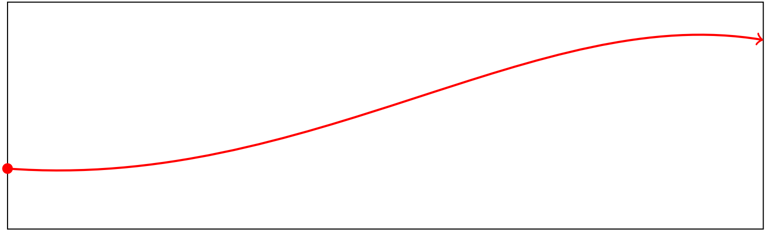
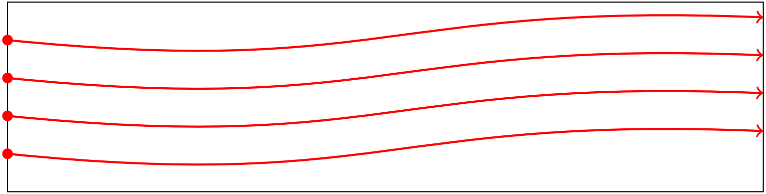
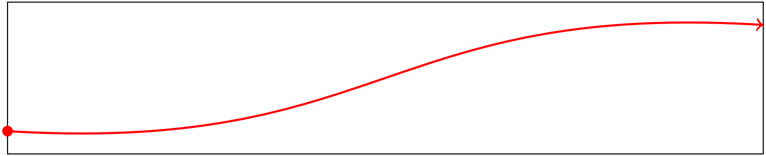


$t = 0$ $t = 1$

Together, we get all 5 solutions of $Q(x) = 0$.

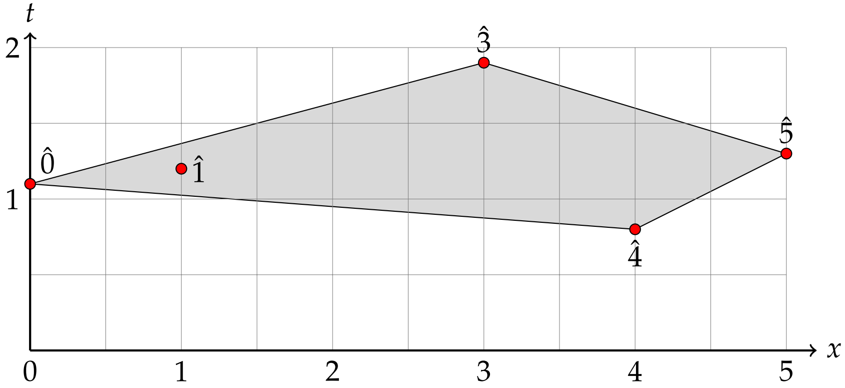


$t = 0$ $\alpha = 0.075$ $t = 1$

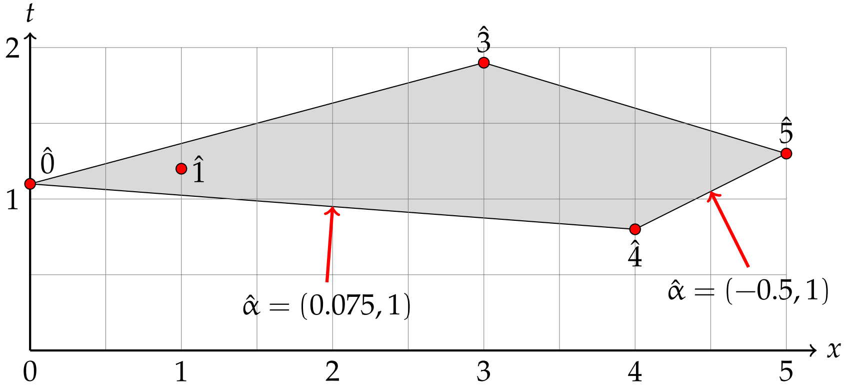


$t = 0$ $\alpha = -0.5$ $t = 1$

How to find α



How to find α



General Construction (to solve $P(x) = 0$)

To solve a system of polynomial equations $P(x) = 0$

$$\begin{cases} p_1(x_1, \ldots, x_n) = \sum_{a \in S_1} c_{1,a} x_1^{a_1} \ldots x_n^{a_n} = \sum_{a \in S_1} c_{1,a} x^a = 0 \\ \vdots \\ p_n(x_1, \ldots, x_n) = \sum_{a \in S_n} c_{n,a} x_1^{a_1} \ldots x_n^{a_n} = \sum_{a \in S_n} c_{n,a} x^a = 0 \end{cases}$$

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$$P(x) = x^5 + 2x^4 - 4x^3 + x - 5 = 0$$

$$\downarrow$$

$$Q(x) = c_1 x^5 + c_2 x^4 + c_3 x^3 + c_4 x + c_5$$

$$\downarrow$$

$$\tilde{H}(x,t) = (1-t)\gamma Q(x) + tP(x)$$

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$\downarrow$ $\downarrow$

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$$\downarrow$$

$$H(x,0) \equiv 0$$

$$Q(x) = c_1x^5 + c_2x^4 + c_3x^3 + c_4x + c_5$$

$$\downarrow$$

$$H(x,t) = c_1x^5t^{1.3} + c_2x^4t^{0.8} + c_3x^3t^{1.9} + c_4xt^{1.2} + c_5t^{1.1}$$

$$\downarrow$$

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Binomial system

$$\begin{aligned} \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} &= 0, \\ &\vdots \\ \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} &= 0. \end{aligned}$$

