Vizing's Conjecture and Techniques from Computer Algebra

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Computational and Applied Math, Rice University

joint work in progress with I.V. Hicks[1]

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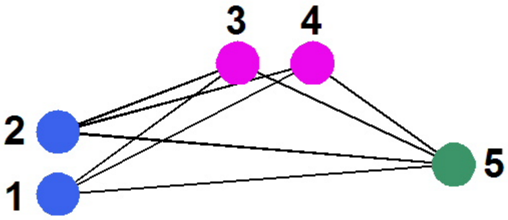

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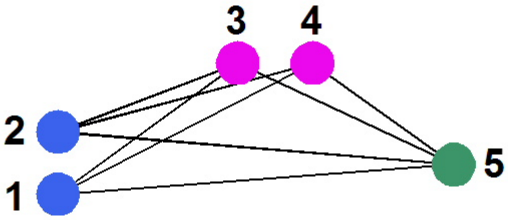

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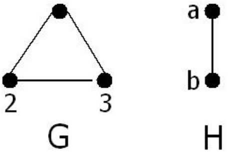

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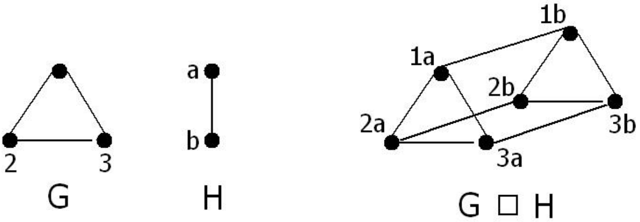

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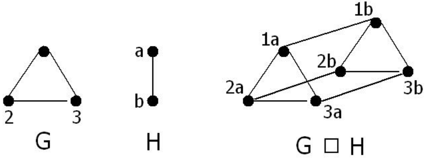

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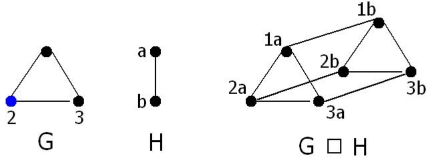

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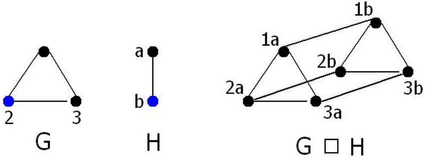

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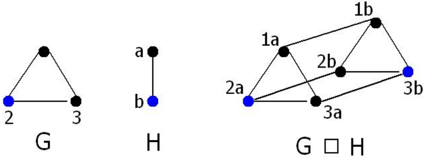

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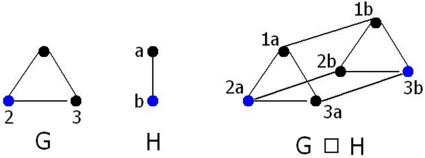

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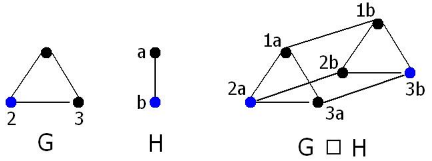

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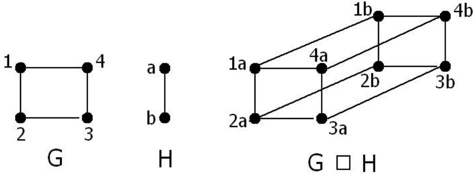

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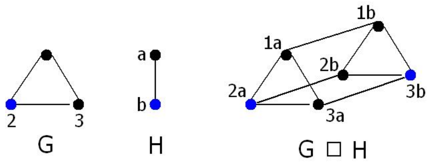

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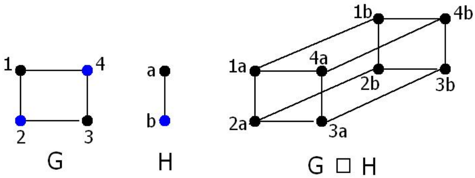