1. It can be solved constructively and efficiently
2. The number of isolated zeros in $(\mathbf{C}^*)^n$

$$= \left| \det \begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{12} \\ \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n2} \end{pmatrix} \right|$$

Binomial system

$$
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&\vdots \\
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How to solve $Q(\mathbf{x}) = (q_1(\mathbf{x}), \ldots, q_n(\mathbf{x}))$

$$\begin{aligned} q_1(\mathbf{x}) &= \sum_{\mathbf{a} \in S_1} \bar{c}_{1,\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \\ &\vdots \\ q_n(\mathbf{x}) &= \sum_{\mathbf{a} \in S_n} \bar{c}_{n,\mathbf{a}} \mathbf{x}^{\mathbf{a}}. \end{aligned}$$

The polyhedral homotopy:

Choose $\omega_j : S_j \to \mathbf{R}, \quad j = 1, \ldots, n$

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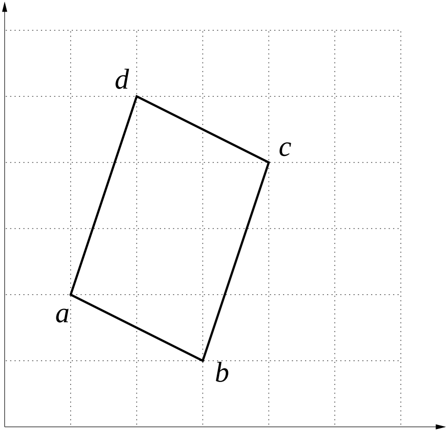
$$S_1,\ S_2,\ \ldots\ S_n \ \subset\ \mathbb{N}_0^n$$



S_i

$$\omega_i : S_i \to \mathbb{R}, \quad i = 1, \ldots, n$$

$$\hat{S}_i = \{\hat{a} = (a, \omega_i(a)) \mid a \in S_i\}$$

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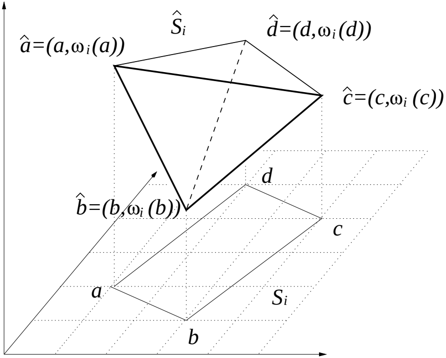
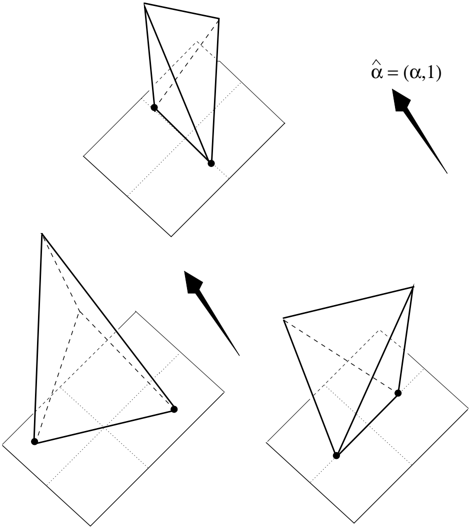


Problem: Look for hyperplane with normal $\hat{\alpha} = (\alpha, 1)$ which supports each $\hat{S}_i$ at exactly 2 points



Looking for $\alpha \in \mathbf{R}^n$, and pairs

$$\begin{aligned} \{\mathbf{a}_{11}, \mathbf{a}_{12}\} &\subset S_1, \\ &\vdots \\ \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\} &\subset S_n \end{aligned}$$

such that

$$\begin{aligned} \langle \hat{\alpha}, \hat{\mathbf{a}}_{11} \rangle &= \langle \hat{\alpha}, \hat{\mathbf{a}}_{12} \rangle < \langle \hat{\alpha}, \hat{\mathbf{a}} \rangle, \forall \mathbf{a} \in S_1 \backslash \{\mathbf{a}_{11}, \mathbf{a}_{12}\}, \\ &\vdots \\ \langle \hat{\alpha}, \hat{\mathbf{a}}_{n1} \rangle &= \langle \hat{\alpha}, \hat{\mathbf{a}}_{n2} \rangle < \langle \hat{\alpha}, \hat{\mathbf{a}} \rangle, \forall \mathbf{a} \in S_n \backslash \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\}. \end{aligned}$$

where $\quad \hat{\alpha} = (\alpha, 1), \quad \hat{\mathbf{a}} = (\mathbf{a}, \omega(\mathbf{a}))$

The **Mixed Volume** computation.

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$$

Consider the coordinate transformation

$$\begin{aligned} x_1 &= y_1 t^{\alpha_1}, \\ &\vdots \\ x_n &= y_n t^{\alpha_n}, \end{aligned} \qquad \mathbf{x} = \mathbf{y} t^{\alpha}.$$

$$\begin{aligned} \mathbf{x}^{\mathbf{a}} t^{\omega_i(\mathbf{a})} &= x_1^{a_1} \cdots x_n^{a_n} t^{\omega_i(\mathbf{a})} \\ &= (y_1 t^{\alpha_1})^{a_1} \cdots (y_n t^{\alpha_n})^{a_n} t^{\omega_i(\mathbf{a})} \\ &= y_1^{a_1} \cdots y_n^{a_n} t^{\alpha_1 a_1 + \cdots + \alpha_n a_n + \omega_i(\mathbf{a})} \\ &= \mathbf{y}^{\mathbf{a}} t^{\langle \alpha, \mathbf{a} \rangle + \omega_i(\mathbf{a})} \\ &= \mathbf{y}^{\mathbf{a}} t^{\langle \hat{\alpha}, \hat{\mathbf{a}} \rangle}. \end{aligned}$$

$$\hat{\alpha} = (\alpha, 1), \;\; \hat{\mathbf{a}} = (\mathbf{a}, \omega_i(\mathbf{a})).$$

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$$\hat{\alpha} = (\alpha, 1), \;\; \hat{\mathbf{a}} = (\mathbf{a}, \omega_i(\mathbf{a})).$$

$$\mathbf{x} = \mathbf{y}t^{\alpha}$$

$$\mathbf{x}^{\mathbf{a}}t^{\omega_i(\mathbf{a})} = \mathbf{y}^{\mathbf{a}}t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle}$$

$$\hat{\alpha} = (\alpha, 1), \quad \hat{\mathbf{a}} = (\mathbf{a}, \omega_i(\mathbf{a}))$$

$$\begin{aligned} q_1(\mathbf{x}, t) &= \sum_{\mathbf{a}\in S_1} \bar{c}_{1,\mathbf{a}}\mathbf{x}^{\mathbf{a}}t^{\omega_1(\mathbf{a})} = 0, \\ &\vdots \\ q_n(\mathbf{x}, t) &= \sum_{\mathbf{a}\in S_n} \bar{c}_{n,\mathbf{a}}\mathbf{x}^{\mathbf{a}}t^{\omega_n(\mathbf{a})} = 0. \end{aligned}$$

$$\begin{aligned} h_1(\mathbf{y}, t) = q_1(\mathbf{y}t^{\alpha}, t) &= \sum_{\mathbf{a}\in S_1} \bar{c}_{1,\mathbf{a}}\mathbf{y}^{\mathbf{a}}t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle} = 0, \\ &\vdots \\ h_n(\mathbf{y}, t) = q_n(\mathbf{y}t^{\alpha}, t) &= \sum_{\mathbf{a}\in S_n} \bar{c}_{n,\mathbf{a}}\mathbf{y}^{\mathbf{a}}t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle} = 0. \end{aligned}$$

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For each $\alpha \in \mathbf{R}^n$, we have pairs

$$\{\mathbf{a}_{11}, \mathbf{a}_{12}\} \subset S_1, \ \{\mathbf{a}_{21}, \mathbf{a}_{22}\} \subset S_2, \ \cdots, \ \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\} \subset S_n$$

such that

$$\begin{aligned} \beta_1 &= \langle \hat{\alpha}, \hat{\mathbf{a}}_{11} \rangle = \langle \hat{\alpha}, \hat{\mathbf{a}}_{12} \rangle < \langle \hat{\alpha}, \hat{\mathbf{a}} \rangle, \ \forall \mathbf{a} \in S_1 \backslash \{\mathbf{a}_{11}, \mathbf{a}_{12}\}, \\ &\vdots \\ \beta_n &= \langle \hat{\alpha}, \hat{\mathbf{a}}_{n1} \rangle = \langle \hat{\alpha}, \hat{\mathbf{a}}_{n2} \rangle < \langle \hat{\alpha}, \hat{\mathbf{a}} \rangle, \ \forall \mathbf{a} \in S_n \backslash \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\}. \end{aligned}$$

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$$\bar{H}(\mathbf{y},t) =$$

$$\begin{cases} \bar{h}_1(\mathbf{y},t) = t^{-\beta_1} h_1(\mathbf{y},t) \\ = \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} + \displaystyle\sum_{\mathbf{a}\in S_1\backslash\{\mathbf{a}_{11},\mathbf{a}_{12}\}} \bar{c}_{1,\mathbf{a}}\mathbf{y}^{\mathbf{a}} t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle-\beta_1} = 0, \\ \quad \vdots \\ \bar{h}_n(\mathbf{y},t) = t^{-\beta_n} h_n(\mathbf{y},t) \\ = \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} + \displaystyle\sum_{\mathbf{a}\in S_n\backslash\{\mathbf{a}_{n1},\mathbf{a}_{n2}\}} \bar{c}_{n,\mathbf{a}}\mathbf{y}^{\mathbf{a}} t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle-\beta_n} = 0. \end{cases}$$

$$\bar{H}(\mathbf{y},0) = \begin{cases} \bar{h}_1(\mathbf{y},0) &=& \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} = 0, \\ & \vdots & \\ \bar{h}_n(\mathbf{y},0) &=& \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} = 0. \end{cases}$$

A binomial system

$$\bar{H}(\mathbf{y},t) =$$

$$\begin{cases} \bar{h}_1(\mathbf{y},t) = t^{-\beta_1} h_1(\mathbf{y},t) \\ = \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} + \displaystyle\sum_{\mathbf{a}\in S_1\setminus\{\mathbf{a}_{11},\mathbf{a}_{12}\}} \bar{c}_{1,\mathbf{a}}\mathbf{y}^{\mathbf{a}} t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle-\beta_1} = 0, \\ \quad\vdots \\ \bar{h}_n(\mathbf{y},t) = t^{-\beta_n} h_n(\mathbf{y},t) \\ = \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} + \displaystyle\sum_{\mathbf{a}\in S_n\setminus\{\mathbf{a}_{n1},\mathbf{a}_{n2}\}} \bar{c}_{n,\mathbf{a}}\mathbf{y}^{\mathbf{a}} t^{\langle\hat{\alpha},\hat{\mathbf{a}}\rangle-\beta_n} = 0. \end{cases}$$

$$\bar{H}(\mathbf{y},0) = \begin{cases} \bar{h}_1(\mathbf{y},0) & = & \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} = 0, \\ & \vdots & \\ \bar{h}_n(\mathbf{y},0) & = & \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} = 0. \end{cases}$$