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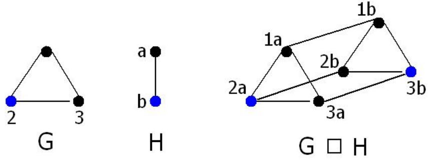

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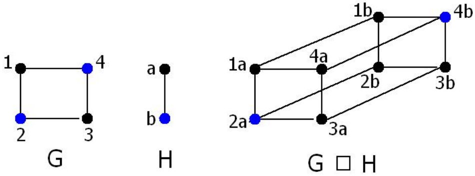

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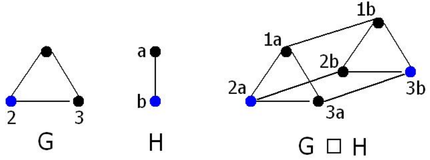

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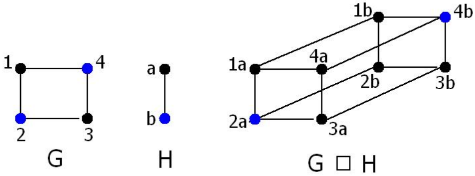

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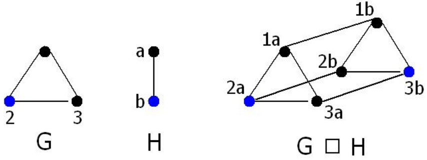

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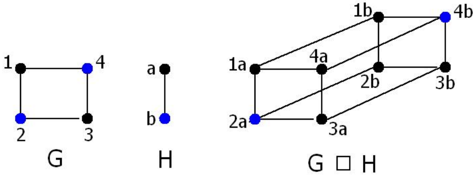

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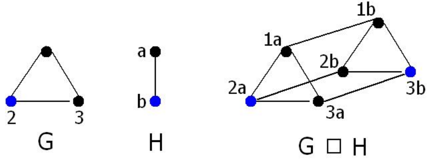

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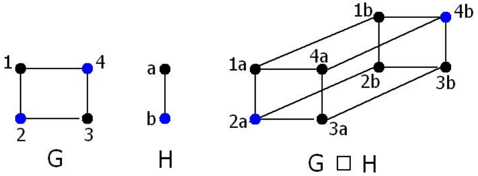

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Vizing's Conjecture

Vizing's Conjecture (1963)

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- In 2003, Sun proves that Vizing's conjecture holds if $\gamma(G) \leq 3$.

A given graph G and a dominating set of size k

Lemma

Given a graph G with n vertices, the following zero-dimensional system of polynomial equations has a solution if and only if there exists a dominating set of size k in G.

$$x_i^2 - x_i = 0 \; , \quad \textit{for } i = 1, \ldots, n \; ,$$

$$(1 - x_i) \prod_{j:(i,j) \in E(G)} (1 - x_j) = 0 \; , \quad \textit{for } i = 1, \ldots, n \; ,$$

$$-k + \sum_{i=1}^{n} x_i = 0 \; .$$

An arbitrary graph G in n vertices and a dominating set of size k

Lemma

The following zero-dimensional system of polynomial equations has a solution if and only if there exists a graph G in n vertices that has a dominating set of size k.

$$\begin{aligned} x_i^2 - x_i = 0 \;, &\quad \text{for } i = 1, \ldots, n \;, \\ e_{ij}^2 - e_{ij} = 0 \;, &\quad \text{for } i, j = 1, \ldots, n \text{ with } i < j \;, \\ (1 - x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - e_{ij} x_j) = 0 \;, &\quad \text{for } i = 1, \ldots, n \;, \\ -k + \sum_{i=1}^{n} x_i = 0 \;. & \end{aligned}$$