A binomial system

$$\bar{H}(\mathbf{y}, t) = 0$$

(1) Solve the binomial system

$$\bar{H}(\mathbf{y}, 0) = \begin{cases} \bar{c}_{11}\mathbf{y}^{\mathbf{a}_{11}} & + \quad \bar{c}_{12}\mathbf{y}^{\mathbf{a}_{12}} = 0, \\ & \vdots \\ \bar{c}_{n1}\mathbf{y}^{\mathbf{a}_{n1}} & + \quad \bar{c}_{n2}\mathbf{y}^{\mathbf{a}_{n2}} = 0. \end{cases}$$

The number of solution in $(\mathbf{C}^*)^n$:

$$k_\alpha = \left| \det \begin{pmatrix} \mathbf{a}_{11} - \mathbf{a}_{12} \\ \vdots \\ \mathbf{a}_{n1} - \mathbf{a}_{n2} \end{pmatrix} \right|$$

(2) Follow paths $\mathbf{y}(t)$ of $\bar{H}(\mathbf{y}, t) = 0$ emanating from the solution in (1) to reach k_α number of $\mathbf{y}(1)$'s. They are solutions of $P(\mathbf{x}) = 0$.

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$$\sum_\alpha k_\alpha = \text{mixed volume of } S = (S_1, \ldots, S_n).$$

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$$S_1 = \{a_1, \ldots, a_{10}\}$$

$$S_2 = \{b_1, \ldots, b_6\} \qquad \subset \ \mathbf{N}_0^3$$

$$S_3 = \{c_1, \ldots, c_7\}$$

$$Q_1 = \mathrm{conv}(S_1),$$

$$Q_2 = \mathrm{conv}(S_2),$$

$$Q_3 = \mathrm{conv}(S_3)$$

$\mathrm{Vol}_3(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3)$ is a 3rd degree homogeneous polynomial in $(\lambda_1, \lambda_2, \lambda_3)$.

Mixed Volume: The coefficient of $\lambda_1 \cdot \lambda_2 \cdot \lambda_3$.

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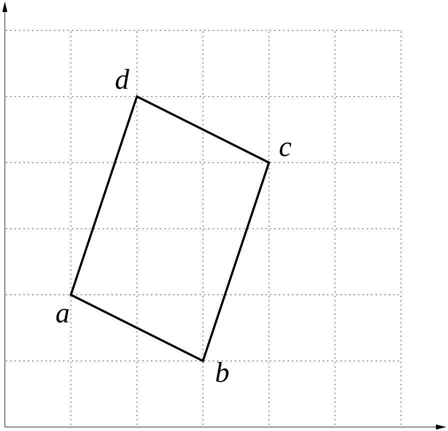
$$S_1,\ S_2,\ S_3 \subset \mathbf{N}_0^3$$



S_i

$$\omega_i : S_i \to \mathbb{R}, \quad i = 1, 2, 3$$

$$\hat{S}_i = \{\hat{a} = (a, \omega_i(a)) \mid a \in S_i\}$$

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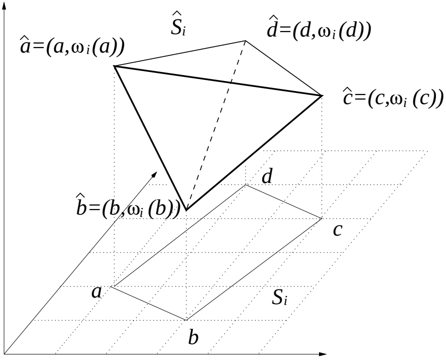
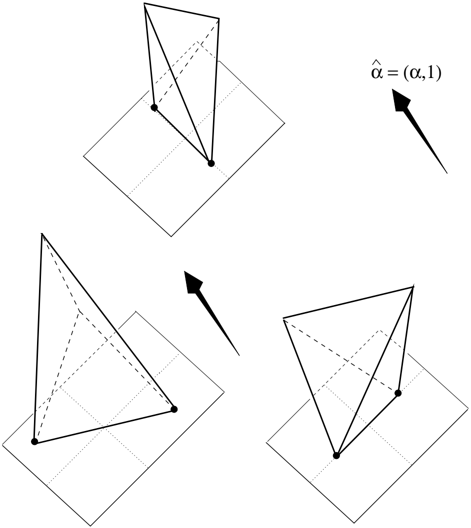


Problem: Look for hyperplane with normal $\hat{\alpha} = (\alpha, 1)$ which supports each $\hat{S}_i$ at exactly 2 points



Look for $\alpha \in \mathbb{R}^3$ satisfying

$$\exists \quad 2 \text{ points in } \hat{S}_1, \quad \text{say } \{\hat{a}_1, \hat{a}_2\}$$

$$2 \text{ points in } \hat{S}_2, \quad \text{say } \{\hat{b}_3, \hat{b}_4\}$$

$$2 \text{ points in } \hat{S}_3, \quad \text{say } \{\hat{c}_5, \hat{c}_6\}$$

such that for $\hat{\alpha} = (\alpha, 1)$

$$\langle \hat{a}_1, \hat{\alpha} \rangle = \langle \hat{a}_2, \hat{\alpha} \rangle < \langle \hat{a}_i, \hat{\alpha} \rangle, \quad i \neq 1, 2$$

$$\langle \hat{b}_3, \hat{\alpha} \rangle = \langle \hat{b}_4, \hat{\alpha} \rangle < \langle \hat{b}_i, \hat{\alpha} \rangle, \quad i \neq 3, 4$$

$$\langle \hat{c}_5, \hat{\alpha} \rangle = \langle \hat{c}_6, \hat{\alpha} \rangle < \langle \hat{c}_i, \hat{\alpha} \rangle, \quad i \neq 5, 6$$

$(\{a_1, a_2\}, \{b_3, b_4\}, \{c_5, c_6\})$ **— A Mixed Cell**

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$(\{a_1, a_2\}, \{b_3, b_4\}, \{c_5, c_6\})$ — **A Mixed Cell**

$$(\{a_1, a_2\}, \{b_3, b_4\}, \{c_5, c_6\})$$

— a mixed cell w.r.t. $\alpha \in \mathbb{R}^3$

$$\sum_{\alpha} \left| \det \begin{pmatrix} a_1 - a_2 \\ b_3 - b_4 \\ c_5 - c_6 \end{pmatrix} \right|$$

= Mixed Volume

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= Mixed Volume

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$$
\begin{array}{ccccc}
\hat{S}_1 & & \hat{S}_2 & & \hat{S}_3 \\
(\hat{a}_1, \hat{a}_2) & \to & (\hat{b}_1, \hat{b}_2) & \to & (\hat{c}_1, \hat{c}_2) \\
(\hat{a}_1, \hat{a}_5) & & (\hat{b}_1, \hat{b}_4) & & (\hat{c}_3, \hat{c}_5) \\
\vdots & & \vdots & & \vdots \\
(\hat{a}_6, \hat{a}_9) & & & &
\end{array}
$$

$(a_1, a_2), (b_1, b_2) : \ \exists\, ? \quad \alpha \in \mathbb{R}^3, \quad \hat{\alpha} = (\alpha, 1)$

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$$\hat{S}_1 \qquad \hat{S}_2 \qquad \hat{S}_3$$

$$(\hat{a}_1, \hat{a}_2) \rightarrow (\hat{b}_1, \hat{b}_2) \rightarrow (\hat{c}_1, \hat{c}_2)$$

$$(\hat{a}_1, \hat{a}_5) \qquad (\hat{b}_1, \hat{b}_4) \qquad (\hat{c}_3, \hat{c}_5)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(\hat{a}_6, \hat{a}_9)$$

$(a_1, a_2), (b_1, b_2)$: $\exists\,?\quad \alpha \in \mathbb{R}^3, \quad \hat{\alpha} = (\alpha, 1)$

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$$\langle \hat{b}_1, \hat{\alpha} \rangle = \langle \hat{b}_2, \hat{\alpha} \rangle \leqslant \langle \hat{b}_k, \hat{\alpha} \rangle$$

$$
\begin{array}{ccccc}
\hat{S}_1 & & \hat{S}_2 & & \hat{S}_3 \\
(\hat{a}_1, \hat{a}_2) & \rightarrow & (\hat{b}_1, \hat{b}_2) & \rightarrow & (\hat{c}_1, \hat{c}_2) \\
(\hat{a}_1, \hat{a}_5) & & (\hat{b}_1, \hat{b}_4) & & (\hat{c}_3, \hat{c}_5) \\
\vdots & & \vdots & & \vdots \\
(\hat{a}_6, \hat{a}_9) & & & &
\end{array}
$$

$$(a_1, a_2), (b_1, b_2) : \ \exists\, ? \quad \alpha \in \mathbb{R}^3, \quad \hat{\alpha} = (\alpha, 1)$$

$$\langle \hat{a}_1, \hat{\alpha} \rangle = \langle \hat{a}_2, \hat{\alpha} \rangle \leqslant \langle \hat{a}_j, \hat{\alpha} \rangle$$

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$$\begin{aligned}\langle \hat{a}_1, \hat{\alpha} \rangle &= \langle \hat{a}_2, \hat{\alpha} \rangle \leqslant \langle \hat{a}_i, \hat{\alpha} \rangle, \; i \neq 1,2 \\ \langle \hat{b}_3, \hat{\alpha} \rangle &= \langle \hat{b}_4, \hat{\alpha} \rangle \leqslant \langle \hat{b}_i, \hat{\alpha} \rangle, \; i \neq 3,4\end{aligned}$$

$$\begin{aligned}\langle a_1 - a_2, \alpha \rangle &= \omega_1(a_2) - \omega_1(a_1) \\ \langle a_1 - a_3, \alpha \rangle &\leqslant \omega_1(a_3) - \omega_1(a_1) \\ &\vdots \\ \langle a_1 - a_{10}, \alpha \rangle &\leqslant \omega_1(a_{10}) - \omega_1(a_1) \\ \langle b_3 - b_4, \alpha \rangle &= \omega_2(b_4) - \omega_2(b_3) \\ \langle b_3 - b_i, \alpha \rangle &\leqslant \omega_2(b_i) - \omega_2(b_3), \; i \neq 3,4\end{aligned}$$

$$
\begin{aligned}
\langle \hat{a}_1, \hat{\alpha} \rangle &= \langle \hat{a}_2, \hat{\alpha} \rangle \leqslant \langle \hat{a}_i, \hat{\alpha} \rangle, \; i \neq 1, 2 \\
\langle \hat{b}_3, \hat{\alpha} \rangle &= \langle \hat{b}_4, \hat{\alpha} \rangle \leqslant \langle \hat{b}_i, \hat{\alpha} \rangle, \; i \neq 3, 4
\end{aligned}
$$