An arbitrary graph G in n vertices and a particular dominating set of size k

Lemma

The following zero-dimensional system has a solution if and only if there exists a graph G in n vertices that has a dominating set of size k consisting of vertices $\{v_1, v_2, \ldots, v_k\}$.

$$e_{ij}^2 - e_{ij} = 0 \ , \quad \textit{for } i,j = 1,\ldots,n \textit{ with } i < j,$$

$$\prod_{j=1}^{k}(1 - e_{ij}) = 0 \ , \quad \textit{for } i = k+1,\ldots,n,$$

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Let $\mathscr{P}_H$ be the set of polynomials representing a graph H in n' vertices with a dominating set of size l:

$$e'^2_{ij} - e'_{ij} = 0 \;, \quad \text{for } 1 \leq i < j \leq n',$$

$$\prod_{S \in S_{n'}^l} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e'_{ij}) \right) \right) = 0 \;.$$

Notation Definitions (continued)

Let $\mathscr{P}_{G\Box H}$ be the set of polynomials representing the cartesian product graph $G\Box H$ with a dominating set of size r:

For $i = 1, \ldots, n$ and $j = 1, \ldots, n'$,

$$z_{ij}^2 - z_{ij} = 0 \ ,$$

$$(1 - z_{ij}) \prod_{k=1}^{n} (1 - e_{ik} z_{kj}) \prod_{k=1}^{n'} (1 - e'_{jk} z_{ik}) = 0 \ ,$$

and

$$-r + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} = 0 \ ,$$

The ideal I_k^l and variety V_k^l

Lemma

The system of polynomial equations $\mathscr{P}_G, \mathscr{P}_H$ and $\mathscr{P}_{G \Box H}$ has a solution if and only if there exist graphs G, H in n, n' vertices respectively with dominating sets of size k, l respectively such that their cartesian product graph $G \Box H$ has a dominating set of size r.

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Let $V_k^l := V(I_k^l)$.
Note that $I(V_k^l) = I_k^l$ since the ideal I_k^l is *radical*.

Unions, Intersections and Vizing's Conjecture

Theorem

Vizing's conjecture is true $\iff V^{l}_{k-1} \cup V^{l-1}_{k} = V^{l}_{k}$.

Proof.

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Corollary

Vizing's conjecture is true $\iff I_{k-1}^{l} \cap I_{k}^{l-1} = I_{k}^{l}$.

Searching for a Counter-Example by Counting Solutions

Recall

$$|V(I)| = \# \text{ of solutions} = \dim\left(\frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{I}\right)$$

Lemma

If

$$\dim\left(\frac{\mathbb{C}[e, e', z]}{I^l_{k-1} \cap I^{l-1}_k}\right) < \dim\left(\frac{\mathbb{C}[e, e', z]}{I^l_k}\right)$$

for any n, n', k, l, then Vizing's conjecture is false. Moreover, there exists a counter-example for Vizing's conjecture for graphs G, H, with n, n' vertices and $\gamma(G), \gamma(H)$ equal to k, l, respectively.

Vizing's Conjecture and Gröbner Bases

Let

$$\mathscr{P}'_{G\Box H} := \mathscr{P}_{G\Box H} \setminus \left\{ -(kl - l) + \sum_{i=1}^{n}\sum_{j=1}^{n'} z_{ij} \right\}$$

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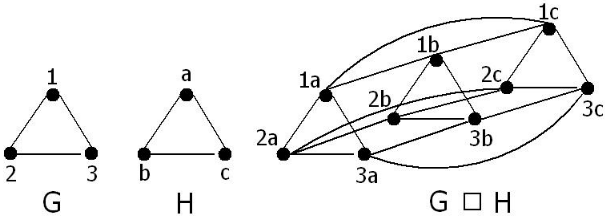
Conjecture

Is the following set of polynomials (described by cases 1 through 6) a graph-theoretic interpretation of the unique, reduced Gröbner basis of $\mathscr{P}'_{G\Box H}$?