$$
\begin{aligned}
\langle a_1 - a_2, \alpha \rangle &= \omega_1(a_2) - \omega_1(a_1) \\
\langle a_1 - a_3, \alpha \rangle &\leqslant \omega_1(a_3) - \omega_1(a_1) \\
&\vdots \\
\langle a_1 - a_{10}, \alpha \rangle &\leqslant \omega_1(a_{10}) - \omega_1(a_1) \\
\langle b_3 - b_4, \alpha \rangle &= \omega_2(b_4) - \omega_2(b_3) \\
\langle b_3 - b_i, \alpha \rangle &\leqslant \omega_2(b_i) - \omega_2(b_3), \; i \neq 3, 4
\end{aligned}
$$

$$\begin{aligned}
\langle \hat{a}_1, \hat{\alpha} \rangle &= \langle \hat{a}_2, \hat{\alpha} \rangle \leqslant \langle \hat{a}_i, \hat{\alpha} \rangle, \; i \neq 1,2 \\
\langle \hat{b}_3, \hat{\alpha} \rangle &= \langle \hat{b}_4, \hat{\alpha} \rangle \leqslant \langle \hat{b}_i, \hat{\alpha} \rangle, \; i \neq 3,4
\end{aligned}$$

$$\begin{aligned}
\langle a_1 - a_2, \alpha \rangle &= \omega_1(a_2) - \omega_1(a_1) \\
\langle a_1 - a_3, \alpha \rangle &\leqslant \omega_1(a_3) - \omega_1(a_1) \\
&\vdots \\
\langle a_1 - a_{10}, \alpha \rangle &\leqslant \omega_1(a_{10}) - \omega_1(a_1) \\
\langle b_3 - b_4, \alpha \rangle &= \omega_2(b_4) - \omega_2(b_3) \\
\langle b_3 - b_i, \alpha \rangle &\leqslant \omega_2(b_i) - \omega_2(b_3), \; i \neq 3,4
\end{aligned}$$

$$
\begin{aligned}
\langle \hat{a}_1, \hat{\alpha} \rangle &= \langle \hat{a}_2, \hat{\alpha} \rangle \leqslant \langle \hat{a}_i, \hat{\alpha} \rangle, \; i \neq 1, 2 \\
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\end{aligned}
$$

$$
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\langle a_1 - a_3, \alpha \rangle &\leqslant \omega_1(a_3) - \omega_1(a_1) \\
&\vdots \\
\langle a_1 - a_{10}, \alpha \rangle &\leqslant \omega_1(a_{10}) - \omega_1(a_1) \\
\langle b_3 - b_4, \alpha \rangle &= \omega_2(b_4) - \omega_2(b_3) \\
\langle b_3 - b_i, \alpha \rangle &\leqslant \omega_2(b_i) - \omega_2(b_3), \; i \neq 3, 4
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{a}_1, \hat{\alpha} \rangle &= \langle \hat{a}_2, \hat{\alpha} \rangle \leqslant \langle \hat{a}_i, \hat{\alpha} \rangle, \ i \neq 1,2 \\
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&\vdots \\
\langle a_1 - a_{10}, \alpha \rangle &\leqslant \omega_1(a_{10}) - \omega_1(a_1) \\
\langle b_3 - b_4, \alpha \rangle &= \omega_2(b_4) - \omega_2(b_3) \\
\langle b_3 - b_i, \alpha \rangle &\leqslant \omega_2(b_i) - \omega_2(b_3), \ i \neq 3,4
\end{aligned}
$$

$$
\begin{array}{ccccc}
\hat S_1 & & \hat S_2 & & \hat S_3 \\
(\hat a_1, \hat a_2) & \rightarrow & (\hat b_1, \hat b_2) & \rightarrow & (\hat c_1, \hat c_2) \\
(\hat a_1, \hat a_5) & & (\hat b_1, \hat b_4) & & (\hat c_3, \hat c_5) \\
\vdots & & \vdots & & \vdots \\
(\hat a_6, \hat a_9) & & & &
\end{array}
$$

$(a_1, a_2), (b_1, b_2), (c_1, c_2)$: $\exists\,?\quad \alpha \in \mathbb{R}^3, \quad \hat\alpha = (\alpha, 1)$

$$\langle \hat a_1, \hat\alpha \rangle = \langle \hat a_2, \hat\alpha \rangle \leqslant \langle \hat a_j, \hat\alpha \rangle$$

$$\langle \hat b_1, \hat\alpha \rangle = \langle \hat b_2, \hat\alpha \rangle \leqslant \langle \hat b_k, \hat\alpha \rangle$$

$$\langle \hat c_1, \hat\alpha \rangle = \langle \hat c_2, \hat\alpha \rangle \leqslant \langle \hat c_l, \hat\alpha \rangle$$

$$
\begin{array}{ccccc}
\hat{S}_1 & & \hat{S}_2 & & \hat{S}_3 \\
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(\hat{a}_1, \hat{a}_5) & & (\hat{b}_1, \hat{b}_4) & & (\hat{c}_3, \hat{c}_5) \\
\vdots & & \vdots & & \vdots \\
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- T. Gao, T. Y. Li, M. Wu

  "**MixedVol**: A software package for mixed volume computation" *ACM Tran. on Math Software*, Vol. 31, No. 4 (2005) pp. 555-560.

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  "Dynamic enumeration of all mixed cells" *Discrete and Computational Geometry*, Vol. 37, (2007) pp. 351-367.

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system	size(n)	DEMiCs-0.95	MixedVol	speed-up
Cyclic-n	12	1m8.8s	4m43.0s	4.11
	13	10m54.7s	49m57.4s	4.58
	14	1h36m37.1s	7h14m24.1s	4.50
	15	15h45m26.0s	-	
Noon-n	16	1m4.9s	33m54.8s	31.38
	17	3m13.1s	2h25m20.8s	45.15
	18	7m38.3s	8h23m19.6	65.90
	19	28m1.0s	-	
	20	1h8m49.6	-	
	21	5h41m54.4s	-	
Eco-n	17	4m56.1s	20m41.8s	4.19
	18	19m31.8s	1h17m56.0s	3.99
	19	1h21m30.4s	4h56m4.6s	3.63
	20	5h41m54.4	-	

Table: 2.4GHz Itanium2 processor, 8GB RAM

eco-n	Total degree $= 2 \cdot 3^{n-2}$

$$
\begin{aligned}
(x_1 + x_1 x_2 + \cdots + x_{n-2} x_{n-1}) x_n - 1 &= 0 \\
(x_2 + x_1 x_3 + \cdots + x_{n-3} x_{n-1}) x_n - 2 &= 0 \\
&\vdots \\
x_{n-1} x_n - (n-1) &= 0 \\
x_1 + x_2 + \cdots + x_{n-1} + 1 &= 0
\end{aligned}
$$

noon-n	Total degree $= 3^n$

$$
\begin{aligned}
x_1 (x_2^2 + x_3^2 + \cdots + x_n^2 - 1.1) + 1 &= 0 \\
x_2 (x_1^2 + x_3^2 + \cdots + x_n^2 - 1.1) + 1 &= 0 \\
&\vdots \\
x_n (x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - 1.1) + 1 &= 0
\end{aligned}
$$

cyclic-n Total degree $= n!$
$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 &= 0 \\ x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{n-1}x_nx_1 + x_nx_1x_2 &= 0 \\ &\vdots \\ x_1x_2\cdots x_n - 1 &= 0 \end{aligned}$$

katsura-n Total degree $= 2^n$
$$\begin{aligned} 2x_{n+1} + 2x_n + \cdots + 2x_2 + x_1 - 1 &= 0 \\ 2x_{n+1}^2 + 2x_n^2 + \cdots + 2x_2^2 + x_1^2 - x_1 &= 0 \\ 2x_nx_{n+1} + 2x_{n-1}x_n + \cdots + 2x_2x_3 + 2x_1x_2 - x_2 &= 0 \\ 2x_{n-1}x_{n+1} + 2x_{n-2}x_n + \cdots + 2x_1x_3 + x_2^2 - x_3 &= 0 \\ &\vdots \\ 2x_2x_{n+1} + 2x_1x_n + 2x_2x_{n-1} + \cdots + 2x_{n/2}x_{(n+2)/2} - x_n &= 0 \\ 2x_2x_{n+1} + 2x_1x_n + 2x_2x_{n-1} + \cdots + x_{(n+1)/2}^2 - x_n &= 0 \end{aligned}$$

reimer-n	Total degree $= (n+1)!$

$$
\begin{aligned}
2x_1^2 - 2x_2^2 + \cdots + (-1)^{n+1} 2x_n^2 - 1 &= 0 \\
2x_1^3 - 2x_2^3 + \cdots + (-1)^{n+1} 2x_n^3 - 1 &= 0 \\
&\vdots \\
2x_1^{n+1} - 2x_2^{n+1} + \cdots + (-1)^{n+1} 2x_n^{n+1} - 1 &= 0
\end{aligned}
$$

Dell PC with a Pentium 4 CPU of 2.2GHz, 1GB of memory

Linear Programming

$$\begin{aligned} \max \quad & y^T b \\ & A^T y \leqslant c \end{aligned}$$

Its dual

$$\begin{aligned} \min \quad & c^T x \\ & Ax = b \\ & x \geqslant 0 \end{aligned}$$

Linear Programming

$$\begin{aligned} \max \quad & y^T b \\ A^T y & \leqslant c \end{aligned}$$

Its dual

$$\begin{aligned} \min \quad & c^T x \\ Ax & = b \\ x & \geqslant 0 \end{aligned}$$

Linear Programming

$$\begin{gathered} \max \quad y^T b \\ A^T y \leqslant c \end{gathered}$$

Its dual

$$\begin{gathered} \min \quad c^T x \\ Ax = b \\ x \geqslant 0 \end{gathered}$$

$$
\begin{array}{ccccc}
\hat{S}_1 & & \hat{S}_2 & & \hat{S}_3 \\
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(\hat{a}_1, \hat{a}_5) & & (\hat{b}_1, \hat{b}_4) & & (\hat{c}_3, \hat{c}_5) \\
\vdots & & \vdots & & \vdots \\
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\end{array}
$$

$(a_1, a_2), (b_1, b_2)$: $\exists\,?\quad \alpha \in \mathbb{R}^3, \quad \hat{\alpha} = (\alpha, 1)$

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$$
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\end{array}
$$

$(a_1, a_2), (b_1, b_2)$: $\exists\, ? \quad \alpha \in \mathbb{R}^3, \quad \hat{\alpha} = (\alpha, 1)$

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$$
\begin{array}{ccccc}
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Dynamic Enumeration