Vizing's Conjecture and Gröbner Bases: Degree



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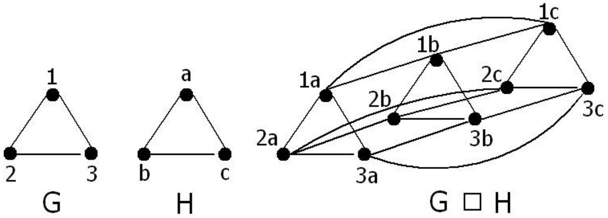

Every polynomial in the Gröbner basis has the following form:

$$(x_{i_1} - 1)(x_{i_d} - 1) \cdots (x_{i_D} - 1) \ ,$$

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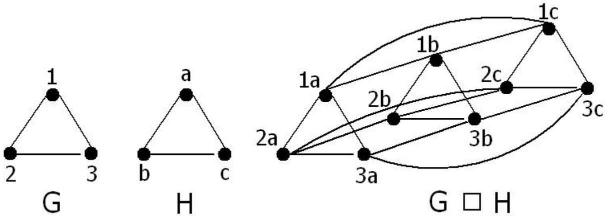

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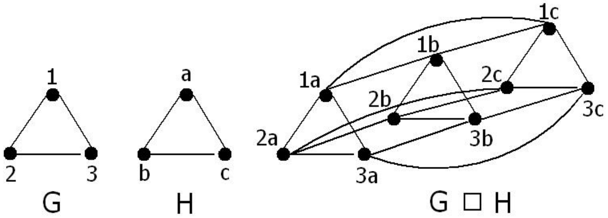
In the $\mathscr{P}'_{\text{tri}\Box\text{tri}}$ example, the degree equals five.

Vizing's Conjecture and Gröbner Bases: Case 1



Notation: Let $\mathscr{G}$ represent the set of G-levels in $G\Box H$. Given a level $l \in \mathscr{G}$, let

$$p(l) := \prod_{i \in V(l)} (x_i - 1) \, .$$

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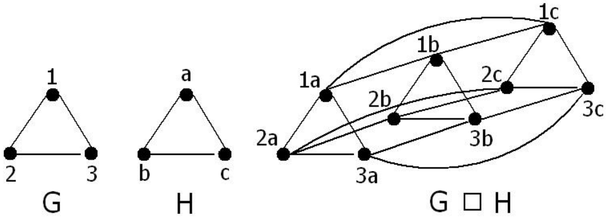

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Example: Consider the a-level in tri□tri. Then,

$$p(a) := (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1) \ .$$

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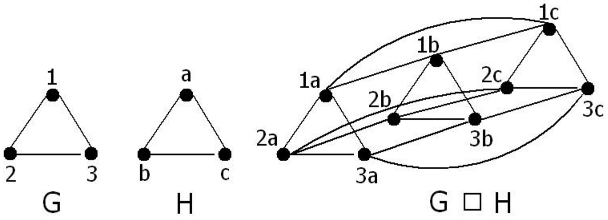

Case 1: There are $|G| \cdot |H|$ polynomials of the form:

$$p(g) \cdot \prod_{\substack{l \in \mathscr{G}: \\ l \neq g}} (x[l_i] - 1) \ , \qquad \text{for each } i \in V(G) \text{ and each level } g \in \mathscr{G}.$$

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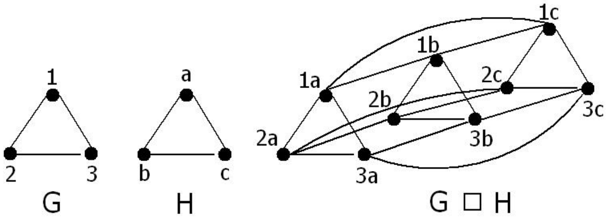