MixedVol-2.0

Dynamic Enumeration

MixedVol-2.0

System	MV	MixedVol-2.0	DEMiCs	Speep-up
Cyclic				
-12	500,352	2.41m	3.31m	1.37
-13	2,704,156	20.9m	29.5	1.41
-14	8,795,976	2.72 hr	4.06 hr	1.49
-15	35,243,520	23.9hr	37.8 hr	1.58
Noon				
-19	1,162,261,429	28.0m	70.6m	2.52
-20	3,486,784,361	1.32 hr	2.69 hr	2.04
-21	10,460,353,161	3.31 hr	9.46 hr	2.86
-22	31,381,059,565	7.12 hr	25.8 hr	3.62
-23	94,143,178,781	21.8 hr	74.4 hr	3.41
Eco				
-18	65,536	32.7m	52.3m	1.60
-19	131,072	2.19 hr	3.31 hr	1.51
-20	262,144	8.53hr	12.0hr	1.41
-21	524,288	28.1 hr	40.2 hr	1.43

Table: 1.6GHz Itanium2 processor, 8G RAM

System	MV	MixedVol-2.0	DEMiCs	Speep-up
Chandra				
-20	524,288	18.6m	76.3m	4.10
-21	1,048,576	46.8m	3.37hr	4.32
-22	2,097,152	2.36hr	8.63hr	3.66
-23	4,194,304	5.75hr	27.8hr	4.83
-24	8,388,608	18.5hr	75.2hr	4.06
Katsura				
-13	8,192	7.10m	11.0m	1.55
-14	16,384	31.5m	60.2m	1.91
-15	32,768	2.58 hr	5.14hr	1.99
-16	65,536	15.8hr	23.2hr	1.47
5 body	133,998,561	71.4s	(113s)	1.58

Table: 1.6GHz Itanium2 processor, 8G RAM

Reliability

vortex-n				
Thel system derived from clearing the denominators of $\sum_{k=1}^{n}[(x_{ik}^{-1}-1)(x_{jk}-x_{ik}-x_{ij})+(x_{jk}^{-1}-1)(x_{ik}-x_{jk}-x_{ij})]=0$ for $1\leqslant i<j\leqslant n$. $\frac{n(n+1)}{2}$ variables: $\{\, x_{ij} \mid 1\leqslant i<j\leqslant n\}$				
	vortex-4	vortex-5	vortex-6	
MV	80	8,333	4,792,772	

MixedVol-2.0 : 8,333
DEMiCs : 8,238, 8,268, 81,54

n-body				
The system derived from clearing the denominators of $\sum_{k=1}^{n}[(r_{ik}^{-3}-1)(r_{jk}^2-r_{ik}^2-r_{ij}^2)+(r_{jk}^{-3}-1)(r_{ik}^2-r_{jk}^2-r_{ij}^2)]=0$ for $1\leqslant i<j\leqslant n$. $\frac{n(n+1)}{2}$ variables: $\{\, r_{ij} \mid 1\leqslant i<j\leqslant n\}$				
	3-body	4-body	5-body	
MV	99	33,201	133,998,561	

gridanti-n
$$\left\{\begin{array}{l} s_{i,j}(c_{i-1,j}+c_{i+1,j}+c_{i,j-1}+c_{i,j+1}) \\ -c_{i,j}(s_{i-1,j}+s_{i+1,j}+s_{i,j-1}+s_{i,j+1})=0 \\ c_{n+1,j}=c_{1,j},\, c_{0,j}=c_{n,j},\, c_{i,n+1}=c_{i,1},\, c_{i,0}=c_{i,n} \\ s_{n+1,j}=-s_{1,j},\, s_{0,j}=-s_{n,j},\, s_{i,n+1}=-s_{i,1},\, s_{i,0}=-s_{i,n} \\ s_{i,j}^2+c_{i,j}^2-1=0 \end{array}\right.$$ for $1 \leqslant i,j \leqslant n$. $2n^2$ variables: $\{\, s_{i,j},\ c_{i,j} \mid 1 \leqslant i,j \leqslant n\}$

	gridanti-3	gridanti-4		
mixed volume	147,456	704,380,928		

sonic-n					
$$\begin{cases} P(x_{10}+x_{11}+x_{12})+x_kx_9+1=0 \\ Qx_4x_5+X(x_4^2+P)+(x_7+1)x_4+x_kx_6+x_{9+k}x_9+1=0 \\ QXx_4+(q_1^2+q_2^2+\cdots+q_n^2)x_5+(x_7+1)Q+x_5x_8+1=0 \\ QXx_5+X^2x_4+(x_7+1)X+x_4x_8+1=0 \\ x_{10}x_1+x_{11}x_2+x_{12}x_3-1=0 \\ Qx_5+Xx_4+x_7+1=0 \\ x_1^2+x_2^2+x_3^2-1=0 \\ x_4^2+x_5^2-1 \\ q_j(p_j+x_5^2)+Xx_4x_5+(x_7+1)x_5+1=0 \\ q_j^2+X^2+(x_{10}+1)^2+(x_{11}+1)^2+(x_{12}+1)^2=0 \end{cases}$$ for $k=1,2,3$ and $j=1,2,\cdots,n$, where $P=\sum_{i=1}^{n} p_i, Q=\sum_{i=1}^{n} q_i$ and $X=\sum_{i=1}^{3} x_i$. $2n+12$ variables: $\{p_j, q_j \mid 1 \leqslant j \leqslant n\} \cup \{x_1, x_2, \cdots, x_{12}\}$					
MV	sonic-1	sonic-2	sonic-3	sonic-4	sonic-5
	1,304	8,032	29,696	96,256	293,120
MV	sonic-6	sonic-7	sonic-8	sonic-9	sonic-10
	852,992	2,395,136	6,533,120	17,395,712	45,383,680
MV	sonic-11	sonic-12	sonic-13	sonic-14	sonic-15
	116,342,784	293,732,352	731,709,440	1,801,191,424	4,386,979,840

A private company "BBN Technologies"

Thank you!