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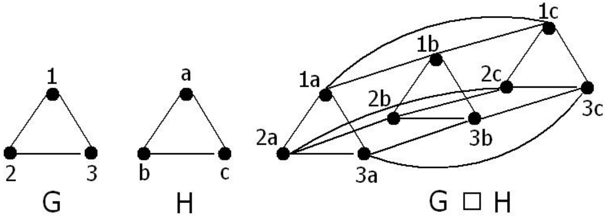
Example: For $g =$ a-level and $i = 1$, then

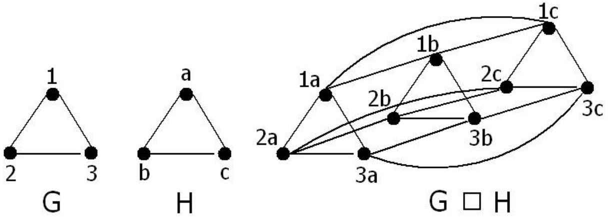
$$(z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1)(z_{1b} - 1)(z_{1c} - 1)$$

Vizing's Conjecture and Gröbner Bases: Case 2



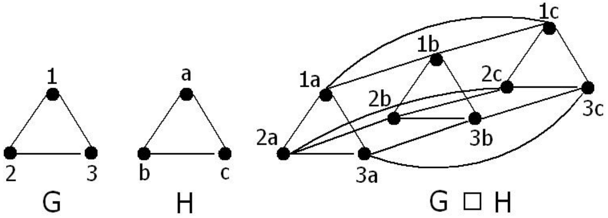
Notation: Let $e \in E[H]$. In $G \Box H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \Box H$.

Vizing's Conjecture and Gröbner Bases: Case 2



Notation: Let $e \in E[H]$. In $G \square H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \square H$. In particular, let $h(e)$ define the G-level that where the edge originates (according to the lexicographic order), and let $t(e)$ denote the G-level where the edge terminates.

Vizing's Conjecture and Gröbner Bases: Case 2



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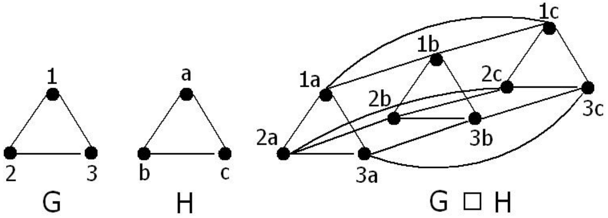
Example: Consider the edge e'_{ac} and the c-level in tri□tri. Then,

$$p(h(e)) := (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1) \ ,$$
$$p(t(e)) := (z_{1c} - 1)(z_{2c} - 1)(z_{3c} - 1) \ .$$

Vizing's Conjecture and Gröbner Bases: Case 2



Case 2: There are $2||H|| \cdot |G| + 2||G|| \cdot |H|$ polynomials of the following form:

$$(x_e - 1)p(h(e)) \prod_{\substack{g \in \mathcal{G}:\ g \neq \mathcal{G}[t(e)] \\ \text{and } g \neq \mathcal{G}[h(e)]}} (g_i - 1)\ , \qquad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

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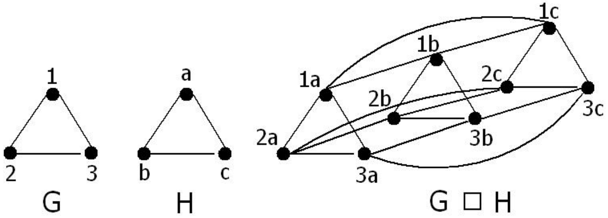

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Example: For $e = e'_{ac}$ and $i = 1$, then

$$(e'_{ac} - 1)(z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1)(z_{1b} - 1)\ ,$$
$$(e'_{ac} - 1)(z_{1c} - 1)(z_{2c} - 1)(z_{3c} - 1)(z_{1b} - 1)\ .$$

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Thank you for your kind attention!
Questions, comments, thoughts and suggestions are most welcome.